Combinatorial groupoids and their applications

Rade T. Živaljević
Mathematical Institute SANU, Belgrade

Key words and phrases:

Groups of projectivities (M. Joswig[1]), combinatorial parallel transport, combinatorial holonomy, Lovász conjecture, etc.

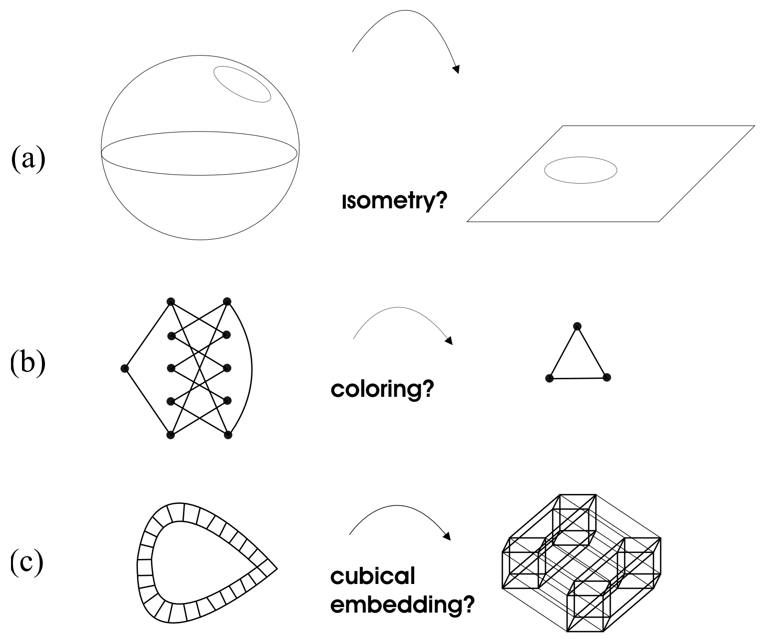

Figure 1: Is there a common point of view?!

[1]Projectivities in simplicial complexes and coloring of simple polytopes, Math. Z. 240 (2002)

A positive answer is provided by the theory of groupoids!

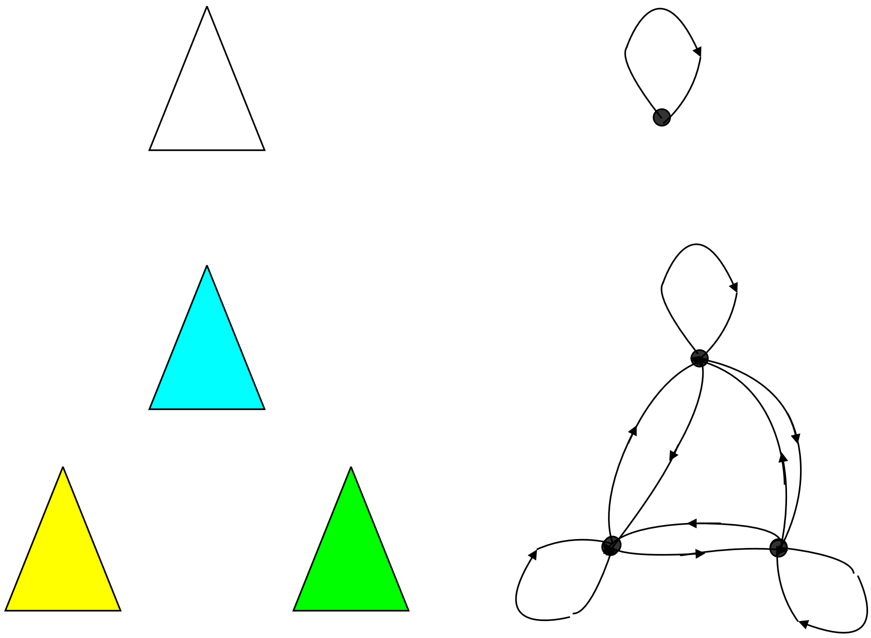

Figure 2: Groupoids are groups with many objects.

The concept of a *group* is sometimes not sufficient to deal with the concept of symmetry in general. *Groupoids* alow us to handle objects which exhibit what is clearly recognized as symmetry although they admit no global automorphism whatsoever. Unlike groups, groupoids are capable of describing reversible processes which can pass through a number of states. For example according to A. Connes, Heisenberg discovered quantum mechanics by considering the groupoid of quantum transitions rather than the group of symmetry.

First example: Cubical embeddings

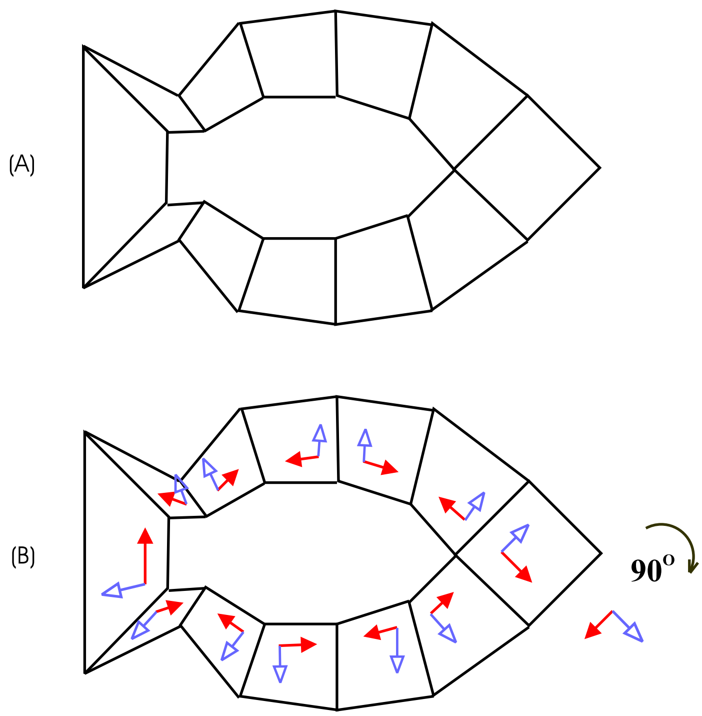

Figure 3: Cubical complex non-embeddable into a cubical lattice.

Theorem 0.1. *Suppose that K is a k-dimensional cubical complex which is embeddable/mappable to $\{\mathbb{R}^n\}_{(k)}$, the k-dimensional skeleton of the standard cubical decomposition $\{\mathbb{R}^n\} = \{\mathbb{R}^1\}^n$ of $\mathbb{R}^n$. Then $\Pi(K,\sigma) \subset BC_k^{even}$ for each cube $\sigma \in K$. Moreover, if $\sigma \in L \subset K$ where $L \cong \{I^{k+1}\}_{(k)}$ is a subcomplex of K isomorphic to the k-skeleton of the $(k+1)$-dimensional cube $\{I^{k+1}\}$, then*

$$\Pi(K,\sigma) \cong BC_k^{even}.$$

Corollary 0.2. *The complex K depicted in Figure 3 is not embeddable (mappable) to a cubical lattice (hypercube) of any dimension. Indeed,*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \Pi(K, \sigma)$$

while by Theorem 0.1 only signed permutation matrices with even number of (-1)-entries can arise as holonomies of subcomplexes of cubical lattices!

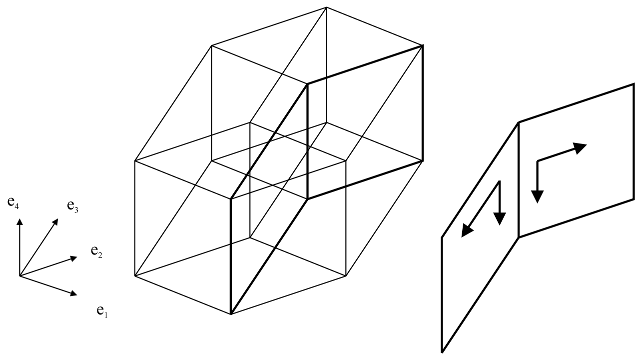

Figure 4: The effect of a "flip" on the sign characteristic.

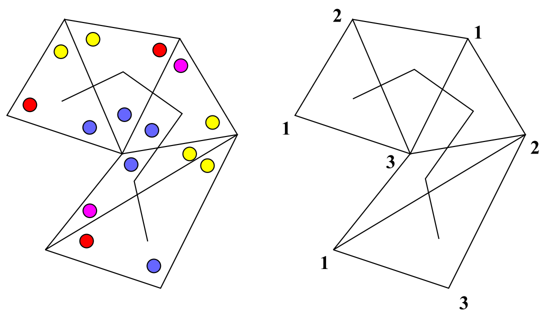

Figure 5: Parallel transport (flips) of triangles.

Problems about cubulations

Problem 1 (S.P. Novikov) Characterize k-dimensional complexes that admit a (cubical) embedding (or "immersion") into the standard cubical lattice of $\mathbb{R}^d$ for some d.

Reference: V.M. Buchstaber, T.E. Panov, Torus actions and their applications in topology and combinatorics, A.M.S. 2002.

Problem 2: (N. Habegger) Suppose we have two cubulations of the same manifold. Are they related by the bubble moves?
Reference: R. Kirby, Problems in low dimensional topology (Problem no. 15), A.M.S. 1995.

Definition 0.3. *Suppose that K is a k-dimensional cubical complex and $\Pi(K, \sigma)$ its combinatorial holonomy group based at $\sigma \in K$. Let $I(K) = 0$ if $\Pi(K, \sigma) \subset BC_k^{even}$ for all σ, and $I(K) = 1$ in the opposite case.*

Theorem 0.4. *$I(K)$ is invariant with respect to bubble moves!*

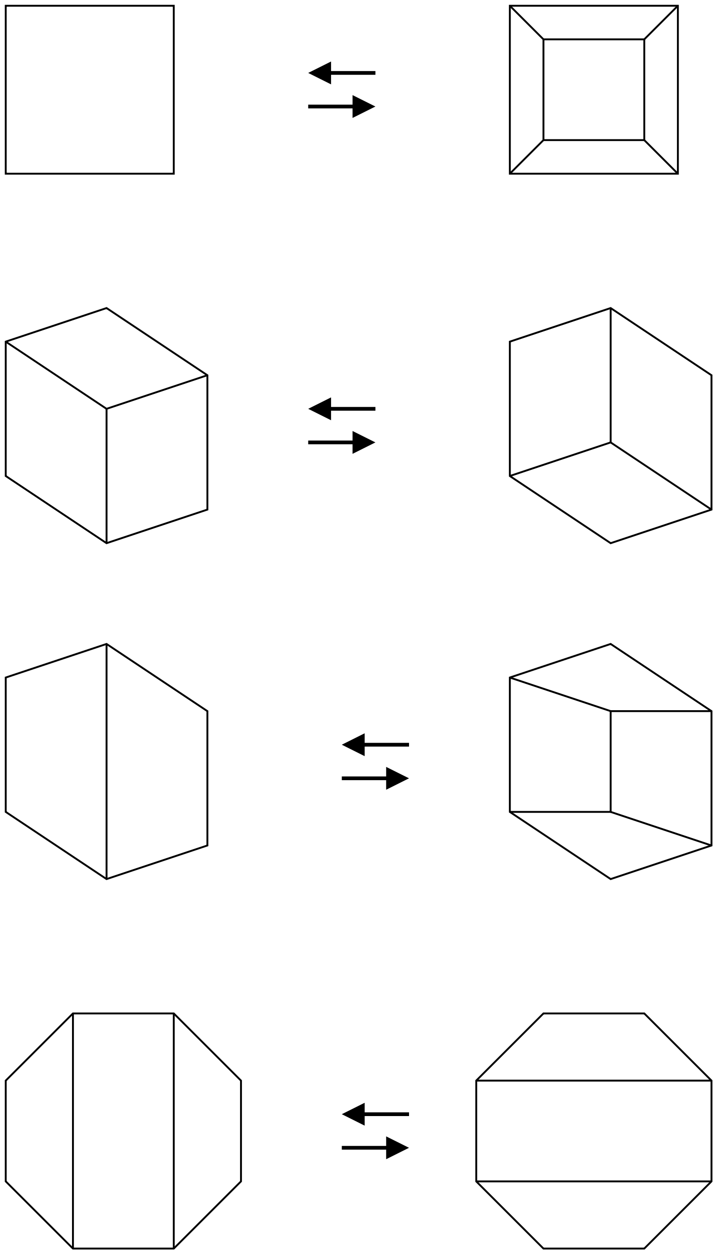

Figure 6: Bubble moves.

Second example: The Lovász conjecture

The general problem is to explore how the topological complexity of a graph complex $X(G)$ reflects in the combinatorial complexity of the graph G itself. The results one is usually interested in come in the form of implications

$$\alpha(X(G)) \geq p \Rightarrow \xi(G) \geq q,$$

where $\alpha(X(G))$ is a topological invariant of $X(G)$, while $\xi(G)$ is a combinatorial invariant of the graph G.

The famous result of Lovász is today usually formulated in the form of an implication

$$Hom(K_2, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k + 3, \quad (1)$$

where $Hom(K_2, G)$ is isomorphic to the so called "box complex" of G. The box complex is a special case of a general graph complex $Hom(H, G)$ (also introduced by L. Lovász), a cell complex which functorially depends on the input graphs H and G.

An outstanding conjecture in this area, refereed to as "Lovász conjecture", was that one obtains a better bound if the graph K_2 in (1) is replaced by an odd cycle C_{2r+1}. More precisely Lovász conjectured that

$$Hom(C_{2r+1}, G) \text{ is } k\text{-connected } \Rightarrow \chi(G) \geq k + 4. \quad (2)$$

This conjecture was confirmed by E.Babson and D.Kozlov (math.CO/0402395, to appear in Annals of Mathematics).

Generalizations

$$Hom(\Gamma, K) \text{ is } k\text{-connected } \Rightarrow \chi(K) \geq k + d + 3 \quad (3)$$

Under suitable assumption on the **test** complex Γ and the assumption that integer k is odd, this implication extends the result of Babson and Kozlov to the case of pure d-dimensional simplicial complexes. (R.Ž, Parallel transport of *Hom*-complexes and the Lovász conjecture, arXiv:math.CO/0506075 v1 Jun 2005.)

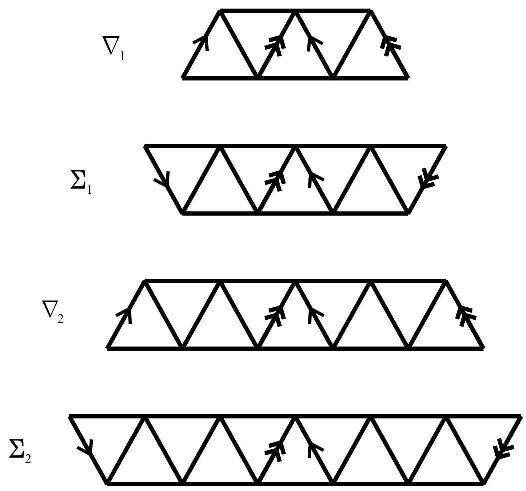

Figure 7: Examples of Γ-complexes.

Proof: Parallel transport of maps between *Hom*-complexes over the groupoid $J(\Gamma)$ introduced by M. Joswig.

Update (Anogia, August 20–26, 2005)

The *complete unfoldings* of simplicial complexes (I. Izmestiev, M. Joswig[2]) are put into the context of coverings of groupoids (R. Brown[3]).

Example (S. Čukić, R.Ž) Chessboard complex $\Delta_{k,n}$ is the complete unfolding of the $(k-1)$-skeleton of a $(n-1)$-dimensional simplex.

[2] Branched coverings, triangulations, and 3-manifolds, Adv. Geom. 3 (2003).

[3] From groups to groupoids; a brief surway, Bull. Lond. Math. Soc., 19 (1987), 113-134.