Dissections, Hom-complexes and the Cayley Trick

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Work in progress

Outline

Introduction

The associahedron

Dissections of polygons

Results in this talk

The Cayley trick

Joins and projections

. . . and the Cayley trick

Hom-complexes

Hom-complexes and the Cayley trick

Symmetry classes

The case Hom(K_g, H)

Dissection complexes

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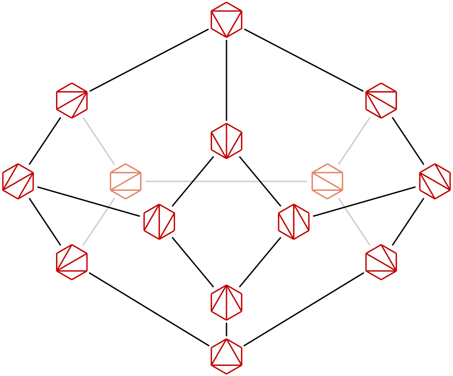
The associahedron



Its face lattice records the incidence structure of the dissections of a convex $(n+2)$-gon into $(j+2)$-gons, $j = 1, 2, \ldots, n$

Polygon dissections

Let's dissect into k-gons instead of triangles:

Definition

Let $k \geq 3$ and $m \geq 1$.

(a) An *allowable* diagonal of a convex N-gon is one that can be completed to a dissection of the N-gon into m convex k-gons. (So $N = m(k-2)+2$.)

(b) [Vic Reiner] Let $\mathrm{T}(k, m)$ be the simplicial complex on the allowable diagonals whose faces are the partial dissections.

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Dissections of polygons

Theorem (Tzanaki 2005)

The complex $\mathrm{T}(k,m)$

(a) *is vertex-decomposable, hence shellable;*

(b) *has the homotopy type of a wedge of* $\frac{1}{m}\binom{(k-2)m}{m-1}$ *spheres of dimension* $m-2$.

In particular, for $k=3$ it really *is* a sphere:
the boundary complex of the polar of the associahedron!

But for $k>3$, the complex $\mathrm{T}(k,m)$ *cannot* be the boundary complex of a convex polytope.

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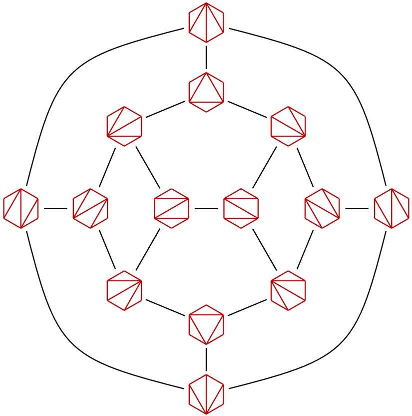
Flip graphs



$D(k,m)$ is the dual graph of $\mathrm{T}(k,m)$:
two dissections are connected if they only differ in one diagonal.

Preliminary results

1. We relate Hom-complexes to the Cayley trick, and study $\mathrm{Hom}(G, H)/S_G$, where S_G is the symmetry group of G
2. We focus on $\mathrm{Hom}(K_g, H)$.
   (For coloring problems, people look at $\mathrm{Hom}(G, K_h)$.)
3. We obtain results on $\mathrm{T}(k, m)$ and $D(k, m)$:
   (a) $\mathrm{T}(k, m) = \mathrm{sk}^{m-2}\,\mathrm{Hom}^t_+(K_{m-1}, I(k, m))/S_{m-1}$.
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4. Also: $\mathrm{T}(k, m) = \mathrm{sk}^{m-2}\,\diamondsuit(k, m)$
   (motivated by a question of Fomin & Zelevinski)
5. $\mathrm{Hom}(K_{m-1}, I(k, m))$ contains copies of $\mathrm{sk}^{d/2}\,(C_d^{\downarrow}(n))^{\Delta}$
6. $D(k, m)$ contains copies of $C(r, s)$, the “flip graph” of weak compositions of r into s parts

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The polyhedral Cayley trick

Let $P_1, \dots, P_g \subset \mathbb{R}^d$ be polytopes.

The *Cayley embedding* $\mathcal{C}(P_1, \dots, P_g)$ is

$$\mathcal{C}(P_1, \dots, P_g) = \operatorname{conv} \bigcup_{i=1}^{g} P_i \times e_i \subset \mathbb{R}^d \times \mathbb{R}^g$$

Theorem (The polyhedral Cayley trick)

$$\{\textit{polyhedral subdivisions of } \mathcal{C}(P_1, \dots, P_g)\} \cong \{\textit{mixed subdivisions of } P_1 + \cdots + P_g\}$$

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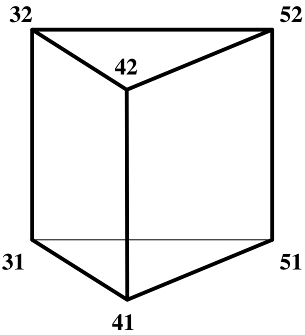

$$\mathcal{C}(\Delta_{\{3,4,5\}}, \Delta_{\{3,4,5\}}) = \Delta_{\{3,4,5\}} \times \Delta_{\{1,2\}}$$

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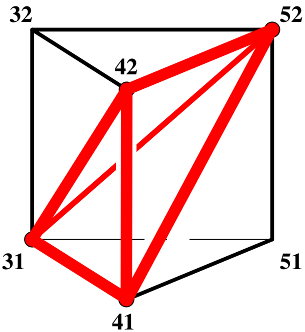

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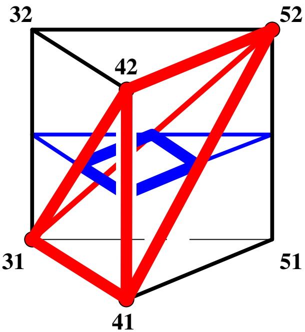

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Joins and the Cayley trick

- A, B sets of cardinality $a = |A|$ and $b = |B|$.
- $\Delta_A := \operatorname{conv}\{e_i : i \in A\} \subset \mathbb{R}^a$, so that $\dim \Delta_A = a - 1$.
- $\star_{i\in A} \Delta_B$ is a simplex of dimension $ab - 1$.
- $\mu_i : \mathbb{R}^b \hookrightarrow \mathbb{R}^{ab} = \mathbb{R}^b \times \cdots \times \mathbb{R}^b$ inclusion into i-th factor

Observation

$$\mathop{\star}_{i\in A} \Delta_B = \operatorname{conv} \bigcup_{i=1}^{a} \mu_i(\Delta_B) \times e_i = \mathcal{C}(\mu_1(\Delta_B), \dots, \mu_a(\Delta_B))$$

$$\mathop{\star}_{i\in A} \sigma_i = \mathcal{C}(\mu_1(\sigma_1), \dots, \mu_a(\sigma_a))$$

for faces $\sigma_1, \dots, \sigma_a$ of Δ_B.

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Projections

Define the projections

- $\pi_\square : \mathbb{R}^{ab} \times \mathbb{R}^a \to \mathbb{R}^b \times \mathbb{R}^a$ with matrix

$$\begin{pmatrix} \mathbb{1}_b & \cdots & \mathbb{1}_b & 0 \\ 0 & \cdots & 0 & \mathbb{1}_a \end{pmatrix},$$

  so that $\pi_\square\big((\boldsymbol{x}_1, \dots, \boldsymbol{x}_a, \boldsymbol{y})^T\big) = (\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_a, \boldsymbol{y})^T$

- $\pi_\Delta : \mathbb{R}^b \times \mathbb{R}^a \to \mathbb{R}^b$ is the projection onto the first factor.

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$$\pi_\square(\sigma) = \pi_\square\, \mathcal{C}\big(\mu_1(\sigma_1), \dots, \mu_a(\sigma_a)\big) = \mathcal{C}(\sigma_1, \dots, \sigma_a)$$

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Joins and the Cayley trick

Set $L = \mathbb{R}^{ab} \times \frac{\mathbf{1}}{\mathbf{a}} \subset \mathbb{R}^{ab} \times \mathbb{R}^{a}$, with $\frac{\mathbf{1}}{\mathbf{a}} = (\frac{1}{a}, \dots, \frac{1}{a})$.
Let $\sigma = \star_{i\in A}\sigma_i = \mathcal{C}\big(\mu_1(\sigma_1), \dots, \mu_a(\sigma_a)\big)$ be a face of $\star_{i\in A}\Delta_B$.

Proposition

The following diagram commutes:

$$\begin{array}{ccccc}
\star_{i\in A}\Delta_B & \supset & \sigma & \xrightarrow{\iota_L} & \frac{1}{a}\big(\mu_1(\sigma_1)+\dots+\mu_a(\sigma_a)\big)\times\frac{\mathbf{1}}{\mathbf{a}} \\
 & & \downarrow \pi_\square & & \downarrow \pi_\square \\
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This is exactly the Cayley trick!

This is again the Cayley trick!

Joins and the Cayley trick

Set $L = \mathbb{R}^{ab} \times \frac{\mathbf{1}}{\mathbf{a}} \subset \mathbb{R}^{ab} \times \mathbb{R}^{a}$, with $\frac{\mathbf{1}}{\mathbf{a}} = (\frac{1}{a}, \dots, \frac{1}{a})$.
Let $\sigma = \star_{i\in A}\sigma_i = \mathcal{C}\big(\mu_1(\sigma_1), \dots, \mu_a(\sigma_a)\big)$ be a face of $\star_{i\in A}\Delta_B$.

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Dissection complexes

Hom-complexes

Let G and H be graphs on $g = |V(G)|$ and $h = |V(H)|$ vertices.

Definition (Lovász; Babson & Kozlov)

- A homomorphism from G to H is a map $\varphi : V(G) \to V(H)$ such that for any edge (x, y) of G, $(\varphi(x), \varphi(y))$ is an edge of H.
- $\mathrm{Hom}(G, H)$ is the polytopal subcomplex of $\times_{x \in G} \Delta_{V(H)}$ of all cells $\times_{x \in V(G)} \sigma_x$ such that if $(x, y) \in E(G)$, then (σ_x, σ_y) is a complete bipartite subgraph of H.
- $\mathrm{Hom}_+(G, H)$ is the simplicial subcomplex of $\star_{x \in G} \Delta_{V(H)}$ of all simplices $\star_{x \in V(G)} \sigma_x$ such that if $(x, y) \in E(G)$ and both σ_x and σ_y are nonempty, then (σ_x, σ_y) is a complete bipartite subgraph of H.

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Hom-complexes and the Cayley trick

Definition

The simplicial complex $\mathrm{Hom}^t_+(G,H)$ of transversal faces is the subcomplex of $\mathrm{Hom}_+(G,H)$ induced by the set $\{\star_{x\in V(G)}\sigma_x : |\sigma_x| > 0 \text{ for all } x \in V(G)\}$.

Proposition

Set $L = \mathbb{R}^{gh} \times \frac{1}{g}$. Then

$$\iota_L \mathrm{Hom}^t_+(G,H) = \iota_L \mathrm{Hom}_+(G,H) = \mathrm{Hom}(G,H).$$

In particular, we obtain an embedding of all these complexes into Euclidean space.

Proof.

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Here

$$\mathrm{Hom}(G,H)/S_G \text{ ``}:=\text{''} \pi_\Delta \pi_\square \mathrm{Hom}(G,H)\,.$$

Symmetry classes of Hom-complexes

Let S_G be the symmetry group of G.

Definition

- $\mathrm{Hom}(G, H)/S_G$ is the union of the cells $\{\pi(\sigma) : \sigma \in \mathrm{Hom}(G, H)\}$, where $\pi = \pi_\Delta \pi_\Box$

  This is not necessarily a polytopal complex!

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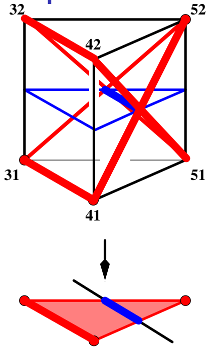

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(a) *Each cell* $\pi(\sigma)$ *of* $\mathrm{Hom}(G,H)/S_G$ *represents an* S_G*-equivalence class of faces of the polytopal complex* $\mathrm{Hom}(G,H)$.

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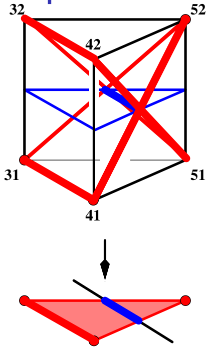

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The case $\mathrm{Hom}(K_g, H)$

Theorem

(a) *$\pi_\square \mathrm{Hom}_+(K_g, H)$ is a simplicial immersion of $\mathrm{Hom}_+(K_g, H)$ into $\Delta_{V(H)} \times \Delta_{[g]}$.*

Another way of expressing this is to say that $\mathrm{Hom}_+(K_g, H)$ is a "horizontal" complex, i.e., it has no faces in $\ker \pi_\square$.

(b) *Each cell of $\pi\,\mathrm{Hom}^t_+(K_g, H)$ represents an S_g-equivalence class of faces of the simplicial complex $\mathrm{Hom}^t_+(K_g, H)$.*

(c) $\pi\,\mathrm{Hom}_+(K_g, H) = \Delta_H$,
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(b) $\mathrm{Hom}(K_g, H)/S_g$ *is a polytopal complex if and only if any complete g-partite subgraph of H is induced. This is the case if and only if $\omega(H) = g$, i.e., the size of a largest clique in H is g.*

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Let $I(k,m)$ be the independence graph on the set of allowable diagonals:
join two diagonals if they do not cross

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(c) *For all $k \geq 3$ and $m \geq 1$, the simplicial complex $\mathrm{T}(k,m)$ is the $(m-2)$-skeleton of a non-pure $(m-1)$-dimensional polytopal complex $\diamondsuit(k,m)$ whose cells are iterated cones over cross-polytopes.*

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T$(4,3)$ is the 1-skeleton of a polyhedral decomposition of $\mathbb{RP}^2$ into four squares and one octagon.

Question (Fomin & Zelevinski, 2005)

Is it true in general that T(k,m) *is the* $(m-2)$*-skeleton of a polyhedral manifold of dimension* $k+m-5$*?*

Probably not:

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Open problems

- Invariants of $(\phi)\,\mathrm{Hom}_{(+)}(G,H)/S_G$: homotopy type, etc.
- So far, we are dealing with the A_n case. Can we also find polytopal complexes for the Lie groups of type B_n and D_n?
- ...