

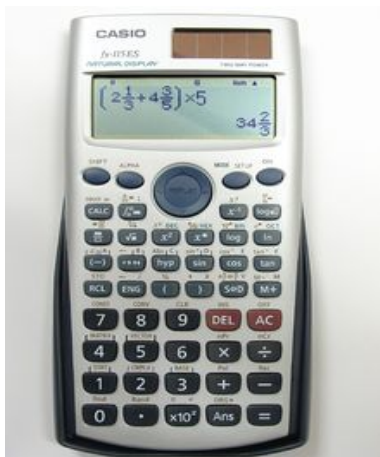
# Funciones matemáticas y cómo (no) calcularlas.

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**Herramientas del matemático:** papel, bolígrafo, libros, revistas, el cerebro y algunas extensiones:



## ¿Qué podemos hacer con una calculadora?

Podemos evaluar:

- 1 Operaciones elementales:  $+$ ,  $\times$ ,  $/$ ,  $\sqrt{\quad}$
- 2 Funciones elementales: exponenciales, trigonométricas y sus inversas.

Y poco más. ¿Es eso suficiente? Pues no, y por eso utilizamos más extensiones:



Hay muchas “extensiones” con variadas funciones (no solo hacer números): Maple, Mathematica, Matlab, Octave, Sage,...

Este *software* proporciona, además de las funciones elementales, otras muchas que son importantes. Por mencionar algunas que no suelen estar en calculadoras:

1 La integral seno:  $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$

2 La función error:  $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$

3 La función Gamma:  $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$

4 Funciones de Airy: soluciones de  $y''(x) - xy(x) = 0.$

5 Funciones del cilindro parabólico: soluciones de  $y''(x) - \left(\frac{x^2}{4} + a\right) y(x) = 0.$

(aunque estas últimas no están en todas las plataformas y cuando están... veremos)

El *software* disponible no es suficiente y queda un buen número de funciones matemáticas útiles que necesitan nuevo *software*, y otras que pueden calcularse mejor.

Éste es uno de los objetivos principales de nuestro trabajo, que tiene varias vertientes:

- 1 Análisis matemático: obtención de nuevas fórmulas, aproximaciones (relaciones de recurrencia, representaciones integrales, aproximaciones asintóticas) y cotas.
- 2 Análisis numérico: desarrollo de algoritmos numéricos, análisis de estabilidad, estimación de errores.
- 3 Ciencias de la computación: desarrollo de *software numérico* y librerías numéricas.

No solo es que se necesiten métodos para funciones nuevas, es que además no todo el *software* disponible funciona siempre bien.

$U(a, x)$ : solución recesiva de  $y''(x) - \left(\frac{x^2}{4} + a\right)y(x) = 0$  cuando  $x \rightarrow +\infty$ .

```

> restart; f:=CylinderU(100, x);
                                     f:= CylinderU(100, x)
(1)

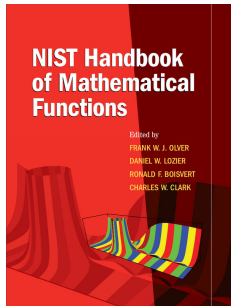
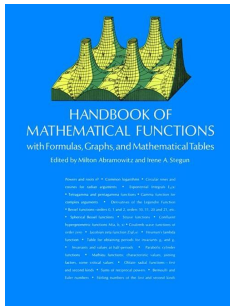
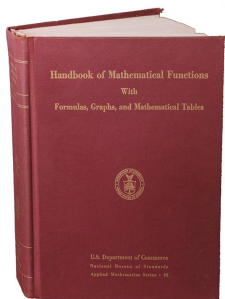
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(2)

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6 10612
(3)

> Digits:=1000;evalf(subs(x=100, f));
                                     Digits:= 1000
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284714275197587407858298122758686633955523264272302698688425232159430312175844299906968954873951,
4086215093774041499820468877 10-1287
(4)
  
```

Para obtener 10 dígitos correctos se necesita trabajar con más de 482 dígitos!!

## Compendios “taxonómicos” (necesariamente incompletos)



## 1964: Mathematics Handbook Becomes Best Seller

From making a military map to explaining the knock in gasoline engines or the light scattering that produces a rainbow, many technical and scientific challenges are best solved with the aid of mathematical functions. Such functions are so important that a national project devoted to compiling tables of them was established in the late 1930s. Thus began a NIST tradition of publishing mathematics reference data.

The invention of computers threatened to make the tables obsolete. But at a national conference in 1954, it was agreed that computers merely changed how the tables should be designed. For instance, for use in programming computers, the figures would need to be more accurate than before. So Milton Abramowitz and Irene A. Stegun of NIST produced an updated compendium, an effort that took eight years.

More than 1,000 pages long, the Handbook of Mathematical Functions was first published in 1964 and reprinted many times, with yet another reprint in 1999. Its influence on science and engineering is evidenced by its popularity. In fact, when *New Scientist* magazine recently asked some of the world's leading scientists what single book they would want if stranded on a desert island, one distinguished British physicist said he would take the Handbook.

The Handbook is likely the most widely distributed and most cited NIST technical publication of all time. Government sales exceed 150,000 copies, and an estimated three times as many have been reprinted and sold by commercial publishers since 1965. During the mid-1990s, the book was cited every 1.5 hours of each working day. And its influence will persist as it is currently being updated in digital format by NIST.



## 19. Parabolic Cylinder Functions

## Mathematical Properties

## 19.1. The Parabolic Cylinder Functions

## Introductory

These are solutions of the differential equation

$$19.1.1 \quad \frac{d^2 y}{dx^2} + (ax^2 + bx + c)y = 0$$

with two real and distinct standard forms

$$19.1.2 \quad \frac{d^2 y}{dx^2} - (\frac{1}{2}x^2 + a)y = 0$$

$$19.1.3 \quad \frac{d^2 y}{dx^2} + (\frac{1}{2}x^2 - a)y = 0$$

The functions

$$19.1.4 \quad y(a, x) \quad y(a, -x) \quad y(-a, ix) \quad y(-a, -ix)$$

are all solutions either of 19.1.2 or of 19.1.3 if any one is such a solution.

Replacement of  $a$  by  $-ia$  and  $x$  by  $xe^{i\pi/4}$  converts 19.1.2 into 19.1.3. If  $y(a, x)$  is a solution of 19.1.2, then 19.1.3 has solutions:

$$19.1.5 \quad y(-ia, xe^{i\pi/4}) \quad y(-ia, -xe^{i\pi/4}) \\ y(ia, -xe^{-i\pi/4}) \quad y(ia, xe^{-i\pi/4})$$

## 19.2.1

$$y_1 = e^{-1/2 x^2} M(\frac{1}{2}a + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}x^2)$$

$$= e^{-1/2 x^2} \left\{ 1 + (a + \frac{1}{2}) \frac{x^2}{2!} + (a + \frac{1}{2})(a + \frac{1}{2}) \frac{x^4}{4!} + \dots \right\}$$

$$= e^{-1/2 x^2} F_1(\frac{1}{2}a + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}x^2)$$

## 19.2.2

$$= e^{1/2 x^2} M(-\frac{1}{2}a + \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}x^2)$$

$$= e^{1/2 x^2} \left\{ 1 + (a - \frac{1}{2}) \frac{x^2}{2!} + (a - \frac{1}{2})(a - \frac{1}{2}) \frac{x^4}{4!} + \dots \right\}$$

## 19.2.3

$$y_2 = xe^{-1/2 x^2} M(\frac{1}{2}a + \frac{3}{2}, \frac{3}{2}, \frac{1}{2}x^2)$$

$$= e^{-1/2 x^2} \left\{ x + (a + \frac{1}{2}) \frac{x^3}{3!} + (a + \frac{1}{2})(a + \frac{1}{2}) \frac{x^5}{5!} + \dots \right\}$$

## 19.2.4

$$= xe^{1/2 x^2} M(-\frac{1}{2}a + \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}x^2)$$

$$= e^{1/2 x^2} \left\{ x + (a - \frac{1}{2}) \frac{x^3}{3!} + (a - \frac{1}{2})(a - \frac{1}{2}) \frac{x^5}{5!} + \dots \right\}$$

these series being convergent for all values of  $x$  (see chapter 13 for  $M(a, c, x)$ ).

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## PARABOLIC CYLINDER FUNCTIONS

Table 19.1

$x$	$U(1.5, x)$	$U(2.0, x)$	$U(2.5, x)$	$U(3.0, x)$	$U(3.5, x)$	$U(4.0, x)$	$U(4.5, x)$	$U(5.0, x)$
0.0	(0) 1.0000	(-1) 8.1085	(-1) 6.2666	(-1) 4.6509	(-1) 3.3333	(-1) 2.3167	(-1) 1.5666	(-1) 1.0335
0.1	(-1) 6.8187	(-1) 7.0232	(-1) 5.3409	(-1) 3.9060	(-1) 2.7615	(-1) 1.8950	(-1) 1.2662	(-2) 8.2588
0.2	(-1) 7.7700	(-1) 6.0787	(-1) 4.5492	(-1) 3.2786	(-1) 2.2067	(-1) 1.5494	(-1) 1.0230	(-2) 6.5971
0.3	(-1) 6.8389	(-1) 5.2566	(-1) 3.8719	(-1) 2.7501	(-1) 1.8924	(-1) 1.2662	(-2) 8.2604	(-2) 5.2673
0.4	(-1) 6.0120	(-1) 4.5410	(-1) 3.2925	(-1) 2.3050	(-1) 1.5650	(-1) 1.0340	(-2) 6.6663	(-2) 4.2032
0.5	(-1) 5.2778	(-1) 3.9182	(-1) 2.7969	(-1) 1.9302	(-1) 1.2931	(-2) 8.4374	(-2) 5.3758	(-2) 3.3518
0.6	(-1) 4.6262	(-1) 3.3763	(-1) 2.3731	(-1) 1.6146	(-1) 1.0674	(-2) 6.8788	(-2) 4.3316	(-2) 2.6777
0.7	(-1) 4.0482	(-1) 2.9051	(-1) 2.0109	(-1) 1.3490	(-2) 8.9019	(-2) 5.6025	(-2) 3.4869	(-2) 2.1262
0.8	(-1) 3.5360	(-1) 2.4957	(-1) 1.7015	(-1) 1.1256	(-2) 7.2491	(-2) 4.5579	(-2) 2.8040	(-2) 1.6910
0.9	(-1) 3.0825	(-1) 2.1403	(-1) 1.4375	(-2) 9.3785	(-2) 5.9624	(-2) 3.7035	(-2) 2.2523	(-2) 1.3434
1.0	(-1) 2.6816	(-1) 1.8321	(-1) 1.2124	(-2) 7.8022	(-2) 4.8971	(-2) 3.0053	(-2) 1.8068	(-2) 1.0660
1.1	(-1) 2.3276	(-1) 1.5651	(-1) 1.0208	(-2) 6.4802	(-2) 4.0160	(-2) 2.4351	(-2) 1.4475	(-3) 8.4479
1.2	(-1) 2.0157	(-1) 1.3343	(-1) 0.8373	(-2) 5.3727	(-2) 3.2880	(-2) 1.9701	(-2) 1.1579	(-3) 6.6856
1.3	(-1) 1.7412	(-1) 1.1330	(-1) 0.7192	(-2) 4.4461	(-2) 2.6872	(-2) 1.5913	(-3) 9.2486	(-3) 5.2831
1.4	(-1) 1.5003	(-1) 0.95317	(-1) 0.6190	(-2) 3.6721	(-2) 2.1922	(-2) 1.2831	(-3) 7.3749	(-3) 4.1663
1.5	(-1) 1.2893	(-1) 0.81541	(-1) 0.50255	(-2) 3.0265	(-2) 1.7849	(-2) 1.0327	(-3) 5.8705	(-3) 3.2833
1.6	(-1) 1.1049	(-1) 0.68857	(-1) 0.41862	(-2) 2.4890	(-2) 1.4503	(-3) 8.2953	(-3) 4.6645	(-3) 2.5816
1.7	(-1) 0.94412	(-1) 0.57994	(-1) 0.34786	(-2) 2.0423	(-2) 1.1759	(-3) 6.6500	(-3) 3.6991	(-3) 2.0262
1.8	(-1) 0.80438	(-1) 0.48712	(-1) 0.28833	(-2) 1.6718	(-3) 9.5127	(-3) 5.3198	(-3) 2.9276	(-3) 1.5873
1.9	(-1) 0.68324	(-1) 0.40801	(-1) 0.23837	(-2) 1.3652	(-3) 7.6780	(-3) 4.2463	(-3) 2.3122	(-3) 1.2409
2.0	(-2) 5.7853	(-2) 3.4076	(-2) 1.9653	(-2) 1.1120	(-3) 6.1823	(-3) 3.3818	(-3) 1.8222	(-4) 9.6810
2.1	(-2) 4.8830	(-2) 2.8375	(-2) 1.6159	(-3) 9.0339	(-3) 4.9656	(-3) 2.6869	(-3) 1.4328	(-4) 7.5364
2.2	(-2) 4.1080	(-2) 2.3556	(-2) 1.3248	(-3) 7.3193	(-3) 3.9782	(-3) 2.1796	(-3) 1.1240	(-4) 5.8538
2.3	(-2) 3.4444	(-2) 1.9495	(-2) 1.0829	(-3) 5.9138	(-3) 3.1787	(-3) 1.6837	(-3) 8.7960	(-4) 4.5364
2.4	(-2) 2.8782	(-2) 1.6082	(-2) 0.8260	(-3) 4.7646	(-3) 2.5331	(-3) 1.3277	(-4) 6.8665	(-4) 3.5071
2.5	(-2) 2.3966	(-2) 1.3223	(-3) 7.1710	(-3) 3.8275	(-3) 2.0129	(-3) 1.0442	(-4) 5.3467	(-4) 2.7047
2.6	(-2) 1.9886	(-2) 1.0837	(-3) 5.8081	(-3) 3.0655	(-3) 1.5951	(-4) 8.1895	(-4) 4.1523	(-4) 2.0806
2.7	(-2) 1.6441	(-3) 8.8509	(-3) 4.6891	(-3) 2.4478	(-3) 1.2603	(-4) 6.4052	(-4) 3.2161	(-4) 1.5964
2.8	(-2) 1.3544	(-3) 7.2040	(-3) 3.7734	(-3) 1.9484	(-4) 9.9277	(-4) 4.9954	(-4) 2.4881	(-4) 1.2216
2.9	(-2) 1.1116	(-3) 5.8431	(-3) 3.0264	(-3) 1.5460	(-4) 7.7967	(-4) 3.8845	(-4) 1.9134	(-5) 9.3228

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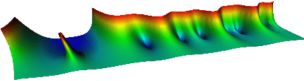
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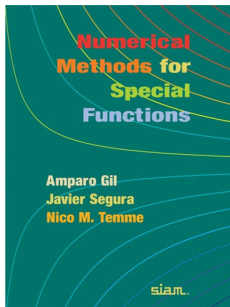
**Project News**

2018-09-15 [DLMF Update: Version 1.0.20](#)  
 2018-06-22 [DLMF Update: Version 1.0.19](#)  
 2018-06-22 [Philip J. Davis, A&S Author, dies at age 95](#)  
 2018-03-27 [DLMF Update: Version 1.0.18](#)  
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No hay, por supuesto, tablas numéricas, pero tampoco *software*.

Nuestra aportación numérica (año 2007):



Extraído de la introducción: “In the present book we are not so formidably optimistic that we claim to describe computational methods or algorithms for all functions described in the old and new version of the Handbook”

Hemos avanzado desde que se publicó el libro, pero queda muchísimo trabajo por delante.

## Otro ejemplo de cómo NO calcular funciones (en este caso un número):

Dado  $\psi = (1 - \sqrt{5})/2$ , comprobamos que las potencias  $y_n = \psi^n$  satisfacen la recurrencia de Fibonacci:

$$y_{n+1} = y_n + y_{n-1},$$

Esto lo podríamos utilizar para calcular de forma sencilla, por ejemplo,  $\psi^{40}$  mediante 39 sumas partiendo de  $y_0 = 1$ ,  $y_1 = \psi$  (que es más económico que hacer el mismo número de multiplicaciones):

$$y_2 = y_1 + y_0, y_3 = y_2 + y_1, \dots, y_{50} = y_{49} + y_{48}$$

Para no calcular mucho, hacemos los cálculos con 8 cifras y para comprobar que todo está bien, podemos por ejemplo ver si

$$y_{40}/y_{39} = \psi^{40}/\psi^{39} = (1 - \sqrt{5})/2 \simeq -0.6180340$$

Pues lo que se obtiene es...

```

> restart;
Digits:=8;
phi[0]:=1;phi[1]:=evalf((1-sqrt(5))/2);
for i from 1 to 39 do;phi[i+1]:=phi[i]+phi[i-1];end;

                                 $\phi_0 := 1$ 
                                 $\phi_1 := -0.61803400$  (1)
> phi[40]/phi[39];
                                1.6180340 (2)
> evalf((1+sqrt(5))/2);
                                1.6180340 (3)
>

```

Un análisis matemático (muy elemental) nos ha permitido encontrar una relación sencilla, pero el análisis numérico (de estabilidad) nos explica por qué no hay que utilizarla (y el *software* utilizando esta relación ni nos lo tenemos que plantear).