Hierarchical Gaussian Topologies

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ABSTRACT

We present in this paper some of the properties of hierarchical Gaussian networks HG_β, an interesting hierarchical-type structure built up with dense Midimews. We also provide the optimal routing algorithm for these structures. Such networks may be considered in the design of efective and optimized interconnection networks.

KEYWORDS: Interconnection Networks; Hierarchical Networks; Routing

1 Introduction

Definition 1 *Given $t \leq b$, a vertex α is said to t-*dominate *a vertex β if $\beta \in B_t(\alpha)$, where*

$$B_t(\alpha) = \{\gamma \in \mathbb{Z}[i]_\pi \mid D(\gamma, \alpha) \leq t\}$$

is the ball of radii t centered in α embedded in G_π. Then, a vertex subset S is called a perfect t-dominating set *if every vertex of G_π is t-dominated by a unique vertex in S.*

In a recent paper, we have introduced new hierarchical-type interconnection networks based in Gaussian networks, an interesting class of degree four Circulant graphs, whose nodes are labeled by means of a subset of the Gaussian integers [5]. In this paper, we analyze these hierarchical networks of Gaussian networks, providing an optimal routing algorithm. Hierarchical Gaussian networks can be considered as good candidates for implementing parallel systems based on chip multiprocessors (M-CMPs). As presented in [6], in such systems two interconnection levels are needed, one for on-chip communications and another joining different chips.

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2 Related Work

Let $\mathbb{Z}[i] = \{x + yi \mid x, y \in \mathbb{Z}\}$ be the Euclidean ring of the Gaussian integers, where $\mathbb{Z}$ denotes the ring of the integers. We denote as $\mathcal{N}(x + yi) = x^2 + y^2$ the norm of $x + yi \in \mathbb{Z}[i]$. If $0 \neq \pi = a + bi \in \mathbb{Z}[i]$, we consider $\mathbb{Z}[i]_\pi$ the ring of the classes of $\mathbb{Z}[i]$ modulo the ideal (π). This is clearly a finite set and in [4] it is proved that if $\gcd(a, b) = 1$, then $\mathbb{Z}[i]_\pi$ and $\mathbb{Z}_N$ are isomorphic rings, where $N = a^2 + b^2$. Next, we define Gaussian networks.

Definition 2 *(From [4]) Let $\pi = a + bi \in \mathbb{Z}[i]$ such that $\gcd(a, b) = 1$. We define the **Gaussian network generated by** π, $G_\pi = (V, E)$, as follows:*

- *$V = \mathbb{Z}[i]_\pi$ is the node set.*
- *$E = \{(\alpha, \beta) \in V \times V \mid \beta - \alpha \equiv \pm 1, \pm i \pmod{\pi}\}$ is the links set.*

These networks are regular graphs of degree four and in [4] we proved that they are isomorphic to $C_{a^2+b^2}(a, b)$, the circulant graph of $N = a^2 + b^2$ nodes and jumps $\pm a$, $\pm b$. The Gaussian network with maximum number of nodes for a given diameter $t > 0$ is generated by $\pi = t + (t + 1)i$, having $N = 2t^2 + 2t + 1$ nodes. We call them **dense** Gaussian networks and are isomorphic to the dense Midimew of N nodes and jumps t and $t + 1$ introduced in [2]. These networks are the base of our hierarchical structure.

3 Hierarchical Gaussian Networks

Currently, the increasing number of transistors per chip is facilitating the design of Chip Multiprocessors (CMPs) as they provide high performance and cost-effective computing for many scientific and commercial workloads. By combining multiple CMPs higher performance can be achieved. Such is the case of Piranha [1] and IBM Power4 [7]. Hence, it is necessary exploring new networks whose topological properties match the new requirements imposed by these emerging architectures. We explore in this section hierarchical Gaussian networks as possible candidates for implementing such two-level interconnection networks.

Definition 3 *Given t a positive integer let $\beta = t + (t + 1)i$. We define the **2-Level hierarchical Gaussian network** $\mathbf{HG}_\beta = (V, E)$ of G_β as follows:*

- *$V = \mathbb{Z}[i]_\beta \times \mathbb{Z}[i]_\beta$*
- *$E = \{((\alpha_{1,1}, \alpha_{1,2}), (\alpha_{1,1}, \alpha_{2,2})) \in V \times V \mid \alpha_{1,2} = \alpha_{2,2}$ and $d(\alpha_{1,1}, \alpha_{2,1}) = 1$, or $\alpha_{1,1} = \alpha_{2,1} = 0$ and $d(\alpha_{1,2}, \alpha_{2,2}) = 1\}$, where d is the graph distance in G_β.*

An intuitive visualization of how to build this network is to take N dense Gaussian networks of N nodes and join their centers following the adjacency pattern of a dense Gaussian network of N nodes. An illustrative simple example with $\beta = 1 + 2i$ can be seen in Figure 1. Thus, we have that $\mathbf{HG}_\beta$ has N^2 nodes and $2N^2 + 2N$ links, where $N = \mathcal{N}(\beta) = 2t^2 + 2t + 1$. Also, it is clear that the diameter of this structure is also $3t$. This is neither a regular graph, as we have nodes of degree four and eight, nor a node-symmetric network.

Despite these facts, $\mathbf{HG}_\beta$ has $N + 1$ subgraphs isomorphic to G_β. The first N subgraphs correspond to the N dense Gaussian networks in the lower level of hierarchy. Their corresponding links are denoted **level-1** links. The last subgraph is the dense Gaussian network

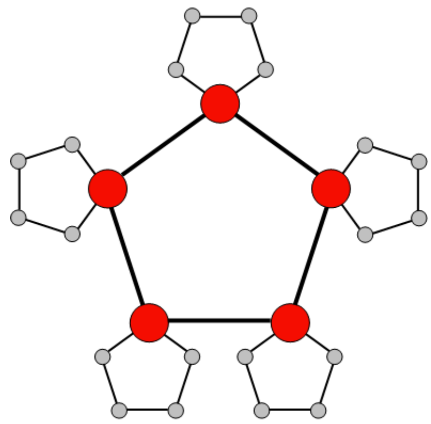

Figure 1: $\mathbf{HG}_{1+2i}$

in the higher level of hierarchy and its links are denoted **level-2** links.

The hierarchical Gaussian network has an edge-connectivity of 4, but a vertex-connectivity of 1. This is because if a vertex of the dense Gaussian network in the higher level of hierarchy falls then it disconnects the dense Gaussian network that is under this vertex from the rest of the network. In [5] we found the formula for the average distance of these networks.

Lemma 4 *Given t a positive integer. Let $\beta = t + (t+1)i$. The average distance of $\mathbf{HG}_\beta$ can be computed as $\frac{3N-1}{N+1}\frac{2t+1}{3}$, where $N = 2t^2 + 2t + 1$.*

4 Routing in Hierarchical Gaussian Networks

Many optimal algorithms for finding minimum paths have been proposed for Circulant graphs; see, for example, [3]. Such algorithms consider graphs whose nodes are labeled by means of integers and implement the Euclidean division algorithm which has a high computational cost. If we consider routing in terms of Gaussian integers, that is, in the isomorphic Gaussian network, we can propose a new algorithm just based on integer sums and comparisons that was introduced in [5].

Lemma 5 *Let t be a positive integer and $\beta = t + (t+1)i \in \mathbb{Z}[i]$. Let $\alpha = x + yi$ and $\alpha' = x' + y'i$ be such that $|x| + |y| \leq t$ and $|x'| + |y'| \leq t$. If $r \equiv \alpha' - \alpha \pmod{\beta}$ is such that $|Re(r)| + |Im(r)|$ minimum, then $r = (\alpha' - \alpha) + \gamma\pi$ with $\gamma \in \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}$.*

If we want to travel from α to α', we have to obtain $X + Yi \equiv (\alpha' - \alpha) \pmod{\pi}$ with $|X| + |Y|$ minimum. This Lemma implies that we only need to compare the graph weight of, at most, nine Gaussian integers to find the pair. We denote the obtained result as $(X, Y) := routing\_record(\alpha, \alpha', t)$. Note that $X + Yi$ gives us all the minimum paths from α to α' in the dense Gaussian network of diameter t.

Now, we give an idea of the routing in hierarchical Gaussian network. Although it is not a node-symmetric graph, we can use the symmetry of our network to obtain all the possible shortest paths from a source node $s = (\alpha_1, \alpha_2)$ to a destination node $d = (\alpha'_1, \alpha'_2)$. The idea of the algorithm is that, if both nodes are in the same dense Gaussian network, then we can follow a direct routing. In other cases, the routing consists in going to the center of the corresponding dense Gaussian network, then go up to the higher level of hierarchy and attain the destination dense Gaussian network. Now, we just have to go to the destination node following the links in the lower level of hierarchy. Next, we give the complete algorithm.

Data: $s = (\alpha_1, \alpha_2)$: Source node; $d = (\alpha'_1, \alpha'_2)$: Destination node; $\beta = t + (t+1)i$: graph generator
Result: $(x_1, y_1, x_2, y_2, x_3, y_3)$ such that: (x_1, y_1) level-1 links; (x_2, y_2) level-2 links; (x_3, y_3) level-1 links

begin
  if $(\alpha_2 = \alpha'_2)$ **then**
    $(x_1, y_1) = routing\_record(\alpha_1, \alpha'_1, t)$ // links to attain d directly;
    $(x_2, y_2) = (x_3, y_3) = (0, 0)$;
  else
    $(x_1, y_1) := routing\_record(\alpha_1, 0, t)$ // links to reach the center;
    $(x_2, y_2) := routing\_record(\alpha_2, \alpha'_2, t)$ // links in the higher level of hierarchy ;
    $(x_3, y_3) := routing\_record(0, \alpha'_1, t)$ // links to attain d;
  end
end

Algorithm 1: Routing Algorithm for $\mathbf{HG}_\beta$.

5 Conclusions

We have introduced hierarchical Gaussian networks $\mathbf{HG}_\beta$, a hierarchical-type structure that may be suitable for implementing parallel systems based on chip multiprocessors. Furthermore, we have solved the minimal routing problem in dense Gaussian and in hierarchical Gaussian networks, using just integer additions and comparisons.

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