On the structure of sets minimizing the rectilinear crossing number

O. Aichholzer, D. Orden, P. Ramos

Crete, August 2005

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  Determine minimum number of crossings of a straight-edge drawing of K_n (vertices in general position).

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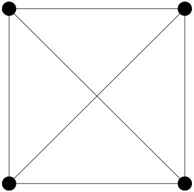

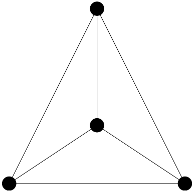

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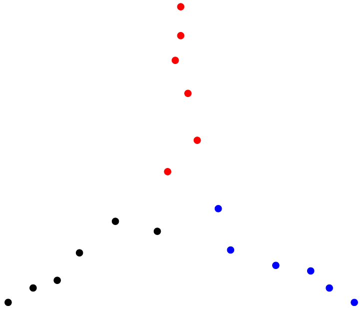

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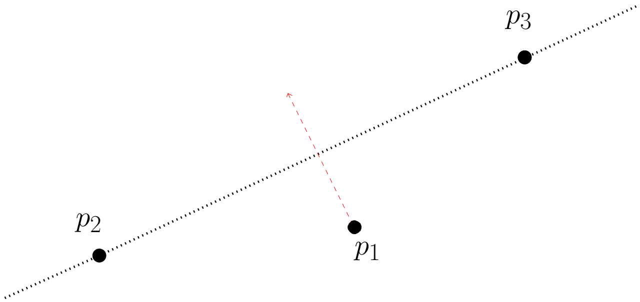

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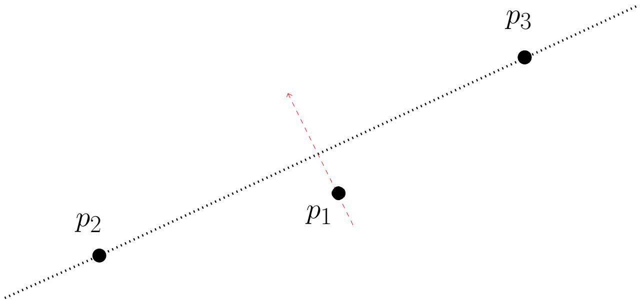

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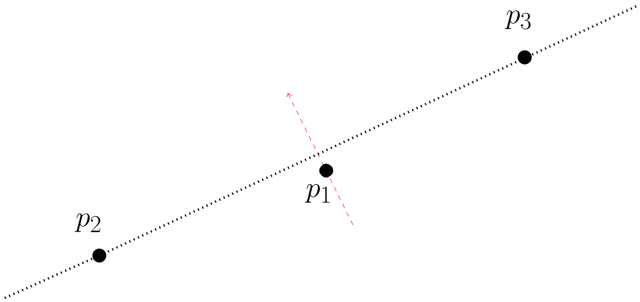

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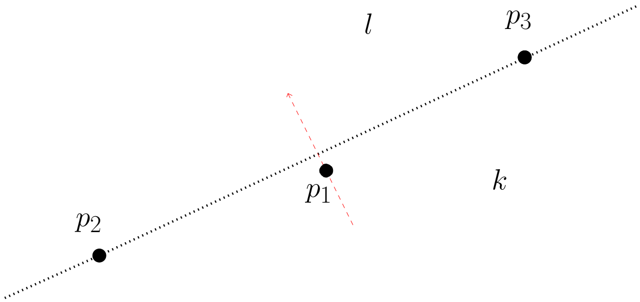
- Consider a set S of n points and move a point p_1 along a line:
  The order type changes precisely when p_1 passes over a line spanned by some $p_2 p_3$.
- We call this a (k, l)-*flip* if p_1 passes from the side of $\overline{p_2 p_3}$ containing k points (p_1 excluded) to the side with l points.



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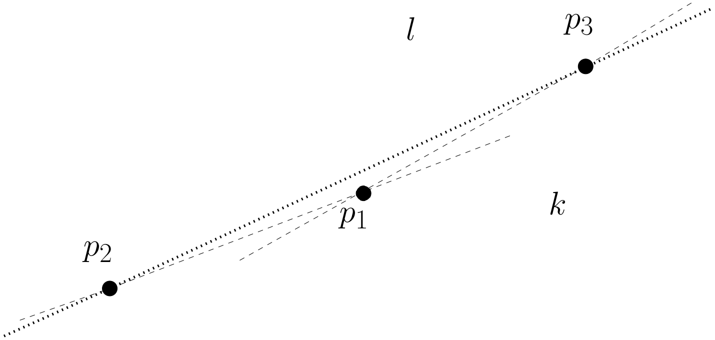

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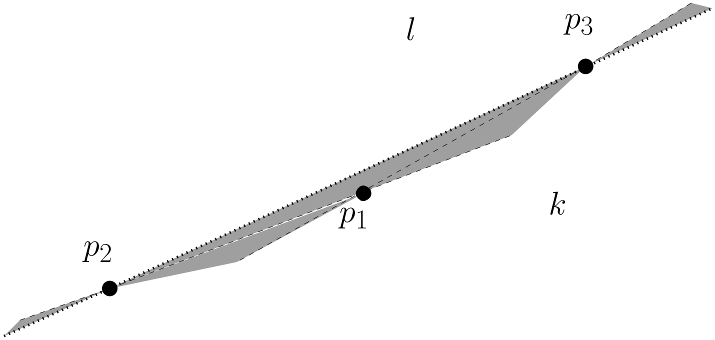

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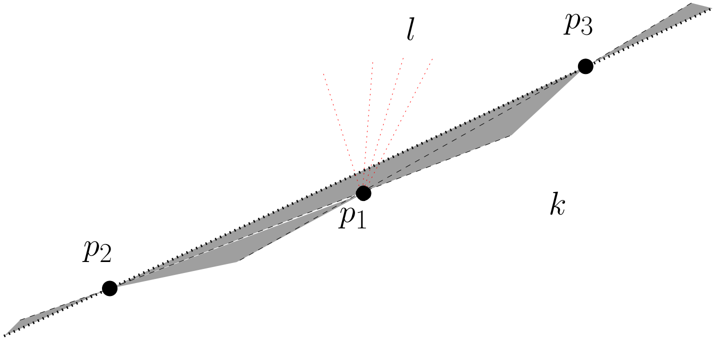

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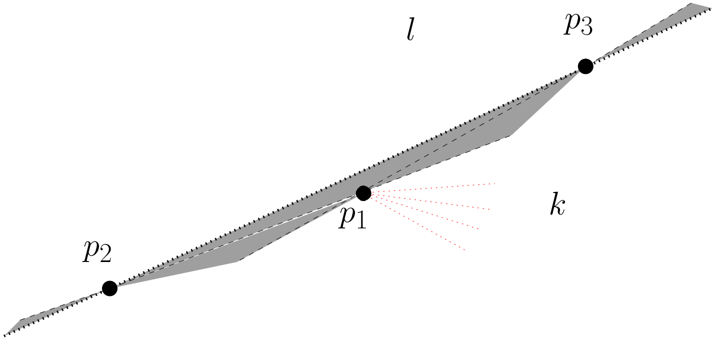

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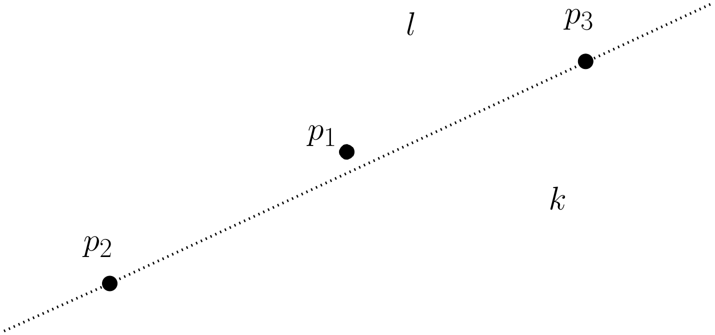

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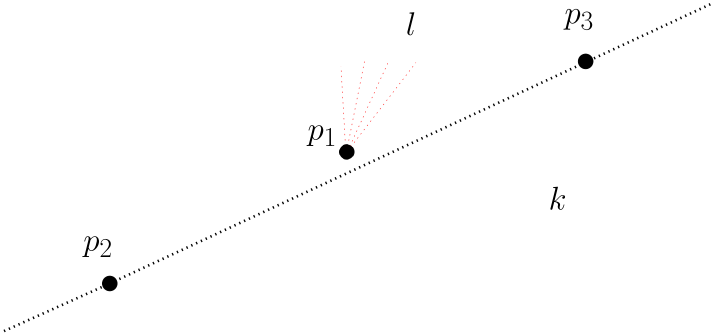

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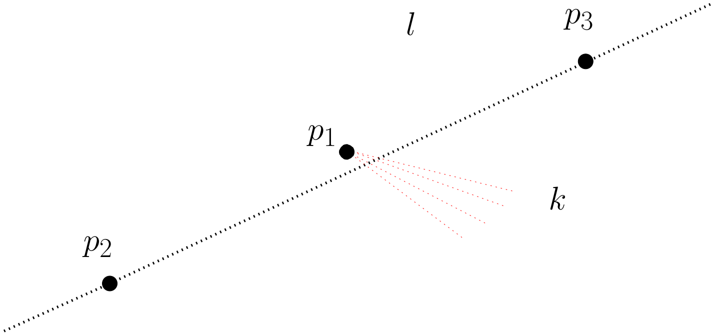

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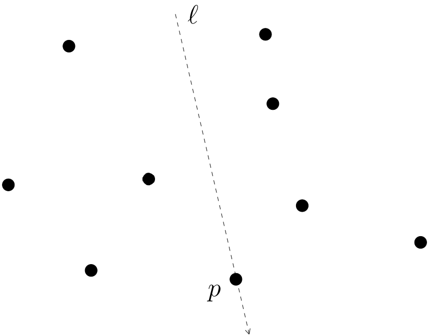

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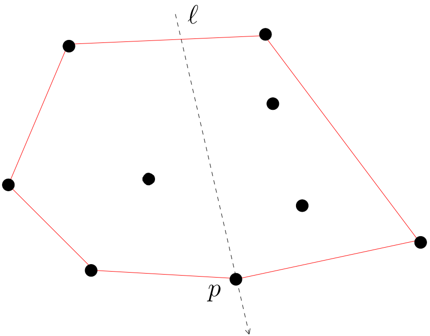

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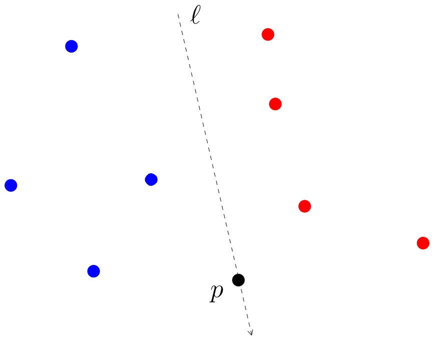

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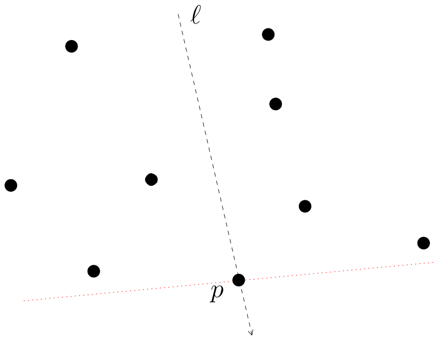

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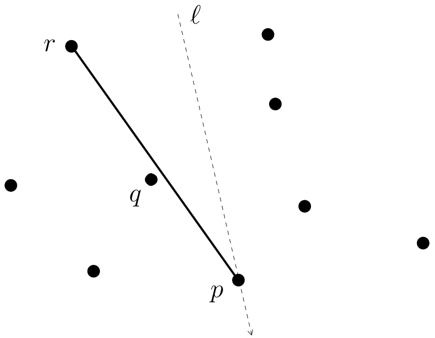

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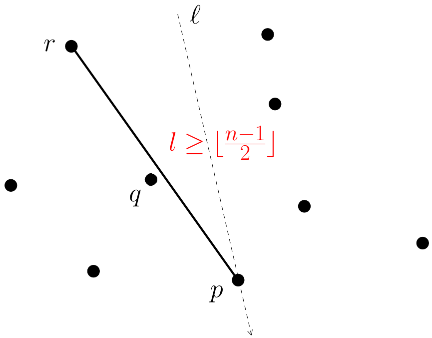

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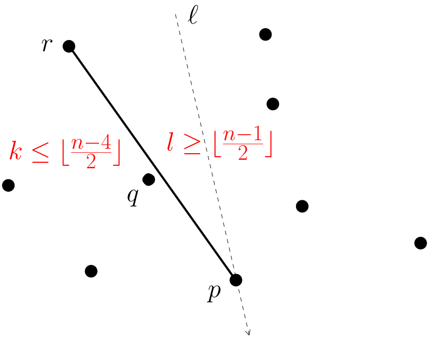

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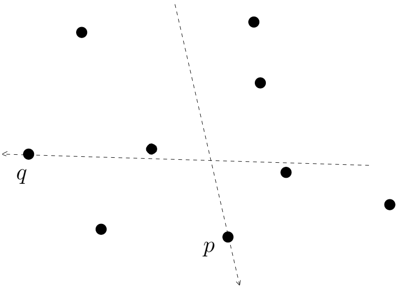

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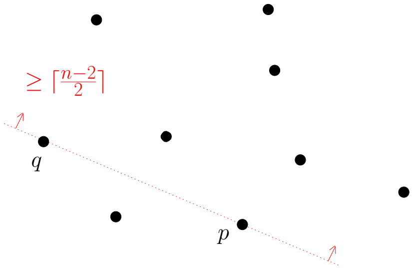

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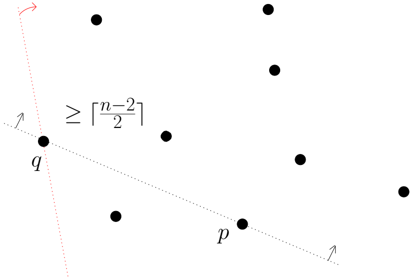

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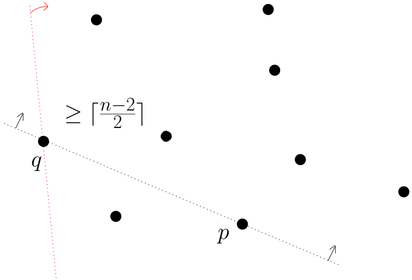

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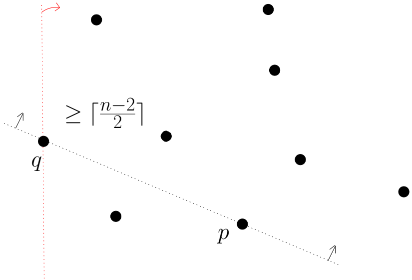

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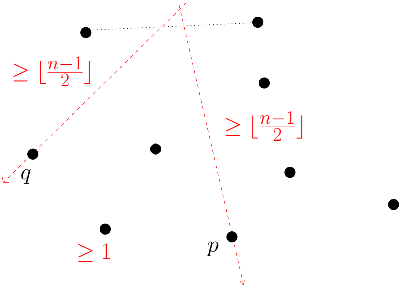

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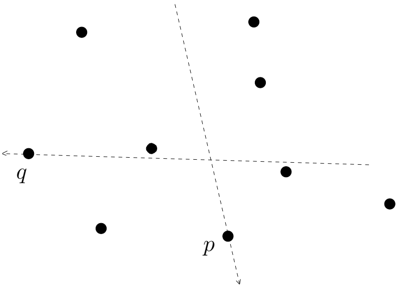

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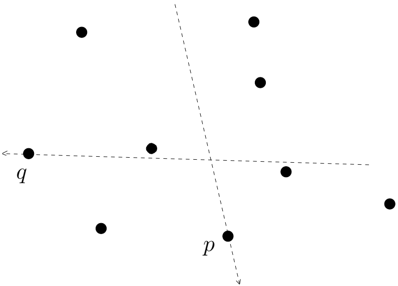

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Let S be a set of n points in the plane in general position with $h > 3$ extreme points. Then there exists a set S' of n points in general position which has a smaller rectilinear crossing number than S and less than h extreme points.

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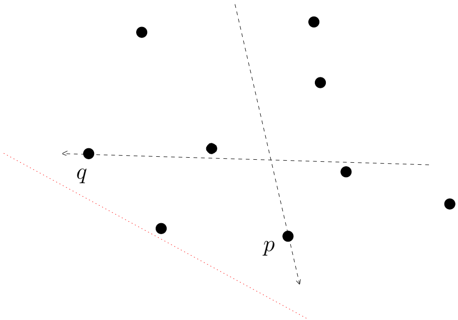

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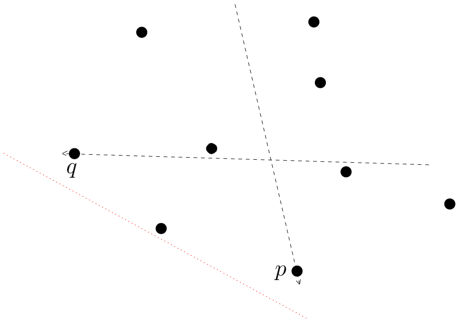

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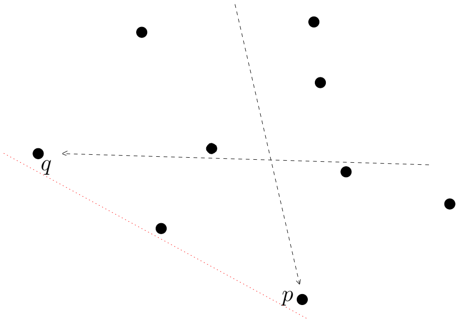

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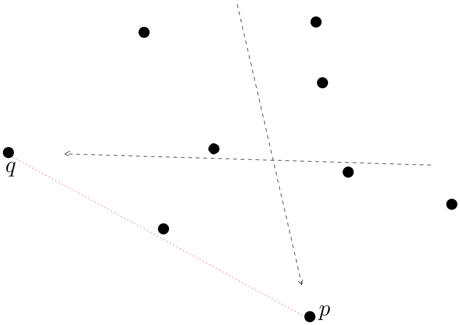

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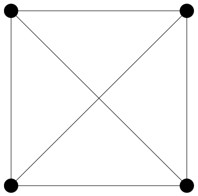

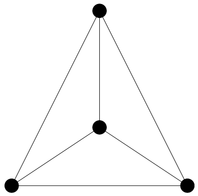

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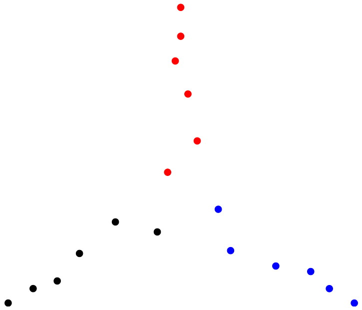

STOP??

YES **NO**

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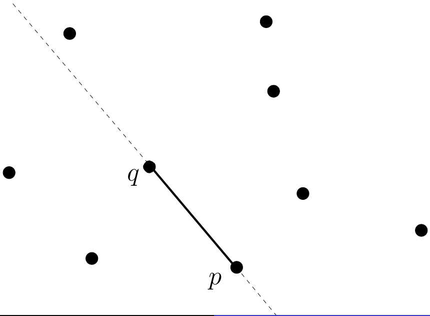

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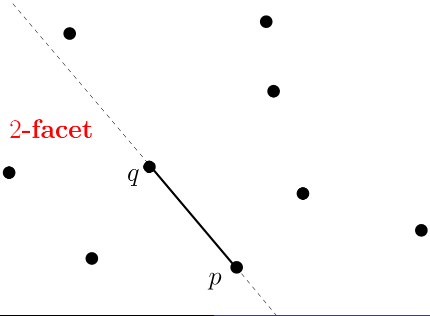

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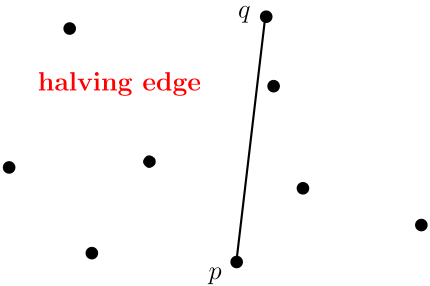

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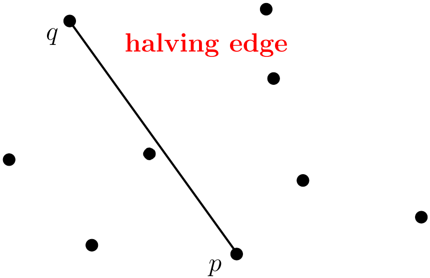

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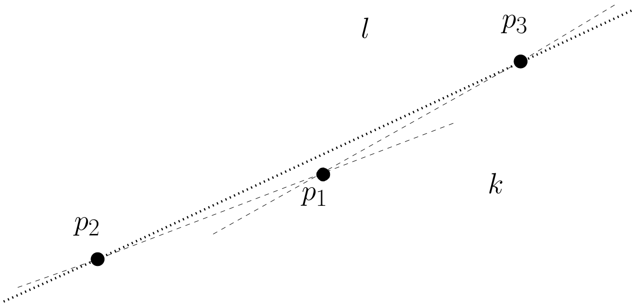

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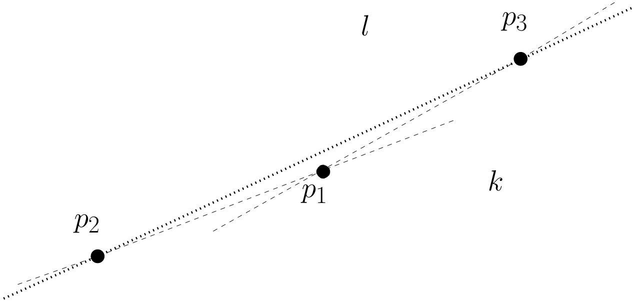

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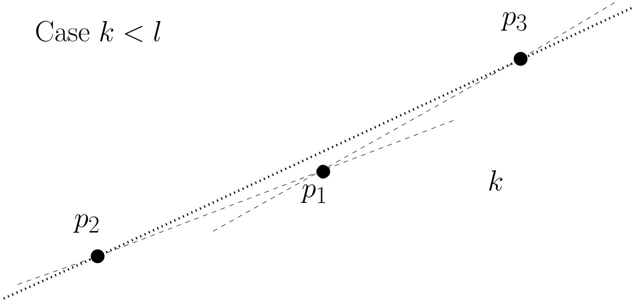

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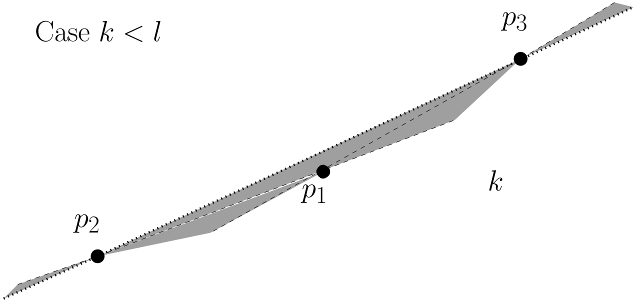

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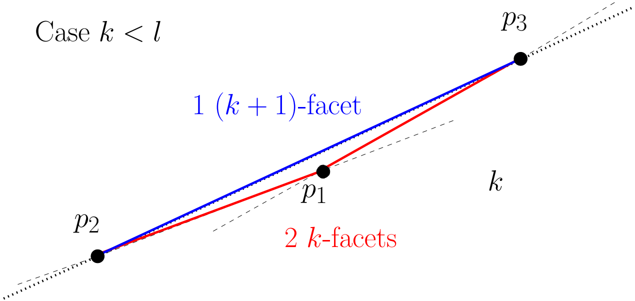

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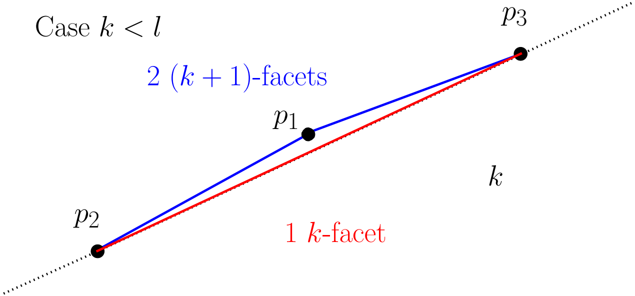

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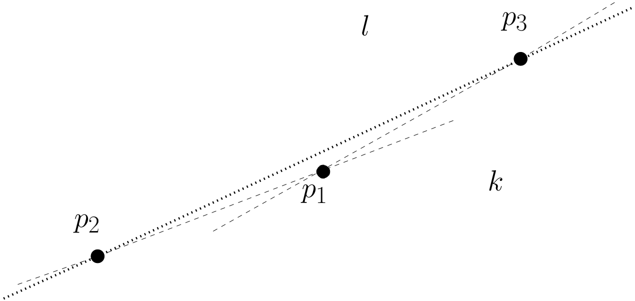
A (k, l)-flip changes the number of j-facets as follows:

$k < l$	k-facets ↓ -1	$(k+1)$-facets ↑ $+1$
$k > l$		
$k = l$		



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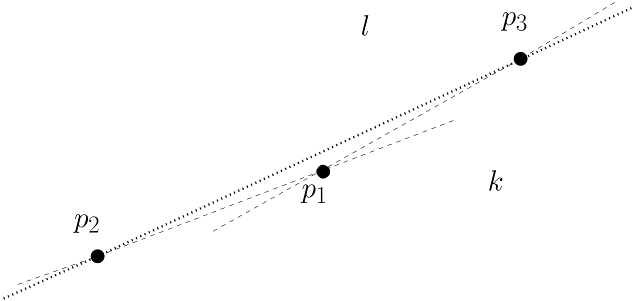

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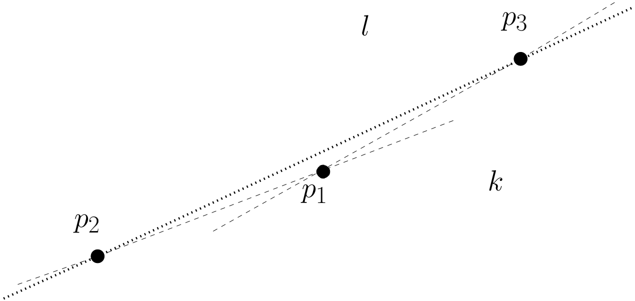

Crossings and j-facets are related

Theorem 7

The rectilinear crossing number $\overline{cr}(S)$ and the numbers of j-facets f_j are related by

$$\overline{cr}(S) + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (j-1) \cdot (n-j-3) \cdot f_j = \frac{1}{8} \cdot (n^4 - 10n^3 + 27n^2 - 18n)$$

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Theorem 8

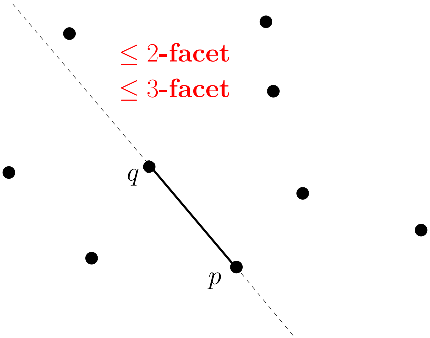
For any fixed cardinality $n \geq 3$ there exist point sets maximizing the number of halving edges and having a triangular convex hull.

STOP??
(last chance)

YES **NO**

Preliminaries

- $\leq k$-**facet:** any j-facet with $j \leq k$ $(0 \leq k \leq \lfloor \frac{n-2}{2} \rfloor)$.



How flips affect $\leq k$-facets and a useful result

Lemma 9

For every S with $(\leq)$-facet vector $f = (f_{(\leq 0)}, \ldots, f_{(\leq \lfloor \frac{n-2}{2} \rfloor)})$ there exists a set S' with a triangular convex hull and facet vector f' s.t. $f'_{(\leq i)} \leq f_{(\leq i)}$ $\forall i$, where at least one inequality is strict.

How flips affect $\leq k$-facets and a useful result

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The number of $(\leq k)$-facets of S is at least $3\binom{k+2}{2}$ for $0 \leq k < \frac{n-2}{2}$.
This bound is tight for $k \leq \lfloor \frac{n}{3} \rfloor - 1$.

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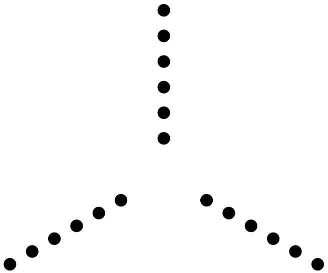

STOP??
(no chance)

YES **YES**

On the structure of sets minimizing the rectilinear crossing number

O. Aichholzer, D. Orden, P. Ramos

Crete, August 2005