Polytopes With Large Signature

Joint work with Michael Joswig

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Outline

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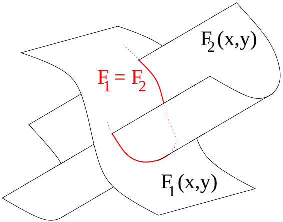
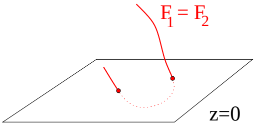
Real Solutions of Polynomial Systems

- A generic system $\mathcal{S}$

$$F_1(t_1,\ldots,t_n) \;=\; \ldots \;=\; F_n(t_1,\ldots,t_n) \;=\; 0$$

  of n real polynomial equations has finitely many real solutions.
- In general it is extremely difficult to compute the real solutions of $\mathcal{S}$.
- Not even the number of real solutions can be computed easily, or in fact if there are any solutions at all.





Real Solutions of Polynomial Systems

Theorem (SOPRUNOVA & SOTTILE '04)

Let $\mathcal{N}$ be a lattice polytope and let $\mathcal{N}_\omega$ be a convex and balanced triangulation of $\mathcal{N}$.

Then there is an associated system $\mathcal{S}(\mathcal{N}_\omega)$ of real polynomial equations and the number of real solutions of $\mathcal{S}(\mathcal{N}_\omega)$ is at least the signature $\sigma(\mathcal{N}_\omega)$ of $\mathcal{N}_\omega$.

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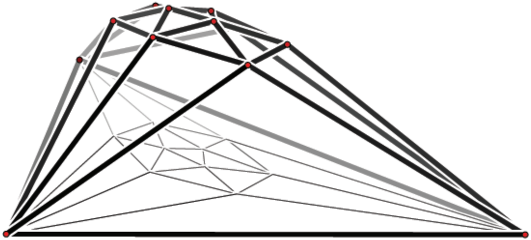

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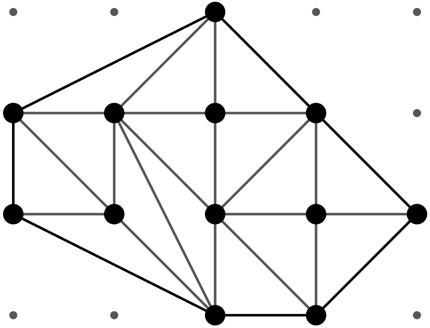

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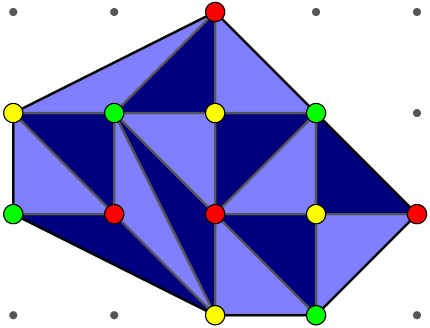

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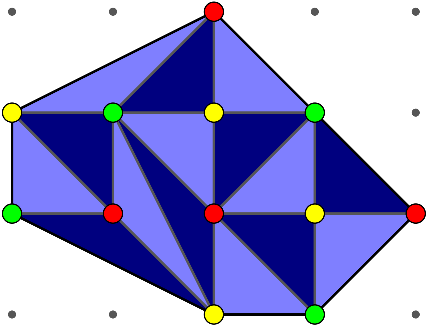

Polytopes With Large Signature

- It is extremely difficult to determine the number of real solutions of a polynomial system.
- SOPRUNOVA & SOTTILE construct non trivial polynomial systems where the number of real solutions is bounded from below by the signature of a triangulation of the Newton Polytope.
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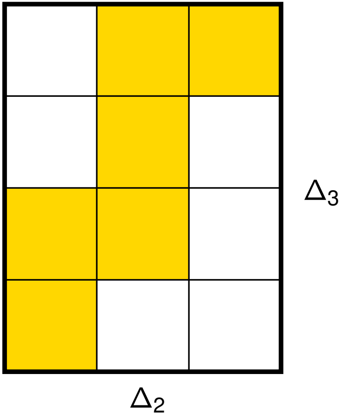
The Staircase Triangulation

The facet $(0, 1, 0, 0, 1)$ of $\mathrm{stc}(\Delta_2 \times \Delta_3)$.

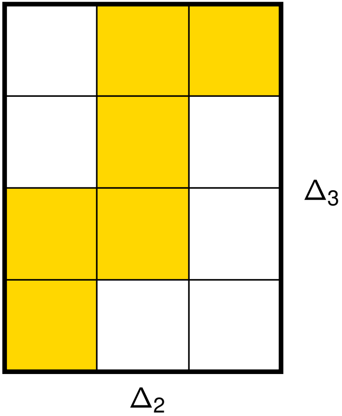
The triangulation $\mathrm{stc}(\Delta_2 \times \Delta_3)$ has $\binom{2+3}{2} = 10$ facets.



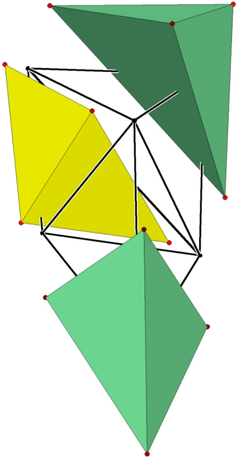
The Staircase Triangulation

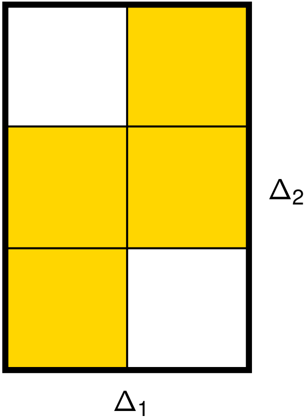
The staircase triangulation is

- a lattice triangulation,
- convex,
- and balanced.



Example: stc($\Delta_1 \times \Delta_2$)





Signature

Theorem (Stanley '97, Soprunova & Sottile '04)

The signature of the staircase triangulation is

$$\begin{aligned}\sigma_{2k,2l} &= \binom{k+l}{k} \\ \sigma_{2k,2l+1} &= \binom{k+l}{k} \\ \sigma_{2k+1,2l+1} &= 0\end{aligned}$$

Outline

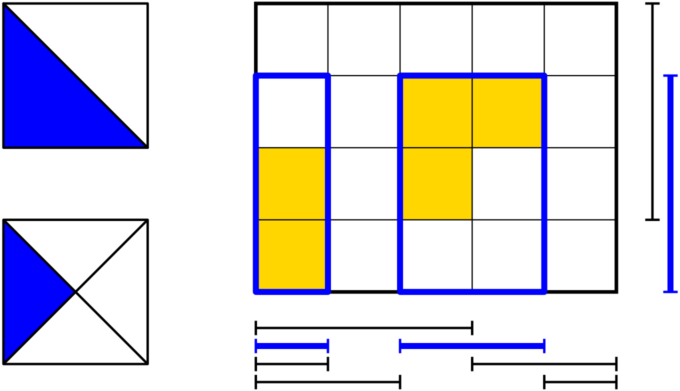
The Simplicial Product

Idea: Triangulating the product $K \times L$ of two abstract simplicial complexes by using staircase triangulations for the cells of $K \times L$.

Problem: How do the triangulated facets fit together?

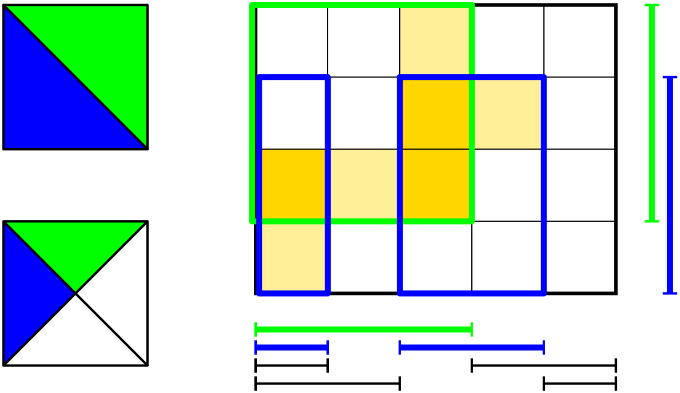
Definition

A facet of the simplicial product $K \times_{\text{stc}} L$.



The Intersection of 2 Facets

The intersection of two facets of $K \times_{\text{stc}} L$.



The Vertex Ordering Does Matter

- Different vertex orderings may yield different triangulations of $K \times L$.
- Given a "wrong" ordering, the simplicial product of two balanced complexes might not be balanced.

Lemma

If K and L are balanced simplicial complexes with color consecutive vertex orderings then $K \times_{\text{stc}} L$ is again balanced.

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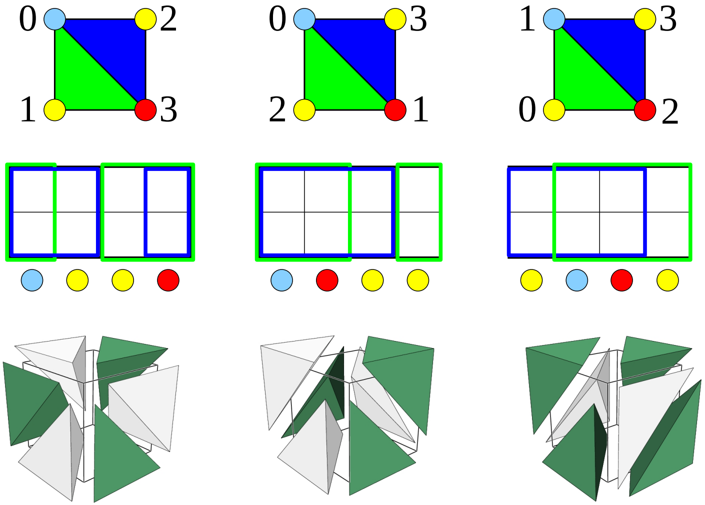
Example: 3 Triangulations of the 3-Cube



Regularity

Lemma

If K and L are regular simplicial complexes then $K \times_{\text{stc}} L$ is regular for any vertex orderings of K and L.

Let $\lambda : \mathbb{R}^m \to \mathbb{R}$ and $\mu : \mathbb{R}^n \to \mathbb{R}$ be lifting functions of K resp. L.

Define a lifting function $\omega : \mathbb{R}^{m+n} \to \mathbb{R}$ by

$$\begin{aligned} \omega : \mathbb{R}^{m+n} &\to \mathbb{R} \\ (v, w) &\mapsto \lambda(v) + \mu(w) + \epsilon(v, w) \end{aligned}$$

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The Product Theorem

Theorem (JOSWIG & W '05)

Let K and L be convex and balanced simplicial complexes of dimension m resp. n. Then $K \times_{\text{stc}} L$ is a convex and balanced triangulation for any color consecutive vertex orderings of K and L. The signature of $K \times_{\text{stc}} L$ is

$$\sigma(K \times_{\text{stc}} L) \;=\; \sigma(K)\,\sigma(L)\,\sigma_{m,n}\,.$$

Corollary

Let P and Q be lattice polytopes of dimension m resp. n. Let the signatures of P and Q be non-negative. Then the signature of $P \times Q$ is at least

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Enumeration up to Dimension 4

Signature of the d-cube for $d \leq 4$. Complete enumeration by TOPCOM and polymake.

dim	# triangulations	# balanced	signature
1	1	1	1
2	1	1	0
3	6	4	4
4	247451	454	2

Lower Bounds

Theorem

The signature of the d-cube for $d \geq 3$ is bounded from below by

$$\sigma(C_d) \;\geq\; \begin{cases} 2^{\frac{d+1}{2}} \left(\frac{d-1}{2}\right)! & \textit{if } d \equiv 1 \mod 2 \\ \left(\frac{d}{2}\right)! & \textit{if } d \equiv 0 \mod 4 \\ \frac{2}{3} \left(\frac{d}{2}\right)! & \textit{if } d \equiv 2 \mod 4\,. \end{cases}$$

Corollary

$$\sigma(C_d) = \Omega\left(\left\lceil \frac{d}{2} \right\rceil ! \right)$$

Lower Bounds

Three cases:

- $d \equiv 1 \mod 2$
  Induction on d: Factorize $C_d = C_{d-2} \times_{\text{stc}} C_2$ with special vertex ordering of C_2.
- $d \equiv 0 \mod 4$
  Induction on d: Factorize $C_d = C_{d-4} \times_{\text{stc}} C_4$.
- $d \equiv 2 \mod 4$
  Factorize $C_d = C_{d-6} \times_{\text{stc}} C_6$ and use special explicit triangulation of C_6.

Triangulations constructed and checked *explicitly* up to dimension 6 using `polymake`.

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Lower Bounds up to Dimension 20

dim	signature	dim	signature
5	16	13	92,160
6	4	14	3,360
7	96	15	129,0240
8	24	16	40,320
9	768	17	20,643,840
10	80	18	241,920
11	7,680	19	371,589,120
12	720	20	3,628,800

Upper Bound

Lemma

The signature of the d-cube is bounded from above by

$$\sigma(C_d) \;\le\; \left\lfloor \frac{d!\,(d+5)}{3(d+3)} \right\rfloor \quad \to \quad \frac{d!}{3}$$

The upper bound is tight in dimension 3.

What's New?

- Definition of the simplicial product.
- The Product Theorem.
- Non-trivial lower bounds for the signature of the *d*-cube.
- Special classes of triangulations such that their simplicial products meet the conditions of the theorem by SOPRUNOVA & SOTTILE.

What's Next?

- Do our triangulations of the *d*-cube with large signature meet the conditions of the theorem by SOPRUNOVA & SOTTILE?

- Does the rectangular grid admit a unimodular and balanced triangulation with a positive signature?