NOTATION

- $\mathbb{F}$ ordered field.
- $\mathbb{V}, \mathbb{W}, \dots$ finite dimensional vector spaces.
- $\mathbb{V}^*$ dual space of $\mathbb{V}$.

- $\mathcal{Q}(\mathbb{V}) = \mathcal{Q}$ non-empty polyhedra in $\mathbb{V}$.
- $\mathcal{P}(\mathbb{V}) = \mathcal{P}$ sub-family of polytopes.
- $\mathcal{C}(\mathbb{V}) = \mathcal{C}$ sub-family of cones with apex o.

- $\eta(P, \cdot)$ support functional of $P \in \mathcal{Q}$.

- $F \leqslant P$ face.
- $F \lhd P$ facet.

POLYHEDRON GROUP

- Addition of classes $[P] \in \Gamma(\mathbb{V}) = \Gamma$ is given by valuation property:

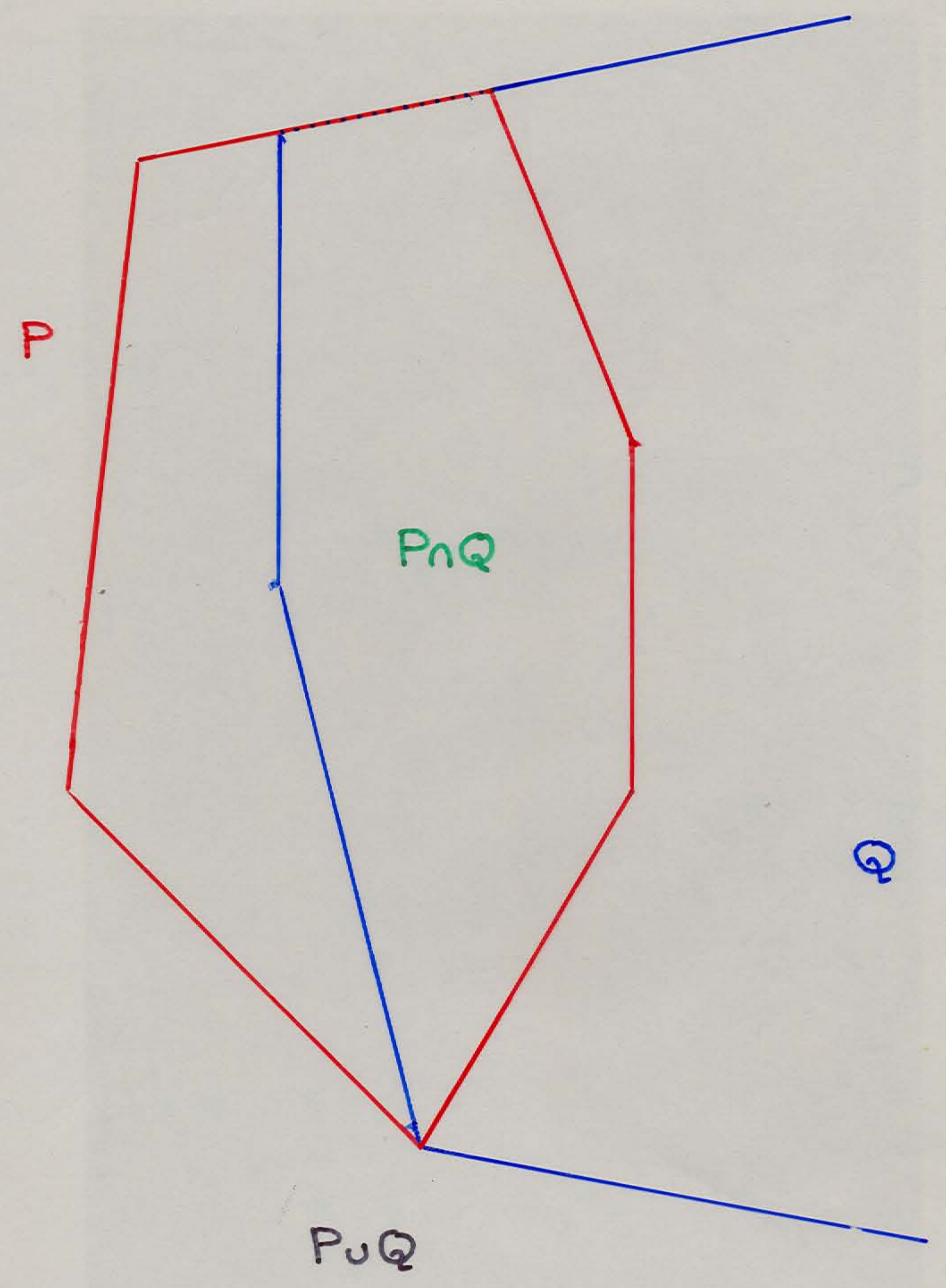
$$[P \cup Q] + [P \cap Q] = [P] + [Q],$$

with $P, Q, P \cup Q \in \mathcal{Q}$. (Convention $[\emptyset] = 0$.)

- Weak addition:

$$[P] + [P \cap H] = [P \cap H^+] + [P \cap H^-],$$

with $P \in \mathcal{Q}$, H hyperplane in $\mathbb{V}$ bounding closed half-spaces $H^{\pm}$. This determines $\langle \Gamma, + \rangle$.


P
P∩Q
Q
P∪Q

CHARACTERISTIC FUNCTIONS

- If $P \in \mathcal{Q}$, then $\delta(P, \cdot)$ is given by

$$\delta(P,x) := \begin{cases} 1, & \text{if } x \in P, \\ 0, & \text{if } x \notin P. \end{cases}$$

- $\mathfrak{X}\mathcal{Q}(\mathbb{V}) = \mathfrak{X}\mathcal{Q}$: abelian group generated by the $\delta(P, \cdot)$.

- $\mathfrak{X}\mathcal{Q} \cong \Gamma$ as abelian groups.

- $\mathfrak{U}\mathcal{Q}$ consists of finite unions in $\mathcal{Q}$;

$$\overline{\mathfrak{U}}\mathcal{Q} := \{X \setminus Y \mid X, Y \in \mathfrak{U}\mathcal{Q}\}.$$

Then $[X] \in \Gamma$ is defined for $X \in \overline{\mathfrak{U}}\mathcal{Q}$.

Euler Map

- $\varepsilon : \Gamma \to \Gamma$ defined by
$$[P]\varepsilon := \sum_{F \leq P} (-1)^{\dim F} [F].$$

- Inverting $[P] = \sum_{F \leq P} [\operatorname{relint} F]$ gives
$$[\operatorname{relint} P] = (-1)^{\dim P} [P]\varepsilon.$$

- Hence ε is involutory automorphism of Γ.

- $\bar{\chi} : \Gamma \to \mathbb{Z}$ defined by
$$[P]\bar{\chi} := 1.$$

- Euler characteristic $\chi := \varepsilon\bar{\chi}$. This is a topological invariant.

RECONSTRUCTION

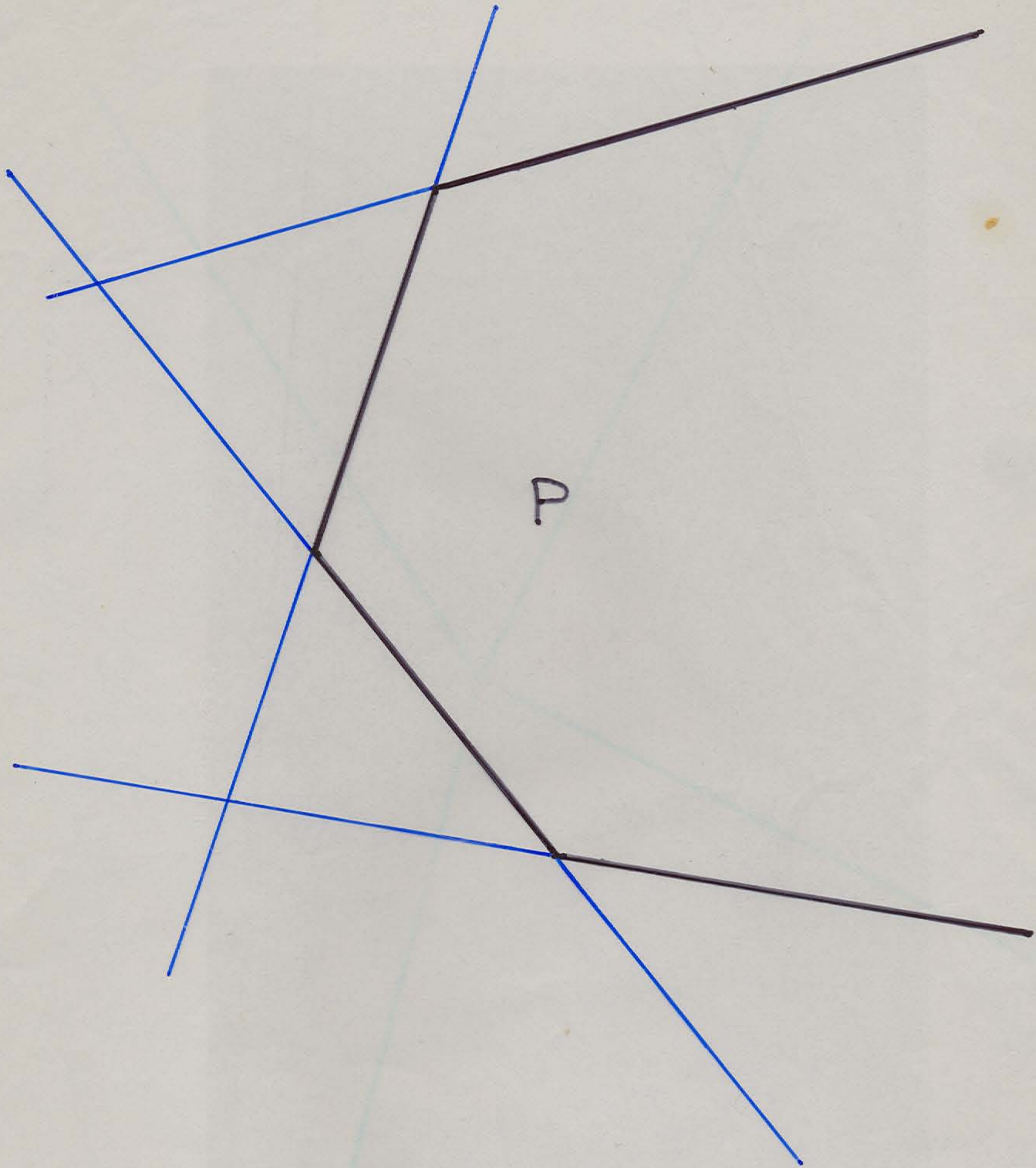
- If $P \in \mathcal{Q}$, $F \leq P$, $a \in \operatorname{relint} F$, then
$$C(F,P) := \operatorname{cone}(a,P).$$

- $$[P] = \sum_{F \leq P} (-1)^{\dim F} \chi(F)\,[C(F,P)].$$

  This is a wide generalization of the Brianchon-Gram angle-sum theorem.

- Define formal exponential
$$\gamma(P,\cdot) := \exp(-\eta(P,\cdot)).$$

  Then $P \longmapsto \gamma(P,\cdot)$ induces separating homomorphism on Γ.


P

Multiplication

- Linear mapping $\Theta : \mathbb{V} \longrightarrow \mathbb{W}$ induces homomorphism $\vartheta : \Gamma(\mathbb{V}) \longrightarrow \Gamma(\mathbb{W})$.
- Note that $\varepsilon\vartheta \neq \vartheta\varepsilon$.

- Sum mapping $\Sigma : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ induces multiplication $\mu : \Gamma \otimes \Gamma \longrightarrow \Gamma$, by

$$[P] \cdot [Q] := [P + Q].$$

- Distribution law is easy:

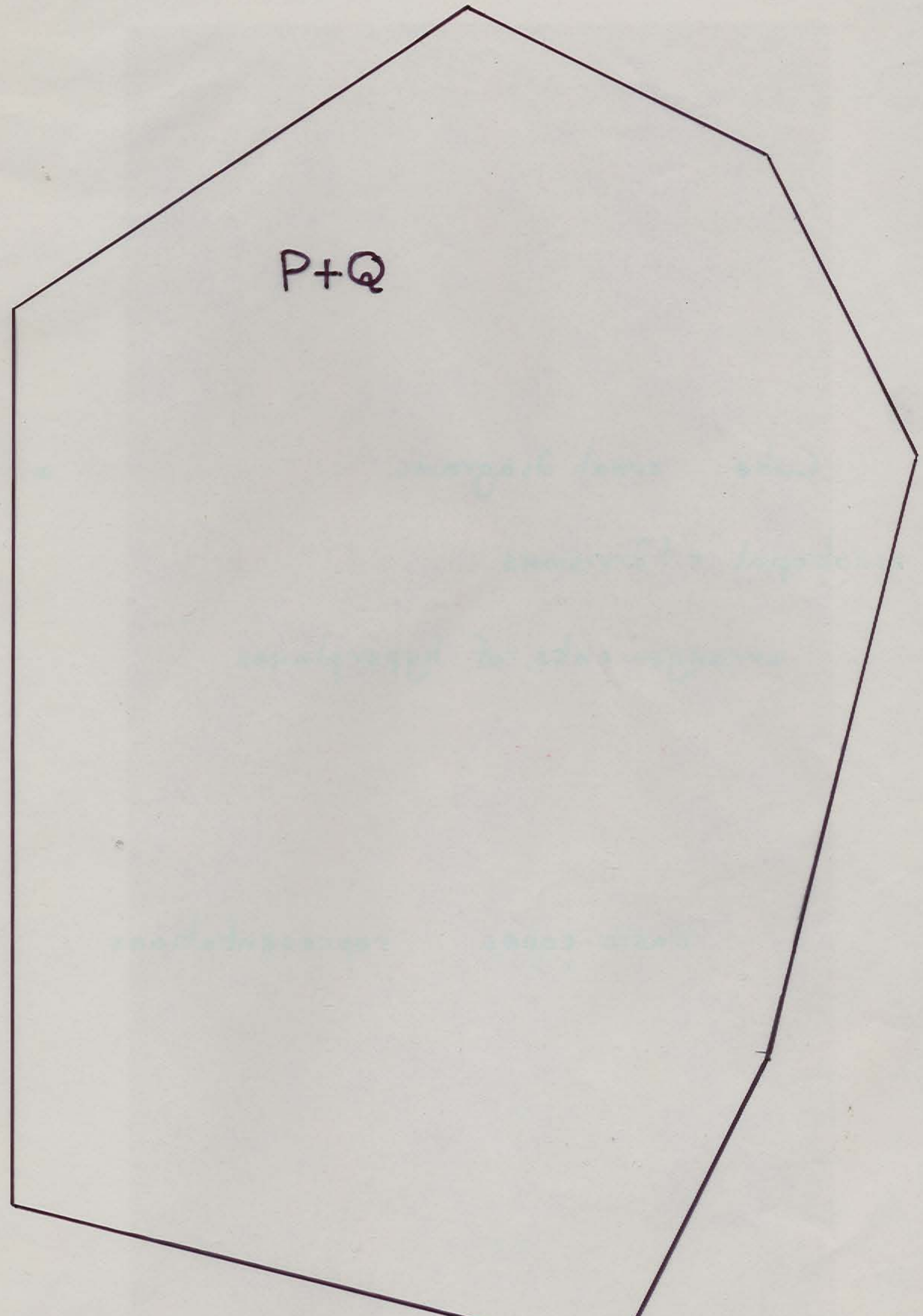
$$(P_1 \cup P_2) \times Q = (P_1 \times Q) \cup (P_2 \times Q)$$

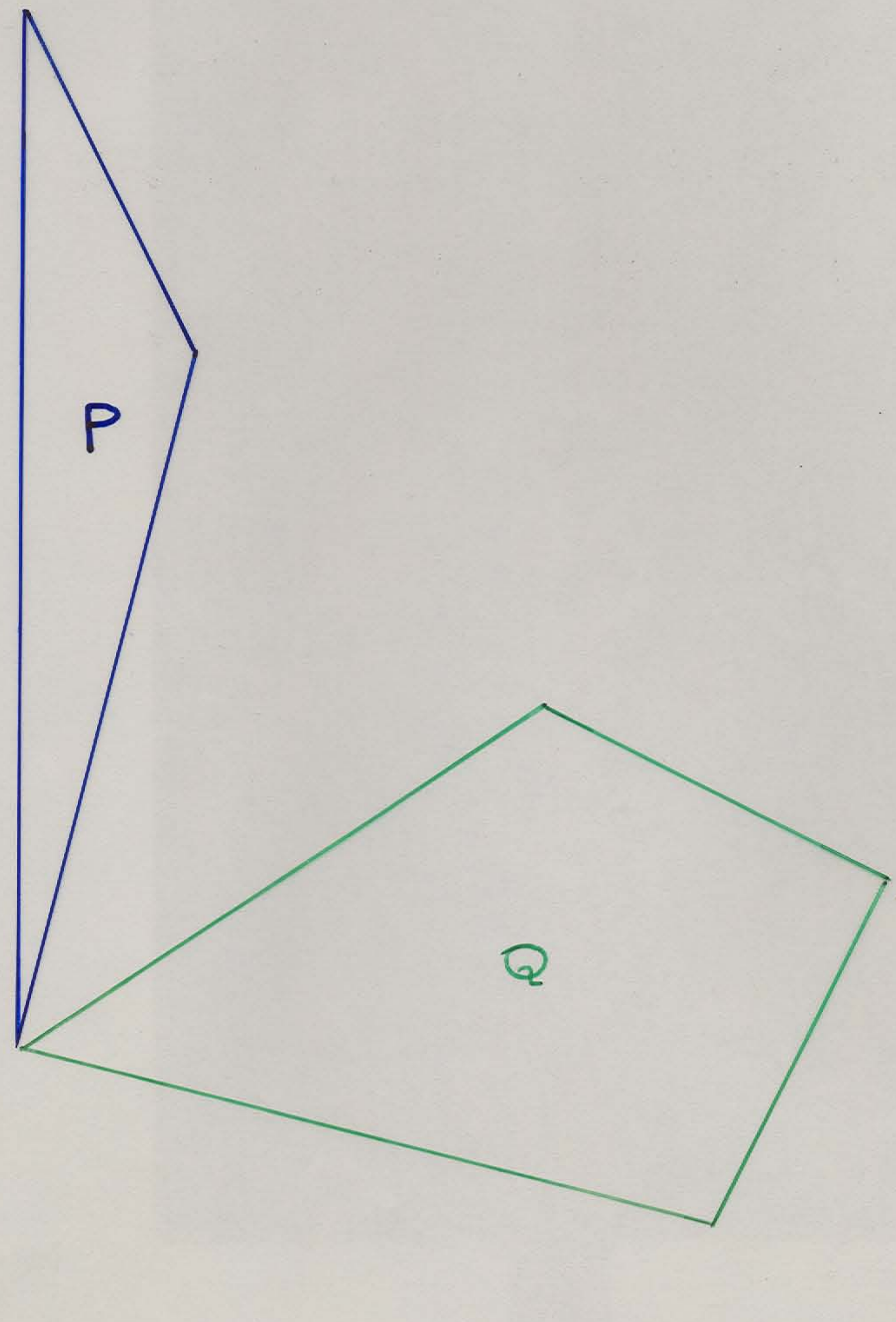
$$(P_1 \cap P_2) \times Q = (P_1 \times Q) \cap (P_2 \times Q)$$

- Unit is

$$\upsilon : k \longmapsto k[0].$$

- Associativity is obvious.


P+Q


P
Q

CONE RING

- $\Xi = \Xi(\mathbb{V}) := \langle [P] \mid P \in \mathcal{C}(\mathbb{V}) \rangle.$

- Polar cone

  $$P^* := \{ y \in \mathbb{V}^* \mid \langle x, y \rangle \leq 0 \text{ for all } x \in P \}.$$

- Polarity induces group homomorphism

  $$\pi : \Xi \longrightarrow \Xi^* := \Xi(\mathbb{V}^*).$$

- If $\bar{\iota}$ is given by $[P]\bar{\iota} := [-P]$, then

  $$\varepsilon\pi = (-1)^{\dim \mathbb{V}} \bar{\iota}\pi\varepsilon.$$

- Hence

  $$(\varepsilon\pi)^4 = \iota,$$

  the identity.

IDEALS AND QUOTIENTS

- Recession cone:

$$\operatorname{rec} P := \{ y \in V \mid x + \lambda y \in P \text{ for all } x \in P \text{ and } \lambda \geq 0 \}.$$

- $P \mapsto \operatorname{rec} P$ induces ring homomorphism $\rho : \Gamma \to \Xi$, given by

$$[P]\rho := [\operatorname{rec} P].$$

- The kernel of ρ is the translation ideal

$$T := \langle [x] - [0] \mid x \in V \rangle.$$

- Thus $\Gamma / T \cong \Xi$.

- As a ring, Γ is generated by the classes of points and (linear) rays.

More on the Euler Map

- Call $q \in \Gamma$ the relative inverse of $[P]$ if $q \cdot [P] = [\operatorname{rec} P]$. Then
$$q = ([\operatorname{rec} P]\varepsilon \cdot [-P])\varepsilon.$$

- In $\Pi := \langle [P] \mid P \in \mathcal{P} \rangle \leq \Gamma$,
$$[P]^{-1} = [-P]\varepsilon.$$

- Thus ε is a ring automorphism of Π.

- If $P \in \mathcal{Q}$ is simple, let α, β be indeterminates, and define
$$h([P];\alpha,\beta) := \sum_{F \leq P} \alpha^{\dim P - \dim F} (\beta - \alpha)^{\dim F} [F].$$
The generalized Dehn-Sommerville equations are
$$h([P];\alpha,\beta)\,\varepsilon = h([P];\beta,\alpha).$$

CO-RING

- Co-multiplication: additive homomorphism

$\kappa : \Gamma \longrightarrow \Gamma \otimes \Gamma$, given by

$$[P]\kappa := \sum_{F \leq P} [\operatorname{relint} F] \otimes [C(F,P)].$$

- Note that (using $\Gamma \cong \mathfrak{X}\mathfrak{Q}$)

$$[P]\kappa \longleftrightarrow \{x \otimes y \mid x \in P,\ y \in \operatorname{cone}(x,P)\}.$$

- Co-associative:

$$\kappa(\kappa \otimes \iota) = \kappa(\iota \otimes \kappa).$$

- Co-unit $\bar{\chi}$:

$$\kappa(\bar{\chi} \otimes \iota) = \iota = \kappa(\iota \otimes \bar{\chi})$$

(identify $\mathbb{Z} \otimes \Gamma \cong \Gamma \cong \Gamma \otimes \mathbb{Z}$).

- Some calculations are easier with $\bar{\kappa}$:

$$[P]\bar{\kappa} := \sum_{F \leq P} (-1)^{\dim F} [F] \otimes [C(F,P)];$$

then use

$$\kappa = \varepsilon \bar{\kappa} (\varepsilon \otimes \varepsilon).$$

BI-RING

- If linear mapping $\omega : V \to W$ induces $\vartheta : \Gamma(V) \to \Gamma(W)$, then

$$\vartheta\kappa = \kappa(\vartheta \otimes \vartheta).$$

- Note that

$$\mathrm{cone}(x\omega, P\omega) = \mathrm{cone}(x, P)\omega.$$

- Multiplication and co-multiplication compatible:

$$\mu\kappa = (\kappa \otimes \kappa)(\iota \otimes \tau \otimes \iota)(\mu \otimes \mu),$$

with $\tau : x \otimes y \mapsto y \otimes x$ twist.

- Unit and co-unit:

$$\nu\bar{\chi} = \iota = \bar{\chi}\nu.$$

- Unit and co-multiplication $(1\kappa = 1 \otimes 1)$:

$$\nu\kappa = \kappa(\nu \otimes \nu).$$

- Co-unit and multiplication:

$$\mu\bar{\chi} = (\bar{\chi} \otimes \bar{\chi})\mu.$$

REMARKS

- $\kappa\mu$ plays no rôle. In fact,

$$\kappa(\bar{\iota}\otimes\iota)\mu = \rho.$$

- If κ^* is the co-multiplication on $\Gamma(\mathbb{V}^*)$, then, under $\pi : \Xi \longrightarrow \Xi^*$,

$$\kappa(\pi\otimes\pi) = \pi\kappa^*\tau,$$

with τ the twist. Thus,

$$\kappa^* = \pi\kappa(\pi\otimes\pi)\tau.$$

WEAK CONTINUITY

- Continuity relative to parallel displacement of support hyperplanes.

- Euler map ε is weakly continuous.
- Since $\bar{\chi}$ is weakly continuous, so is $\chi = \varepsilon\bar{\chi}$.

- Multiplication μ is weakly continuous.

- Co-multiplication κ is weakly continuous.

VALUATIONS

- Any additive homomorphism on Γ is a valuation.

- Let $N(F,P)$ be the normal cone to P at $F \leq P$; thus $[N(F,P)] \in \Xi^*$. Then $\kappa(\iota \otimes p\pi)$ is

$$P \longmapsto \sum_{F \leq P} [\operatorname{relint} F] \otimes [N(F,P)]$$
$$= \sum_{F \leq P} [F] \otimes [\operatorname{relint} N(F,P)].$$

- These two expressions can be identified with

$$\{x \otimes u \in V \otimes V^* \mid x \in P,\ u \in N(x,P)\}$$
$$= \{x \otimes u \in V \otimes V^* \mid u \in (\operatorname{rec} P)^*,\ x \in F(P,u)\}.$$

"Intrinsic Volumes"

- Write Γ_k for the additive subgroup of Γ generated by the classes $[Q]$ with $\dim Q \leq k$.

- Let $\langle P \rangle_r := [P] + \Gamma_{r-1} \in \Gamma / \Gamma_{r-1}$.

- For each $0 \leq r \leq d$,

$$P \mapsto \sum_{F \leq P} \langle F \rangle_r \otimes \langle N(F,P) \rangle_{d-r}$$

is an abstract version of the rth intrinsic volume $V_r(P)$.

- Note that the sum is actually over the r-faces of P.

BACKGROUND

- Volume, mixed volume (polynomials) — Minkowski, etc.
- Translation invariant valuations — Hadwiger (simple case); PM (in general).
- Algebra of polytopes (scissors congruence) — Jessen & Thorup; Sah.
- Polytope algebra — PM (Pukhlikov & Khovanskii).
- Combinatorial application (g-theorem) — PM.
- Weight algebra — PM (Fulton & Sturmfels).
- Piecewise polynomials — Billera; Brion.

CYLINDER AND TRANSLATION IDEALS

- Two ideals in Π:

cylinder ideal $Z := \langle [P] - 1 \mid P \in \mathcal{P} \rangle$,

translation ideal $T := \langle [t] - 1 \mid t \in V \rangle$.

THEOREM For $k \geq 1$, $Z^{d+k} = Z^d T^k$.

- Enough to show: if $P_0, \ldots, P_d$ are simplices with $\dim P_j \geq 1$, then there are $F_j \triangleleft P_j$ such that

$$([P_0] - [F_0]) \cdots ([P_d] - [F_d]) = 0.$$

(Here, use separating homomorphism γ.)

COROLLARY For $k \geq 0$, Π / T^{k+1} is (almost) a graded algebra over $\mathbb{Q}$.

THE ALGEBRA IDEAL

To extend the rational algebra structure to one over $\mathbb{F}$, we do the following.

- Replace $\tilde{\Pi}_0 \cong \mathbb{Z}$ by $\bar{\Pi}_0 := \tilde{\Pi}_0 \otimes \mathbb{F}$.
- Define the algebra ideal $A \subseteq Z^2$ by

$$A := \langle ([\lambda P]-1)([Q]-1) - ([P]-1)([\lambda Q]-1) \mid P, Q \text{ polytopes}, \lambda > 0 \rangle$$

- Define $\bar{\Pi}_1 := \tilde{\Pi}_1$, and

$$\bar{\Pi}_r := Z^r / (Z^{r+1} + A \cap Z^r), \; r \geqslant 2.$$

In geometric terms, this corresponds to confining attention to weakly continuous valuations (continuity relative to parallel displacement of facet hyperplanes).

NOTE $A \leqslant Z^3 + ZT$, and so Π/T is (almost) an $\mathbb{F}$-algebra already.

TRANSLATION COVARIANCE

Call a valuation φ on $\mathcal{P}$ polynomial of degree k if ψ, given by $\psi(P,t) := \varphi(P+t) - \varphi(P)$, is polynomial of degree $k-1$; case $k=0$ is translation invariant ($\psi \equiv 0$).

General term is "translation covariant".

- If φ is polynomial of degree k, then φ induces a homomorphism on Π / T^{k+1}.

- If $k \geq 2$ and φ is weakly continuous, then φ vanishes on A.

- General idea: write $\varphi(P) = \varphi([P])$; then $\psi(P,t) = \varphi([P] \cdot ([t] - 1))$. Now use induction on k.

THE SPACE $\overline{\Pi}_1$

- First note that

$$\big([P+Q]-1\big)-\big([P]-1\big)-\big([Q]-1\big)$$
$$=\big([P]-1\big)\big([Q]-1\big)\in Z^2.$$

Thus $\overline{\Pi}_1 = Z/Z^2$ is naturally a $\mathbb{Q}$-vector space.

- In fact, $\overline{\Pi}_1$ is isomorphic to the space of differences of support functionals; it is thus an $\mathbb{F}$-vector space.

- In any Π/T^{k+1}, we can define

$$\log P := \sum_{j=1}^{d+k} \frac{(-1)^{j-1}}{j}\big([P]-1\big)^j \in \overline{\Pi}_1.$$

Then $[P]=\exp(\log P)$.

NEGATIVE DILATATION

- Since $[\lambda P]\cdot[\mu P] = [(\lambda+\mu)P]$ for $\lambda,\mu \geq 0$, and $[P]^{-1} = [-P]\varepsilon$, it makes sense to define dilatation $x \mapsto \lambda \circ x$ by

$$\lambda \circ [P] := \begin{cases} [\lambda P], & \text{if } \lambda \geq 0, \\ [\lambda P]\varepsilon, & \text{if } \lambda < 0. \end{cases}$$

THEOREM $(\lambda \circ x)\cdot(\mu \circ x) = (\lambda+\mu)\circ x$ for all $\lambda,\mu \in \mathbb{F}$ and $x \in \Pi$.

- In $\overline{\Pi}_1$, the implication is

$$(\log P)\varepsilon = -\log(-P).$$

- Taking rth powers accounts for Euler-type relations for translation covariant valuations.

SUMMAND SUBRING

Let $\mathcal{K} \subseteq \mathcal{P}$ be a strong isomorphism class, and let $\Pi(\mathcal{K})$ be the subring of Π generated by elements $[P]\cdot[Q]^{-1}$ with $P, Q \in \mathcal{K}$.

- For each $P \in \mathrm{cl}\,\mathcal{K}$, $[P] \in \Pi(\mathcal{K})$. ($P \in \mathrm{cl}\,\mathcal{K}$ means that $P \preccurlyeq Q$ for some $Q \in \mathcal{K}$.)

- Passing to $\vec{\Pi}$, if $P \in \mathcal{K}$ has facet normals $u_1, \ldots, u_n$, then $\vec{\Pi}_1(\mathcal{K}) \leq \mathbb{F}^n$. Thus $\vec{\Pi}(\mathcal{K})$ is a standard algebra.

Simple Polytopes 1

Now let $\mathcal{K}$ consist of simple d-polytopes, say with facet normals $U = (u_1, \dots, u_n)$. Then $P \in \mathcal{K}$ is of the form

$$P(U;b) := \{x \in \mathbb{V} \mid \langle x, u_j \rangle \leq \beta_j \text{ (each } j)\},$$

with $b := (\beta_1, \dots, \beta_n) \in \mathbb{F}^n$.

- Any small enough perturbation b' of b gives $P(U;b') \in \mathcal{K}$ also.
- With e_k standard basis vector of $\mathbb{F}^n$, $P(U; \lambda b + e_k) \in \mathcal{K}$ for large enough λ. Hence $E_k := [P(U; \lambda b + e_k)] \cdot [P(U;b)]^{-1} \in \Pi(\mathcal{K})$. This is independent of b and λ.
- Write E_k^{α} if e_k is replaced by αe_k. Then
$$P(U;b) = \prod_{j=1}^{n} E_j^{\beta_j}.$$

SIMPLE POLYTOPES 2

Let $F_j \triangleleft P \in \mathcal{K}$ have facet normal u_j $(j=1,\dots,n)$.

THEOREM If $F_{j(1)} \cap \cdots \cap F_{j(k)} = \emptyset$, then

$$(E_{j(1)} - 1) \cdots (E_{j(k)} - 1) = 0.$$

- To see this, use the separating homomorphism φ.

COROLLARY $\overline{\Pi}(\mathcal{K})$ is a quotient of the face ring (over $\mathbb{F}$) of the dual polytope P^*.

- In fact, $\overline{\Pi}(\mathcal{K})$ is isomorphic to this face ring.

TENSORS

The ring (graded algebra) of symmetric tensors on $\mathbb{V}$ is

$$\mathbb{T} := \mathbb{F}[e_1, \dots, e_d],$$

with $\{e_1, \dots, e_d\}$ (standard) basis of $\mathbb{V}$.

$\mathbb{T}_s$ is space of s-tensors.

NOTATION Ordinary addition and multiplication

- $\dim \mathbb{T}_s = \binom{d+s-1}{s}$
- $\mathbb{T}$ embeds naturally in its field of fractions $\widehat{\mathbb{T}}$. In practice, we only need homogeneous fractions.
- Tensors can be identified with polynomial functions (on $\mathbb{E}^d$ as its own dual).

TENSORIALS

The s-tensorial $\Psi_s(P)$ of a polytope P is:

$$\Psi_s(P) := \frac{1}{s!} \int_P x^s \, dx$$

The calculation is intrinsic — in P.

By direct calculation, if

$$T = \operatorname{conv}\{a_0, \dots, a_k\}$$

is a k-simplex, then

$$\Psi_s(T) = \frac{k!}{(s+k)!} \operatorname{vol}_k T \sum_{s_0 + \dots + s_k = s} a_0^{s_0} \cdots a_k^{s_k}$$

Further define

$$\Psi(P) := \sum_{s \geq 0} \Psi_s(P)$$
$$= \int_P e^x \, dx$$

GREEN-MINKOWSI CONNEXIONS

NOTATION G a polytope :

$G_{\parallel} := \mathrm{lin}(G-G)$ subspace parallel to G;

$G_{\perp} := G_{\parallel}^{\perp}$ subspace perpendicular to G;

$u(F,G)$ is unit outer normal (in $G_{\parallel}$) to G at facet F (define $u(F,G) := 0$ if F is not a facet of G).

GREEN-MINKOWSKI CONNEXION (GMC)

$$\sum_F \Phi_s(F)\langle u(F,G), t\rangle = \Phi_{s-1}(G)\, t$$

for $t \in G_{\parallel}$.

PROOF Apply Green's theorem to $\frac{1}{s!}x^s$ on G.

REMARK Set $\Phi_s \equiv 0$ if $s < 0$.

TENSOR WEIGHTS

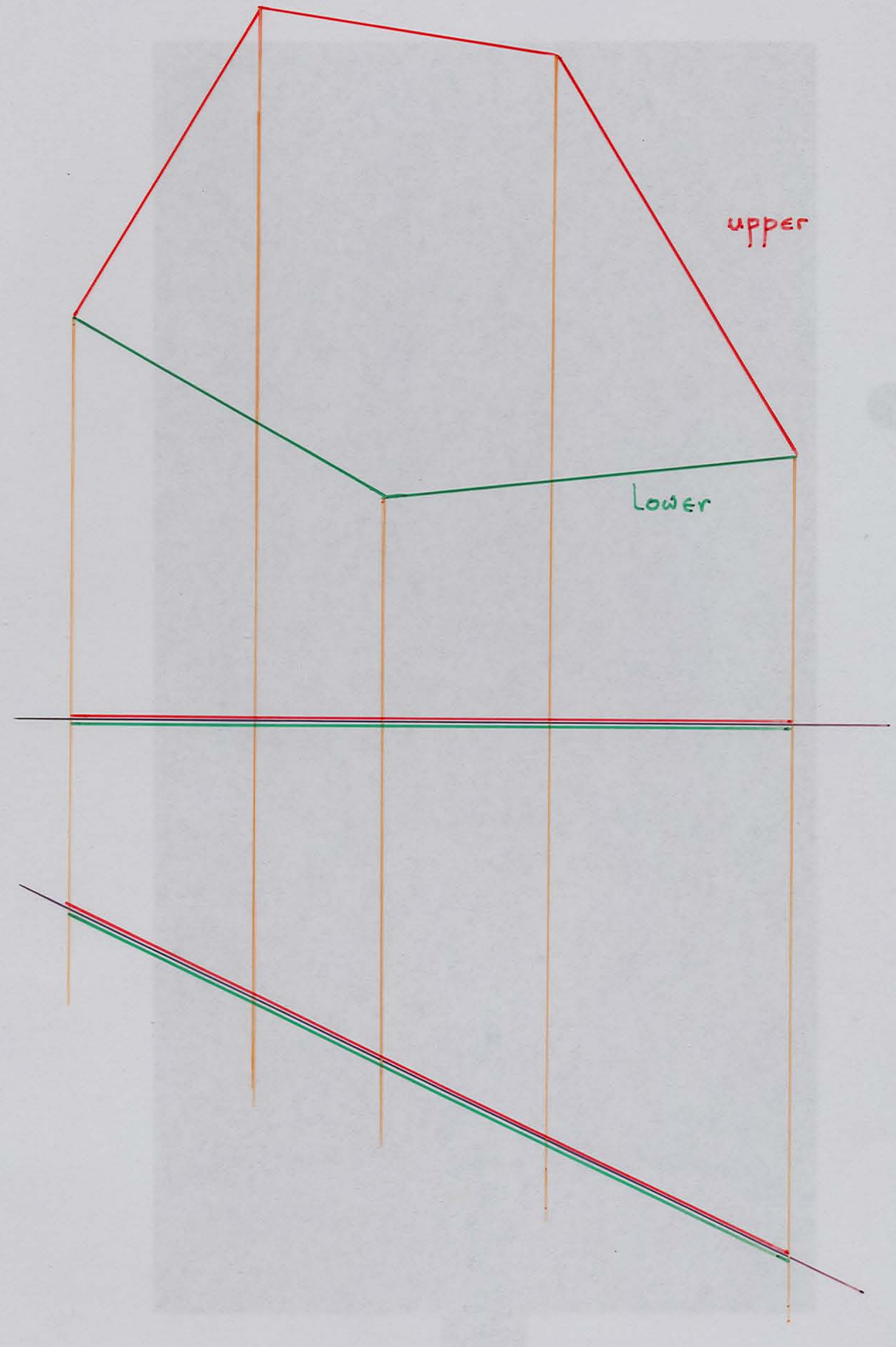
A **weight** a on a polytope P in $\mathcal{P}(\mathbb{V})$ assigns a symmetric tensor $a(F)$ to each face F of P, subject to GMC:

$$\sum_F a(F)\langle u(F,G), t\rangle = a(G)t$$

for $t \in G_{\|}$.

NOTATION $\mathcal{W}(P)$ denotes vector space of weights on P. An s-tensor on a k-face has degree $s+k$. The r-weights (of degree r) form the subspace $\mathcal{W}_r(P)$.

- $t \in G_{\perp}$ makes no contribution to GMC.
- Projecting along t gives Minkowski relation.
- GMC behaves nicely under
  - direct products;
  - (non-singular) linear mappings.


upper
lower

LINEAR MAPPINGS

THEOREM Linear mappings induce (algebra) homomorphisms on weights.

APPLICATION

- $a \in \mathcal{W}(P)$, $b \in \mathcal{W}(Q)$ give product weight $a \times b \in \mathcal{W}(P \times Q)$
- $P \times Q \mapsto P + Q$ induces $a \times b \mapsto ab$.
- associativity is easy.
- behaviour of multiplication under linear mappings is also easy.

REMARK It is exactly the Green-Minkowski connexions which allow these homomorphisms.

TENSORIAL RELATIONS

Polytope P induces r-class $p_r \in \mathcal{W}_r(P)$;

for a k-face F, define

$$p_r(F) := \Psi_{r-k}(F).$$

THEOREM $\quad p_r = \frac{1}{r!} p_1^r$.

- By definition of products, the k-class of $\lambda P + \mu Q$ $(\lambda, \mu \geq 0)$ is

$$\sum_{r+s=k} \lambda^r \mu^s p_r q_s.$$

- Set $P = Q$; compare coefficients of $\lambda^r \mu^s$:

$$p_{r+s} = \binom{r+s}{s} p_r q_s.$$

- Now use easy induction.

COROLLARY If $[P] := \sum_{r \geq 0} p_r$, then

$$[P] = \exp p_1,$$

$$p_1 = \log P \; (:= \log [P]).$$

RESTRICTION

P any polytope, F face of P. Write $a|_F$ for restriction of $a \in W(P)$ to F.

THEOREM Restriction is an algebra homomorphism.

- All calculations are performed locally.

REMARK In general, $W(P)|_F \neq W(F)$.

EXAMPLE P square pyramid with base F.

Then

$$\dim W_1(P) = 4,$$

$$\dim W_1(F) = 5.$$

- P simple, F a face of P, implies that

$$W(P)|_F = W(F).$$

ALTERNATIVE PRODUCT FORMULA

P simple polytope (in $P_{\|}$), facets F_j, unit normals u_j $(j = 1, \dots, n)$. Then

$$P = \{ x \in P_{\|} \mid \langle x, u_j \rangle \le \eta_j \ (j = 1, \dots, n) \}$$

is identified with its support vector

$$p := (\eta_1, \dots, \eta_n).$$

- $\mathcal{W}_1(P) \cong \mathbb{F}^n \oplus P_{\perp}$ in natural way.
- $\mathcal{W}_1(F) = \mathcal{W}_1(P)|_F$ for each face F of P.

THEOREM. If $y + t = (\eta_1, \dots, \eta_n) + t \in \mathcal{W}_1(P)$ and $a \in \mathcal{W}(P)$, then

$$((y+t)a)(P) = \sum_{j=1}^{n} \eta_j a(F_j) + t\, a(P).$$

REMARK This is compatible with Minkowski connexion.

Local Ideals

Let $F \leq P \in \wp$ and $a \in \mathcal{W}(P)$. We say that a is localized to F if $a(J) = 0$ whenever $J \leq P$ is such that $F \cap J = \emptyset$.

- The weights $a \in \mathcal{W}(P)$ localized to $F \leq P$ form an ideal $\mathcal{L}(F, P)$ – the local ideal.

- $\mathcal{L}(F, P) \cap \mathcal{L}(G, P) = \mathcal{L}(F \cap G, P)$. In particular, $G_1 \cap \cdots \cap G_k = \emptyset \Rightarrow \mathcal{L}(G_1, P) \cap \cdots \cap \mathcal{L}(G_k, P) = \{0\}$.

DIMENSIONS OF WEIGHT SPACES

THEOREM If $P \subseteq \wp$ is a simple d-polytope, then the Hilbert function of $\mathcal{W}(P)$ is

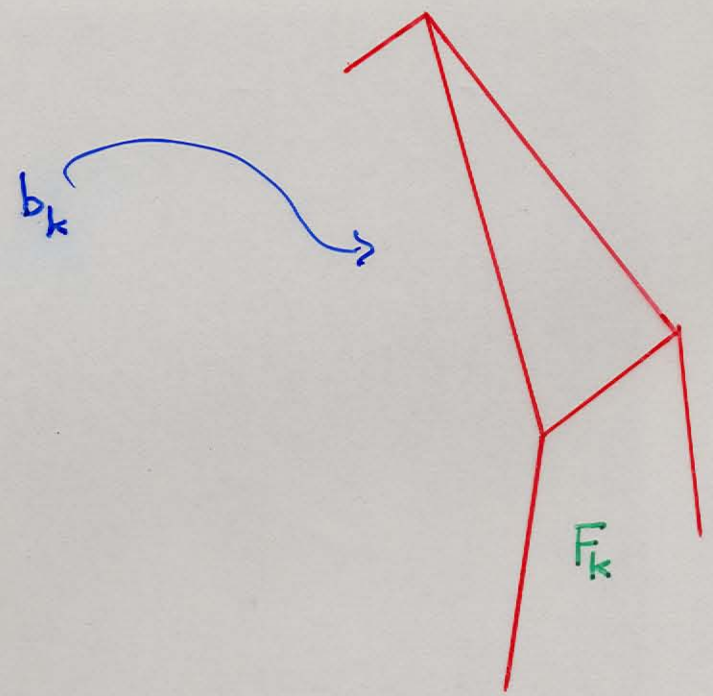
$$(1-\tau)^{-d} \sum_{r \geq 0} h_r(P)\, \tau^r.$$

- $\sum_{r \geq 0} h_r(P)\, \tau^r = \sum_{j \geq 0} f_j(P)\, (\tau - 1)^j$

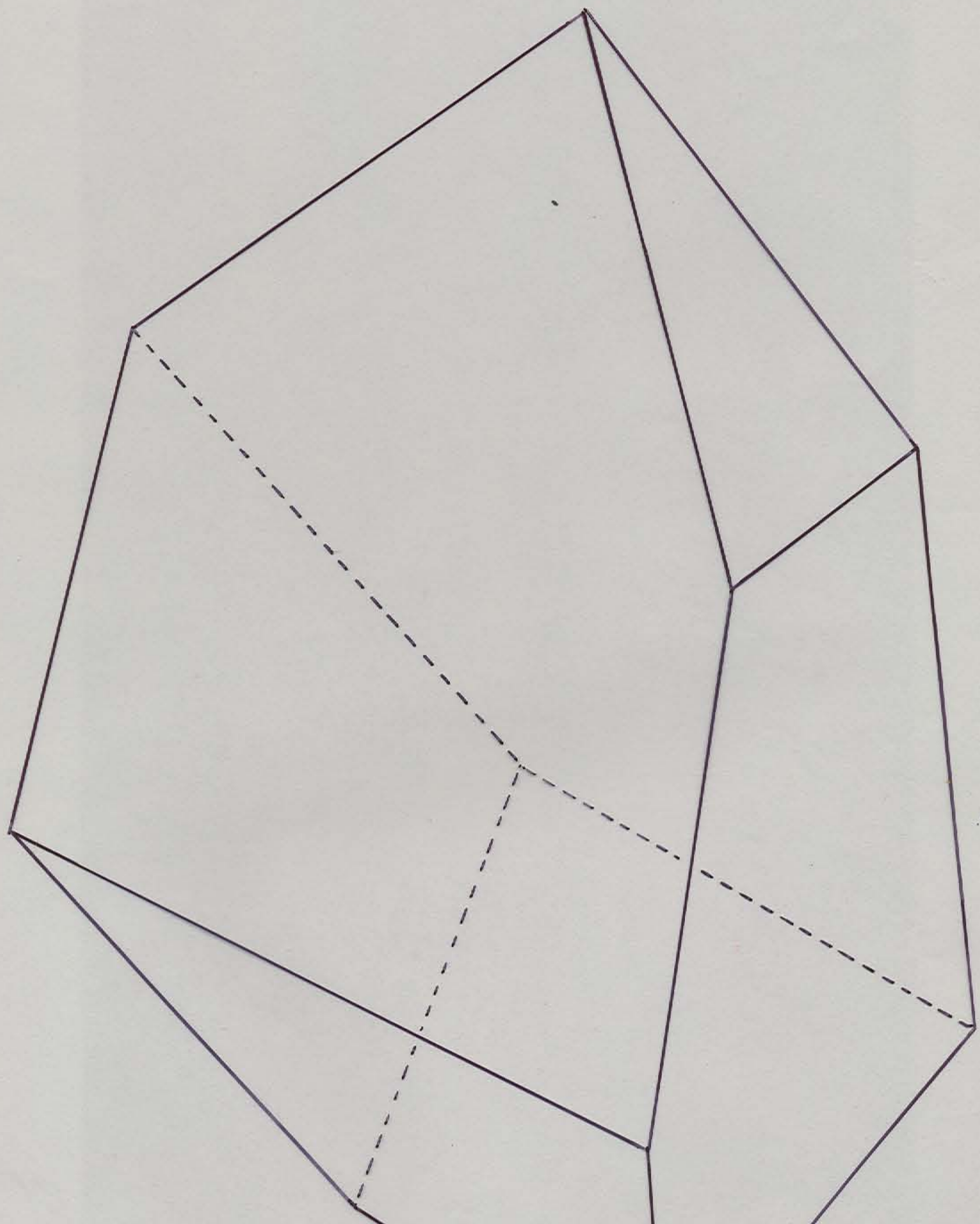
- The face ring of the dual P^* has the same Hilbert function. In fact, they are isomorphic.

IDEA OF PROOF

- Move variable half-space through P.
- At vertex of kind r, pick up generator $b_{j(1)} \cdots b_{j(r)}$.
- Hence also, $\mathcal{W}_1(P)$ generates $\mathcal{W}(P)$.
- $F_{j(1)} \cap \cdots \cap F_{j(r)} = \emptyset \Rightarrow b_{j(1)} \cdots b_{j(r)} = 0.$



Generator b_k is "localized" to facet F_k.



3

2

2

1

2

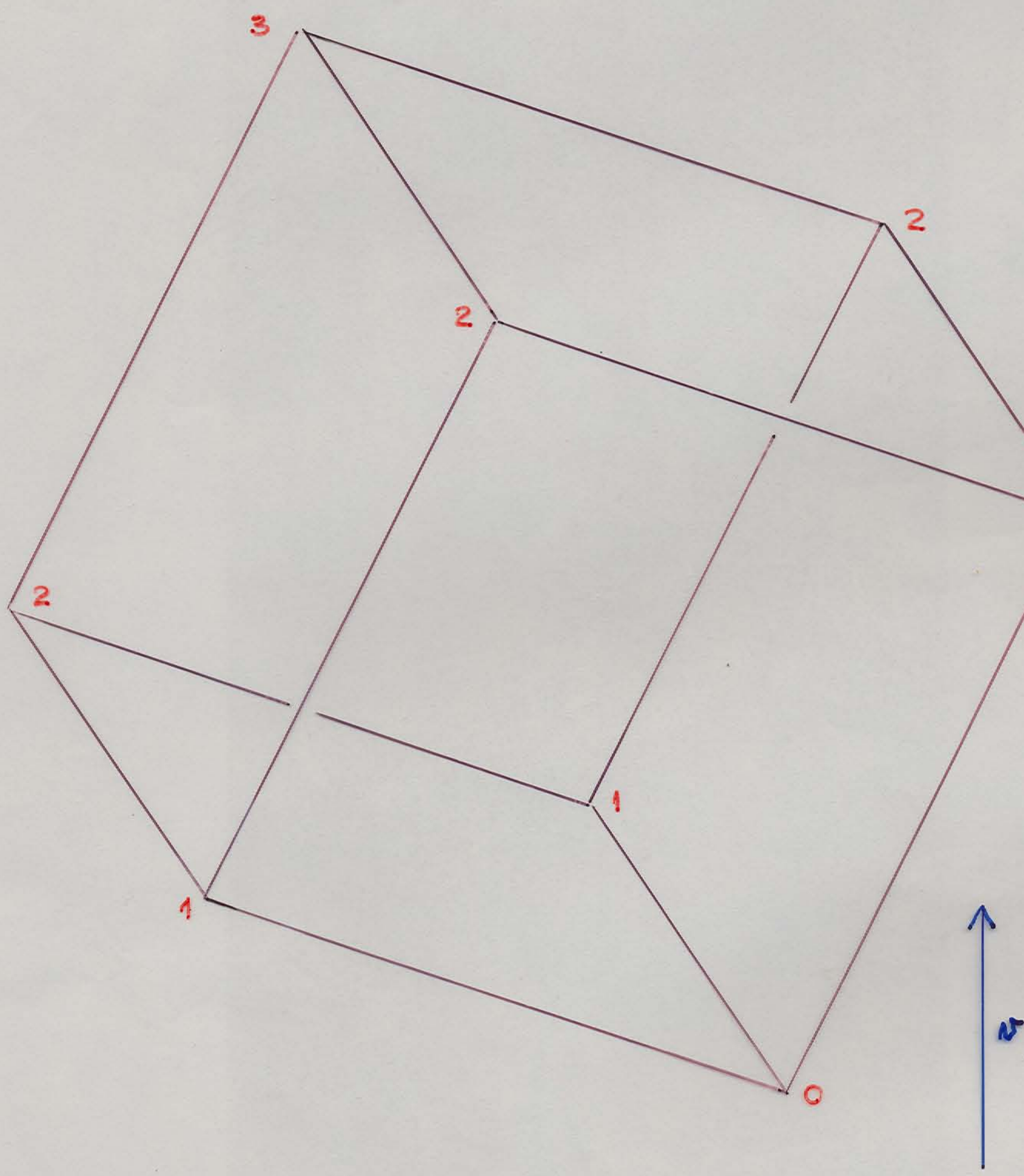
2

1

1

1

0



3
2
2
1
2
1
1
0
v

SWEEP BASES

- Let the sweep hyperplane give basis element b_j at vertex v_j ($1 \leq j \leq k$, say). Reverse the sweep direction to give basis element b_j^* at v_j.

THEOREM The bases $\mathcal{B} = \{b_1, \ldots, b_k\}$ and $\mathcal{B}^* = \{b_1^*, \ldots, b_k^*\}$ satisfy

$$b_i b_j^* = \begin{cases} \text{positive scalar (in } \mathbb{F}), & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}$$

- Thus $\mathcal{B}$ and $\mathcal{B}^*$ are nearly dual bases of $\mathcal{W}(P)$ as a $\mathbb{T}$-algebra.

REMARK All the above shows that $\Pi \cong \mathcal{W}$ is a natural way.

TRUNCATED WEIGHTS

Write $\mathcal{W}^{(j)}$ for the restriction of $\mathcal{W}$ to j-faces of polytopes.

- Translation covariant valuations which are polynomial of degree at most k can be factored through

$$\bigoplus_{j=0}^{d} \mathcal{W}^{(j)} / \mathrm{I}^{k+1}$$

- The case $k = 0$ is most important; it leads to the scalar weight algebra Ω.

SCALAR WEIGHTS

An r-weight on a polytope P assigns a number $a(F)$ to each r-face F of P. These satisfy the Minkowski relations: for each $(r+1)$-face G of P,

$$\sum_F a(F)\, u(F,G) = 0.$$

Here, $u(F,G)$ is the unit outer normal vector to G at its facet F, parallel to G.

- r-weights form vector space $\Omega_r(P)$.
- P induces natural r-weight p_r by
$$p_r(F) := \mathrm{vol}_r(F).$$
- If P is simple, then
$$\dim \Omega_r(P) = h_r(P).$$

MULTIPLICATION OF WEIGHTS

THEOREM There is a multiplication of weights $\Omega_r(P) \otimes \Omega_s(Q) \to \Omega_{r+s}(P+Q)$, induced by Minkowski addition of polytopes P and Q, which is associative and commutative.

Things to note:

- $i \in \Omega_0(Q)$ given by $i(V) = 1$ for each vertex V of Q embeds $\Omega_r(P) \hookrightarrow \Omega_r(P+Q)$ by $x \mapsto xi$;
- if P and Q are strongly isomorphic, then $\Omega(P) = \Omega(Q)$ (in natural way);
- hence $\Omega(P)$ is a graded algebra.

BASIC FACTS ON WEIGHTS

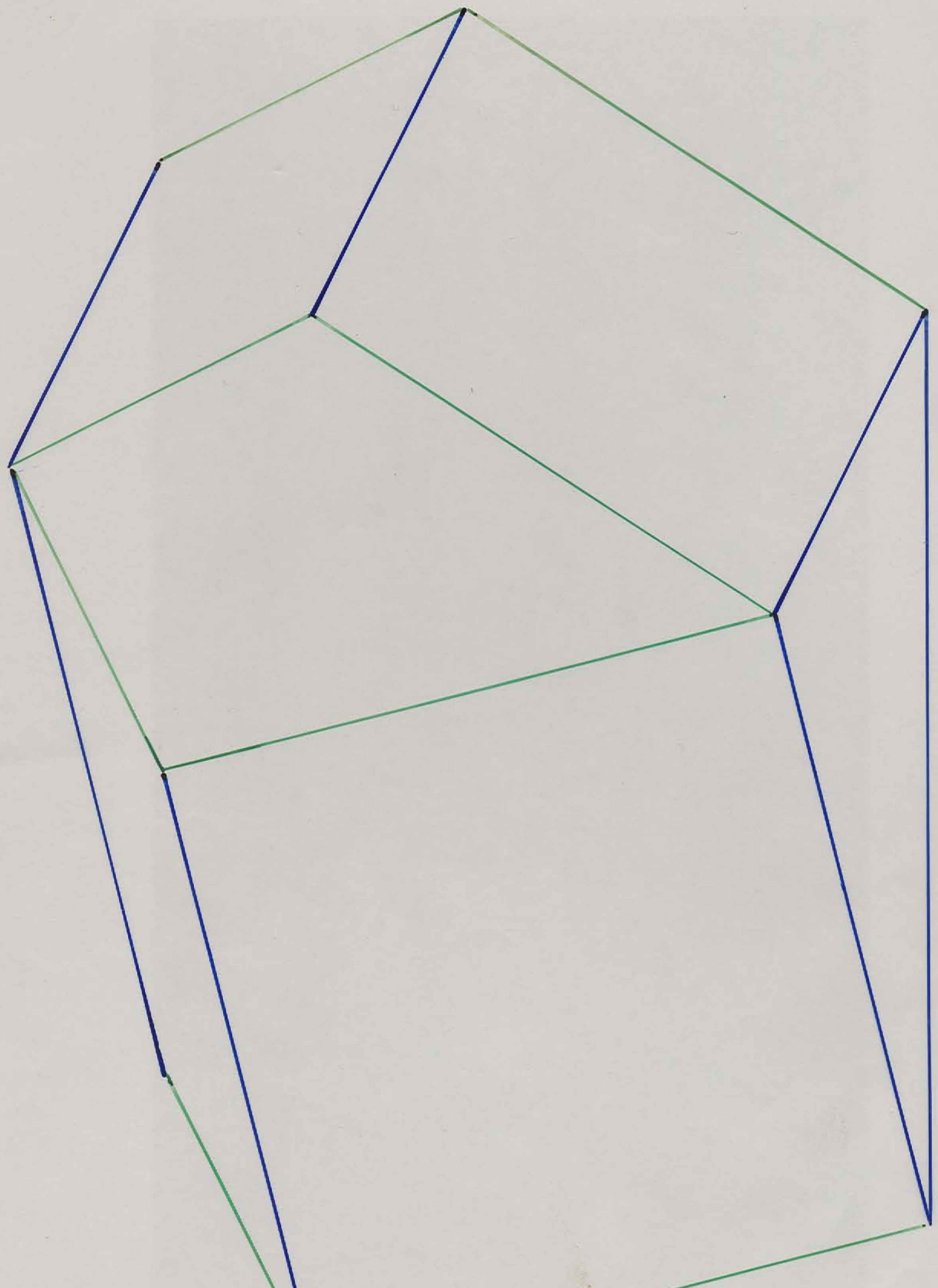
P (simple) d-polytope with n facets.

- We have **weight algebra** $\Omega(P)$; multiplication by "mixed-volume" calculations.
- $\Omega(P) = \bigoplus_{r=0}^{d} \Omega_r(P)$ (graded algebra), and $\Omega(P) = \langle \Omega_1(P) \rangle$.
- $\Omega_r(P)$ and $\Omega_{d-r}(P)$ are in duality.
- Multiplication by $\Omega_1(P)$ can be performed using **support vectors**.
- $P \preccurlyeq Q$ ("summand") $\Rightarrow \Omega(P) \hookrightarrow \Omega(Q)$.

(This means that $Q = P + P'$ for some P'.)

- $p_r = \frac{1}{r!} p^r$ ($p := p_1$). If we identify P with $p_0 + p_1 + \cdots = \exp(p)$, then

$$p = \log P.$$



THE g-THEOREM

P simple d-polytope with $f_j = f_j(P)$ j-faces for $j = 0, \ldots, d$. We define:

- $f(P, \tau) := \sum_{j=0}^{d} f_j \tau^j$.
- $h(P, \tau) = \sum_{r=0}^{d} h_r \tau^r := f(P, \tau - 1)$.
- $g(P, \tau) = \sum_{r=0}^{d+1} g_r \tau^r := (1-\tau) h(P, \tau)$.

$(f_0, \ldots, f_d)$ is the **f-vector** of P, and so on.

g-THEOREM $(f_0, \ldots, f_d)$ is the f-vector of some simple d-polytope if and only if:

a) $g_r = -g_{d-r+1}$, for each $r = 0, \ldots, d+1$;

b) $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is an **M-sequence**, so that there is a graded algebra $R = \bigoplus_{r=0}^{\lfloor d/2 \rfloor} R_r$ over an infinite field $\mathbb{F}$, with $R_0 \cong \mathbb{F}$, $R = \langle R_1 \rangle$ (as an algebra) and $\dim_{\mathbb{F}} R_r = g_r$ for $r = 0, \ldots, \lfloor d/2 \rfloor$.

HISTORICAL NOTES

- Conjectured by McMullen (1970).
  Background Walkup, Kruskal-Katona

- Sufficiency due to Billera-Lee (1980).

- Necessity due to Stanley (1979-80). Proof used hard Lefschetz theorem applied to cohomology ring of toric variety associated with rational simple polytope.

- Necessity reproved by McMullen (1992), using polytope algebra.

- Easier proof in 1994, motivated by queries from Kalai, using weight algebra.

MAIN THEOREMS

P simple d-polytope, $p := \log P \in \Omega_1(P)$.

For $0 \leq r \leq \frac{1}{2}d$, the primitive space is

$$\tilde{\Omega}_r(P) := \{ x \in \mathring{\Omega}_r(P) \mid p^{d-2r+1} x = 0 \}.$$

THEOREM (LD) $\quad p^{d-2r}\,\Omega_r(P) = \Omega_{d-r}(P)$.

COROLLARY $\quad \Omega(P)/\langle p \rangle$ is required algebra

THEOREM (HRM) The quadratic form $(-1)^r p^{d-2r} x^2$ is positive definite on $\tilde{\Omega}_r(P)$.

LEMMA $\quad \mathrm{HRM}(d-1) \Rightarrow \mathrm{LD}(d)$.

REMARK $\quad x \in \Omega_r(P) \Rightarrow x = py + z$, with $y \in \Omega_{r-1}(P)$, $z \in \tilde{\Omega}_r(P)$; then

$$p^{d-2r} x^2 = p^{d-2r+2} y^2 + p^{d-2r} z^2.$$

Thus we work with $\Omega_r(P)$ rather than $\tilde{\Omega}_r(P)$.

REMARKS

- Proof of Lemma. Suppose that $x \in \Omega_r(P)$ satisfies $p^{d-2r} x = 0$. Let $F \triangleleft P$; write $f := p|_F$, $y := x|_F$. Then $f^{d-2r} y = 0$ $\Rightarrow (-1)^r f^{d-2r-1} y^2 \geq 0$ by HRM$(d-1)$.

  Let $p = (\pi_1, \dots, \pi_n) > 0$ as support vector. With $F = F_j$, write f_j, y_j for f, y. Then

$$
\begin{aligned}
(-1)^r p^{d-2r} x^2 &= \sum_{j=1}^{n} (-1)^r f_j^{d-2r-1} y_j^2 \pi_j \\
&\geq 0,
\end{aligned}
$$

  with equality if and only if equality holds above. Hence $x = 0$, otherwise contradiction.

COROLLARY The quadratic form $p^{d-2r} x^2$ is non-singular on $\Omega_r(P)$.

GENERAL POSITION

- The form $p^{d-2r}x^2$ is non-singular on $\Omega_r(P)$ of rank $h_r(P)$.
- Perturbing P preserves non-singularity.
- Hence we can suppose the facet normals $u_1, \ldots, u_n$ of P to be in linearly general position.
- Then P is a general section of some simple $(d+1)$-polytope K with n $(\geq d+2)$ facets.

- We now obtain P from a d-simplex by a sequence of flips; we have an m-flip as the variable section sweeps over a vertex of type m.

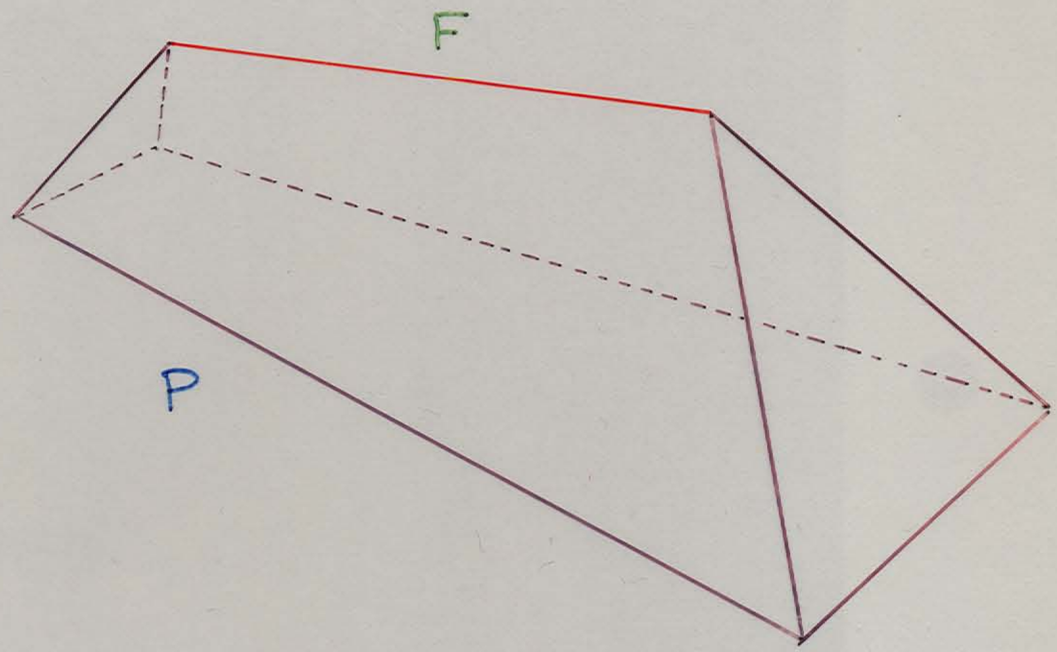
- Our aim is to keep track of how the form $p^{d-2r}x^2$ changes under flips.

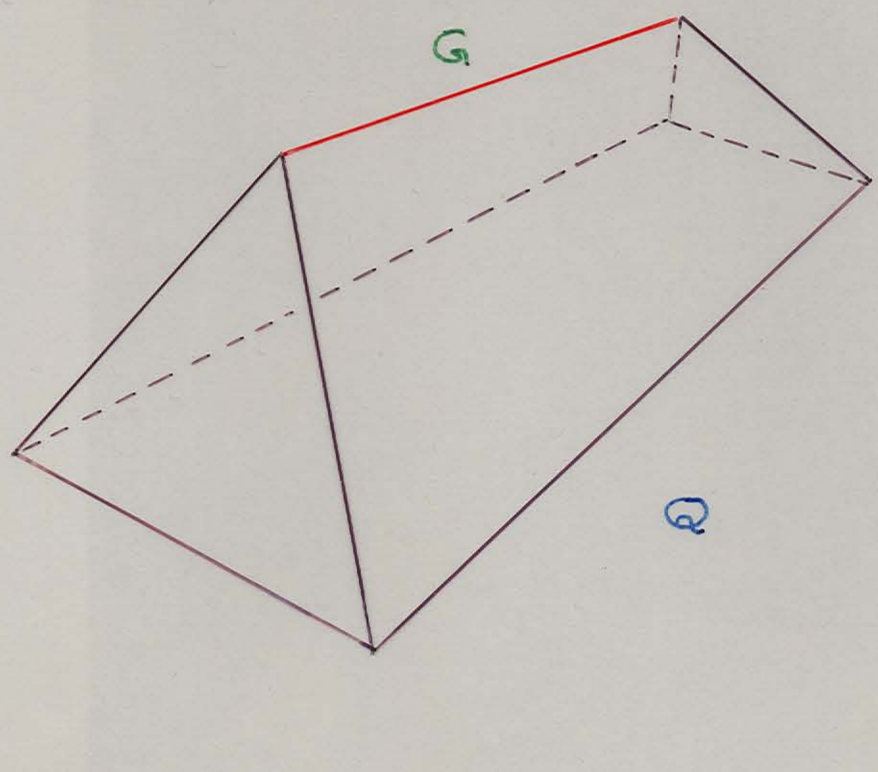
FLIPS

- $P \mapsto Q$ by m-flip. (Flips are dual to bistellar operations.)
- Reverse of m-flip is $(d+1-m)$-flip. So we take $0 \leq m \leq \frac{1}{2}(d+1)$.
- $m = 0$ creates d-simplex from $\emptyset$.
- $m = 1$ introduces new facet — watch notation!

- P: special face $F := F_{m+1} \cap \cdots \cap F_{d+1}$ is $(m-1)$-simplex bounded by $F_1, \ldots, F_m$.
- Q: special face $G := G_1 \cap \cdots \cap G_m$ is $(d-m)$-simplex bounded by $G_{m+1}, \ldots, G_{d+1}$.

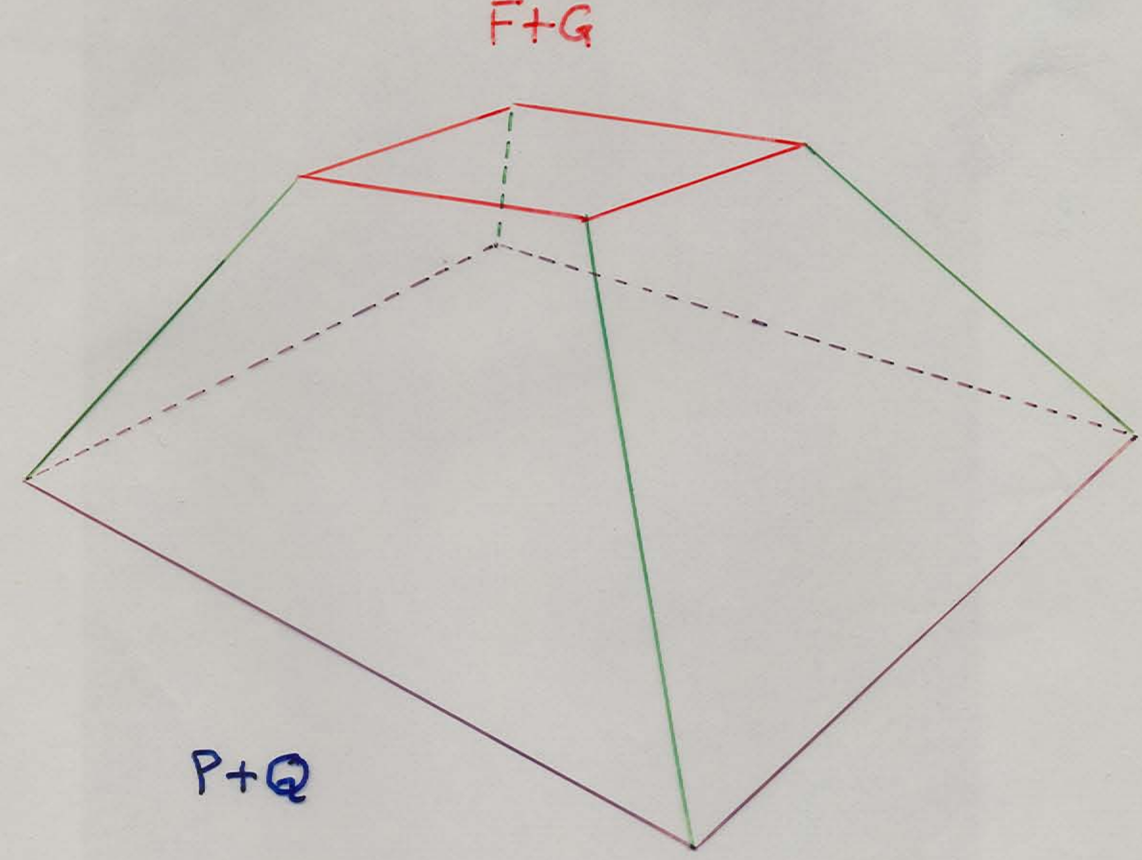
THEOREM $g_r(Q) = g_r(P) + \delta_{rm}$ for $0 \leq r \leq \frac{1}{2}(d+1)$.


F
P


G
Q

MORE REMARKS

- We work with $\Omega(P)$ and $\Omega(Q)$; hence we work in $\Omega(P+Q)$.
- $P+Q$ is also simple. If $m=1$, then $P+Q \approx Q$ (or $P \preccurlyeq Q$). If $m>1$, then $P+Q$ has $n+1$ facets; the extra facet is $F+G$.
- The m-flip $P \mapsto Q$ induces flips $F_j \mapsto G_j$ $(j=1,\dots,d+1)$:
    - $(m-1)$-flip for $j=1,\dots,m$;
    - m-flip for $j=m+1,\dots,d+1$.


F+G
P+Q

EVERTS

Let $H_j := \operatorname{aff} G_j$ $(j = 1, \ldots, d+1)$. Then $H_1, \ldots, H_{d+1}$ bound a d-simplex $\overline{S}$. The outer normals to the facets of $\overline{S}$ are $-u_1, \ldots, -u_m, u_{m+1}, \ldots, u_{d+1}$. For convenience, take $\overline{S}$ to be regular (suitable linear image).

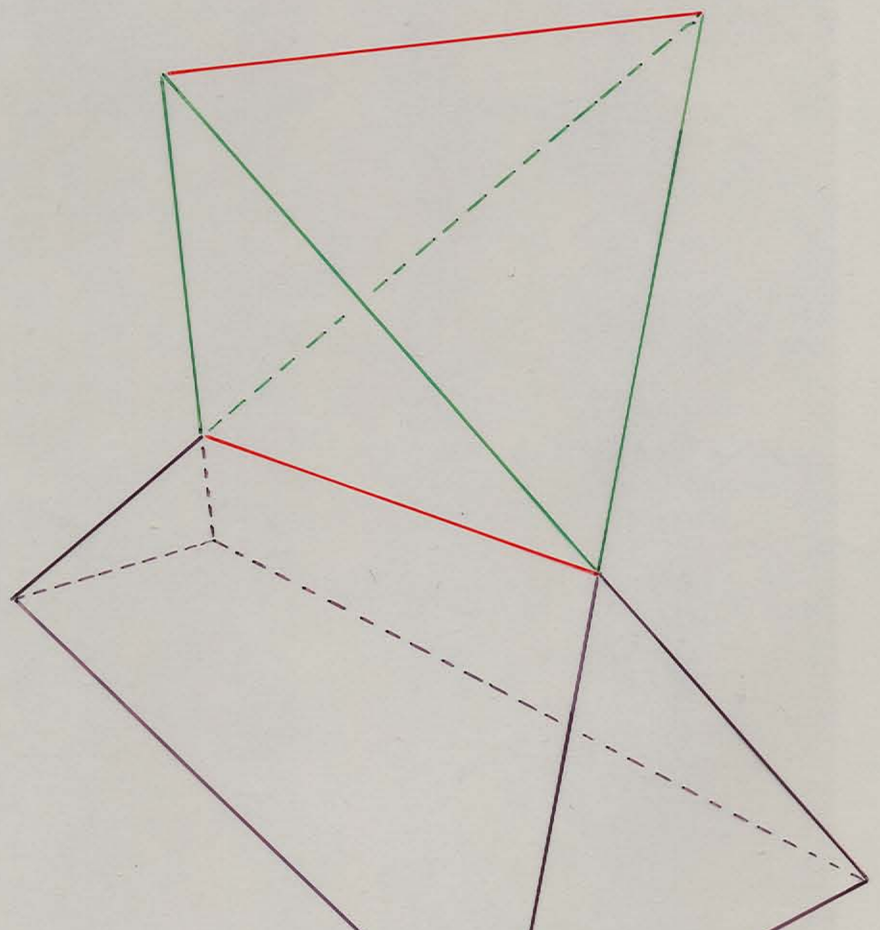
$\overline{S}$ has opposite faces G, $-F$ (after scaling).

r-face $\overline{K} \subseteq \overline{S}$ has r-volume $\frac{1}{r!}\sigma_r$.

Each r-face K of P (Q) which meets F (G) is parallel to unique r-face $\overline{K} \subseteq \overline{S}$. Say that K is of kind k if k vertices of $\overline{K}$ lie in $-F$.

The r-evert $s_r \in E_r(P, Q)$ satisfies

$$s_r(K) = (-1)^k \sigma_r \text{ if } K \text{ is of kind } k.$$



EVERT IDEAL

Define

$E(P,Q) := \{ x \in \Omega(P+Q) \mid x|_J = 0$ if

$J \in \mathcal{F}(P+Q)$ satisfies $J \cap (F+G) = \emptyset \}$.

$E_r(P,Q) := E(P,Q) \cap \Omega_r(P+Q)$.

LEMMA $E(P,Q)$ is (evert) ideal of $\Omega(P+Q)$.

- Calculations involving $E(P,Q)$ and $\Omega(P)$ or $\Omega(Q)$ only use facets F_j or G_j for $j = 1, \ldots, d+1$.
- In such calculations, we can suppose that $0 \in \operatorname{relint} F$ or $0 \in \operatorname{relint} G$.

NOTES

- If $r \geqq m$, then $s_r \in \Omega_r(Q)$.
- If $r > d-m$, then $s_r \in \Omega_r(P)$ also.

THEOREM If $1 \leq r \leq d-m$ and $y \in \Omega_r(Q)$, then there are unique $x \in \Omega_r(P)$ and $\nu \in \mathbb{R}$ such that $y = x + \nu s_r$.

- For proof, use a variable half-space $H^-(v,\alpha)$ on $P+Q$ which meets $F+G$ last.

COROLLARY a) For $r < m$, $\Omega_r(P) \cong \Omega_r(Q)$.

b) For $m \leq r \leq d-m$, $\Omega_r(P) \subseteq \Omega_r(Q)$ with codimension 1 $\left(\Omega_r(Q) = \Omega_r(P) \oplus \langle s_r \rangle\right)$.

c) For $r > d-m$, $\Omega_r(P) = \Omega_r(Q)$.

TRANSITION CALCULATIONS

Flip $P \mapsto Q$ passes through transition polytope T. All $d+1$ (special) facets pass through common vertex o. Write $t := \log T$.

- If $m = 1$, then $T \approx P$.

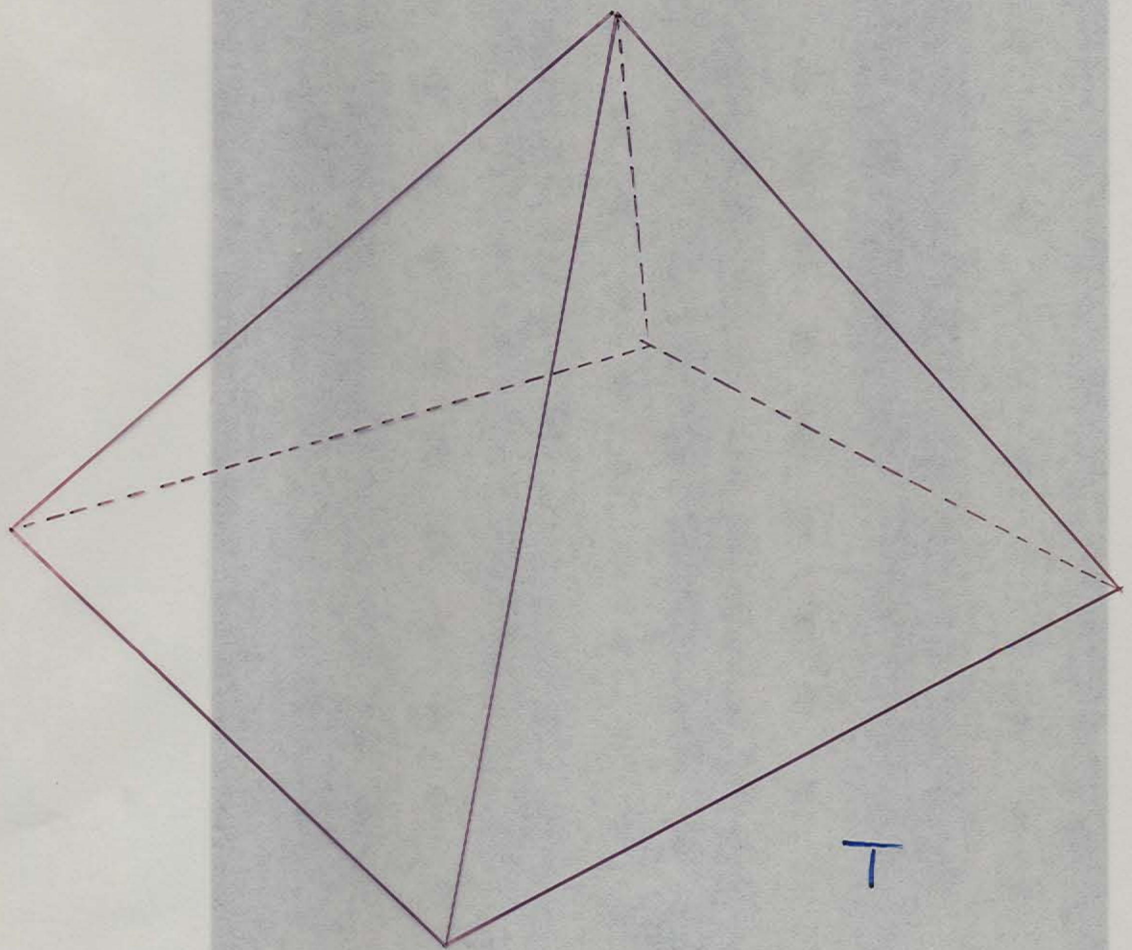
THEOREM a) $t^{d-2r}\Omega_r(P) = \Omega_{d-r}(P)$ $(r \leq \frac{1}{2}d)$;

b) $t^{d-2r}\Omega_r(Q) = \Omega_{d-r}(Q)$ $(r < m$ or $r = \frac{1}{2}d)$.

THEOREM a) For $0 \leq r \leq \frac{1}{2}d$, $(-1)^r t^{d-2r} x^2$ is positive definite on $\{x \in \Omega_r(P) \mid t^{d-2r+1} x = 0\}$;

b) for $r < m$ or $r = \frac{1}{2}d$, $(-1)^r t^{d-2r} y^2$ is positive definite on $\{y \in \Omega_r(Q) \mid t^{d-2r+1} y = 0\}$.

LEMMA The analogues of HRM$(d-1) \Rightarrow$ LD(d) hold.



T

EVERT CALCULATIONS

THEOREM a) $ts_r = 0$.

b) If $k \geq m$, then $\Omega_k(P)\, s_r = \{0\}$.

c) If $m \leq r \leq \frac{1}{2}d$, then $(-1)^m q^{d-2r} s_r^2 > 0$.

- a) This is trivial, since $t|_{F+G} = 0$.

- b) Only $F_1, \ldots, F_m$ contribute. Such F_j is $(m-1)$-flipped to G_j. Now use induction (initial case is (a)).

- c) Only $G_{m+1}, \ldots, G_{d+1}$ contribute; here, F_j is m-flipped to G_j. Induction reduces to case $d = 2r$. Any contribution to s_r^2 comes from $K + K'$, where K, K' are r-faces of kinds $k, m-k$ for some k.

NOTES TO PROOF OF LEMMA

- Leave (b) (of first Theorem) to end – this is just bookkeeping.
- Let K be a facet of P.
  - $K \neq F_1, \ldots, F_{d+1}$ as before.
  - $K = F_1, \ldots, F_{d+1} \Rightarrow t|_K$ corresponds to transition polytope.
    - $m < \frac{1}{2}(d+1)$. We have induced $(m-1)$- or m-flip on K, so use (a) (of second theorem).
    - $m = \frac{1}{2}(d+1)$ (d odd). Use (a) for induced $(m-1)$-flip; (b) for induced m-flip (inverse of an $(m-1)$-flip).

OVER THE TRANSITION

We have $0 \leq r \leq \frac{1}{2}d$, $1 \leq m \leq \frac{1}{2}(d+1)$.

For $y \in \Omega_r(Q)$ write $y = x + \nu s_r$ with $x \in \Omega_r(P)$, $\nu \in \mathbb{F}$. For $0 < \lambda \leq 1$, define

$$p_\lambda := (1-\lambda)t + \lambda p, \qquad q_\lambda := (1-\lambda)t + \lambda q.$$

- $r < m$:

$$\lim_{\lambda \to 0} p_\lambda^{d-2r} x^2 = t^{d-2r} x^2$$

$$= t^{d-2r} y^2 = \lim_{\lambda \to 0} q_\lambda^{d-2r} y^2$$

Non-singular, same rank $\Rightarrow$ same signature.

- $r \geq m$:

$$q_\lambda^{d-2r} y^2 = q_\lambda^{d-2r} x^2 + \lambda^2 \nu^2 q^{d-2r} s_r^2$$

$$\lim_{\lambda \to 0} p_\lambda^{d-2r} x^2 = t^{d-2r} x^2 = \lim_{\lambda \to 0} q_\lambda^{d-2r} x^2$$

Form acquires new eigenvalue, sign $(-1)^m$.

Final Notes

- Remaining parts of two Theorems are easy to check.
- Induction starts from d-simplex P. Here $\Omega_r(P) = \mathrm{lin}\{p^r\}$ $(r = 0, \ldots, d)$.
- Forms $p^{d-2r}x^2$ $(x \in \Omega_r(P),\ 0 \leq r \leq \frac{1}{2}d)$ change in correct way over flips.

- Case $r = 1$ is (essentially) Minkowski's second inequality. The Brunn-Minkowski theorem (with equality for polytopes) and Minkowski's existence and uniqueness theorem then follow.
- Analogues of Aleksandrov-Fenchel inequalities exist for $r > 1$.

STARTING POINT

$\mathcal{P}$ family of convex polytopes in $\mathbb{E}^d$.

V volume; $\varphi = V^{1/d}$.

We assume

WEAK BRUNN-MINKOWSKI THEOREM φ is concave on $\mathcal{P}$, so that, for $P, Q \in \mathcal{P}$, $\lambda, \mu \geq 0$,

$$\varphi(\lambda P + \mu Q) \geq \lambda \varphi(P) + \mu \varphi(Q).$$

We wish to prove

BM EQUALITY CONDITION If P, Q are full-dimensional and $\lambda, \mu > 0$, then equality holds in BM if and only if P, Q are homothetic:

$$Q = \nu P + t$$

with $\nu > 0$, $t \in \mathbb{E}^d$.

MINKOWSKI'S THEOREM

$U = (u_1, \ldots, u_n)$ distinct unit vectors spanning $\mathbb{E}^d$.

THEOREM Let $\alpha_1, \ldots, \alpha_n > 0$ satisfy

$$\sum_{i=1}^{n} \alpha_i u_i = 0.$$

Then there exists $P \in \mathcal{P}$, unique up to translation, whose facets F_j have outer normals u_j and areas $\mathrm{vol}_{d-1}(F_j) = \alpha_j$ for $j = 1, \ldots, n$.

Only uniqueness will be involved here.

REMARK Both existence and uniqueness are trivial for $d = 2$. The familiar proof of uniqueness uses BM with equality.

REPRESENTATIONS

For $b = (\beta_1, \dots, \beta_n) \in \mathbb{E}^n$, write

$$P(U;b) := \{x \in \mathbb{E}^d \mid \langle x, u_j \rangle \leq \beta_j \ (j = 1, \dots, n)\}.$$

If $t \in \mathbb{E}^d$, then $P(U;b) + t = P(U;b')$, where

$$b' = b + (\langle t, u_1 \rangle, \dots, \langle t, u_n \rangle).$$

Thus the linear mapping $\Psi : \mathbb{E}^n \to \mathbb{E}^{n-d}$ with

$$\ker \Psi := \{(\langle t, u_1 \rangle, \dots, \langle t, u_n \rangle) \mid t \in \mathbb{E}^d\}$$

identifies translates.

If $E = (e_1, \dots, e_n)$ is the standard basis of $\mathbb{E}^n$,

$$\overline{U} = (\overline{u}_1, \dots, \overline{u}_n) := E\Psi$$

is a linear transform of U, and

$$p := b\Psi = \sum_{i=1}^{n} \beta_i \overline{u}_i$$

represents $P(U;b)$ (and its translates).

VOLUME AND AREA

If $p \in \operatorname{pos} \overline{U}$ represents $P = P(U; b)$, then we can write $V(p) := V(P)$, $\varphi(p) := \varphi(P)$. The gradient $a = a(p) := \nabla V(p)$ is such that

$$\alpha_j = \alpha_j(p) = \langle a, \bar{u}_j \rangle$$

is the area of the jth facet of P. Moreover,

$$V(p) = \frac{1}{d} \sum_{i=1}^{n} \beta_i \alpha_i = \langle p, a \rangle.$$

If $\mathcal{V} := \{p \mid V(p) \geq 1\}$ (a level set of φ), and $V(p) = 1$, then $a = a(p)$ is an inner normal to the (unique) hyperplane which supports $\mathcal{V}$ at p.

EQUIVALENCE

THEOREM The following are equivalent.

- BM equality condition holds;
- $\mathcal{N}$ is strictly convex where no facets are redundant;
- the mapping $p \mapsto a = a(p)$ is one-to-one where no facets are redundant.

All three conditions say that (modulo non-redundancy) the only line segments in the graph of φ lie in lines through o.

UNIQUENESS

We prove that $p \mapsto a$ is 1-1. If $p \mapsto a$, $q \mapsto a$, we let U form the normal-set to $P+Q$. We may have redundancy, but this will not matter.

Change notation: $p \mapsto p_0$, $q \mapsto p_1$. Then $p_\lambda := (1-\lambda)p_0 + \lambda p_1$ represents $(1-\lambda)P + \lambda Q$. Write $a_\lambda := a(p_\lambda)$, so that $a_0 = a_1 = a$. If $\alpha_{j\lambda} := \langle a_\lambda, \bar{u}_j \rangle$, then $\alpha_{j\lambda} \geq \alpha_j$ $(= \alpha_{j0} = \alpha_{j1})$, with strict inequality for $0 < \lambda < 1$ unless the jth facets of P, Q are congruent (note: $\alpha_{j\lambda} > 0$)

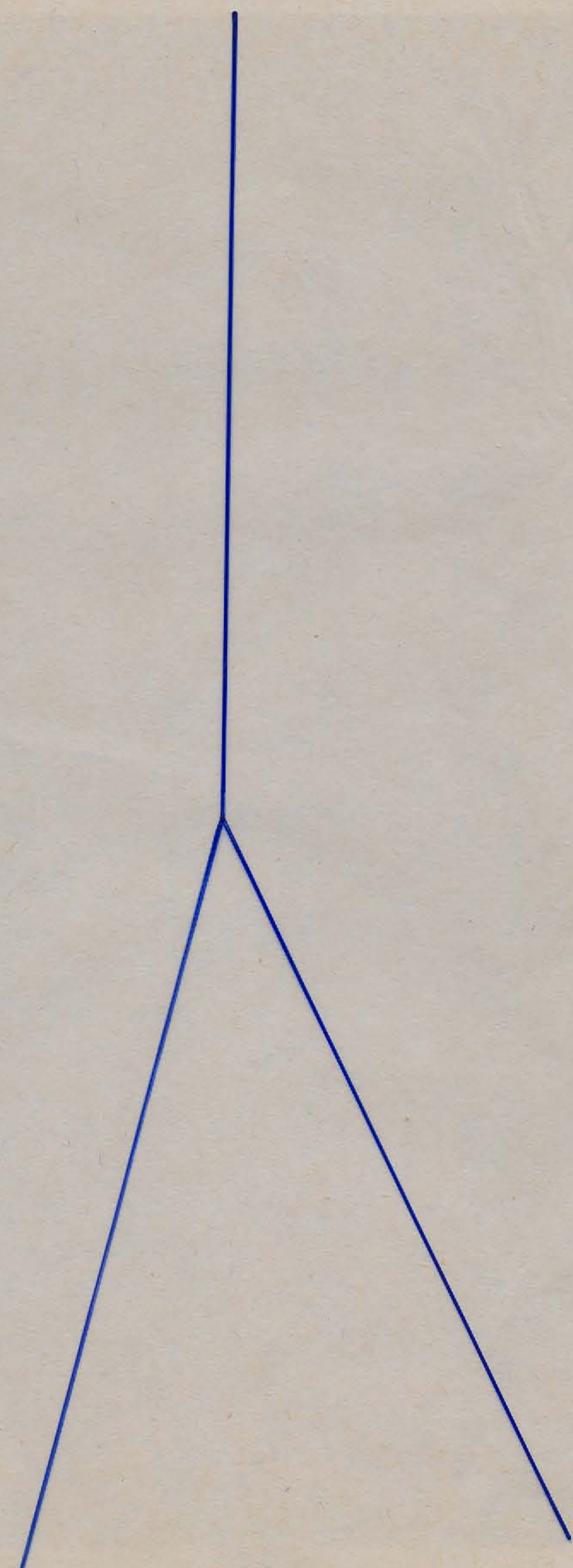
If $P = P(U; b_0)$, $Q = P(U; b_1)$, then, since $\{p_\lambda \mid 0 \leq \lambda \leq 1\}$ is a line-segment in the graph of φ, and we can take all $\beta_{ij} > 0$ $(i = 0, 1;\ j = 1, \ldots, n)$, we have equality in

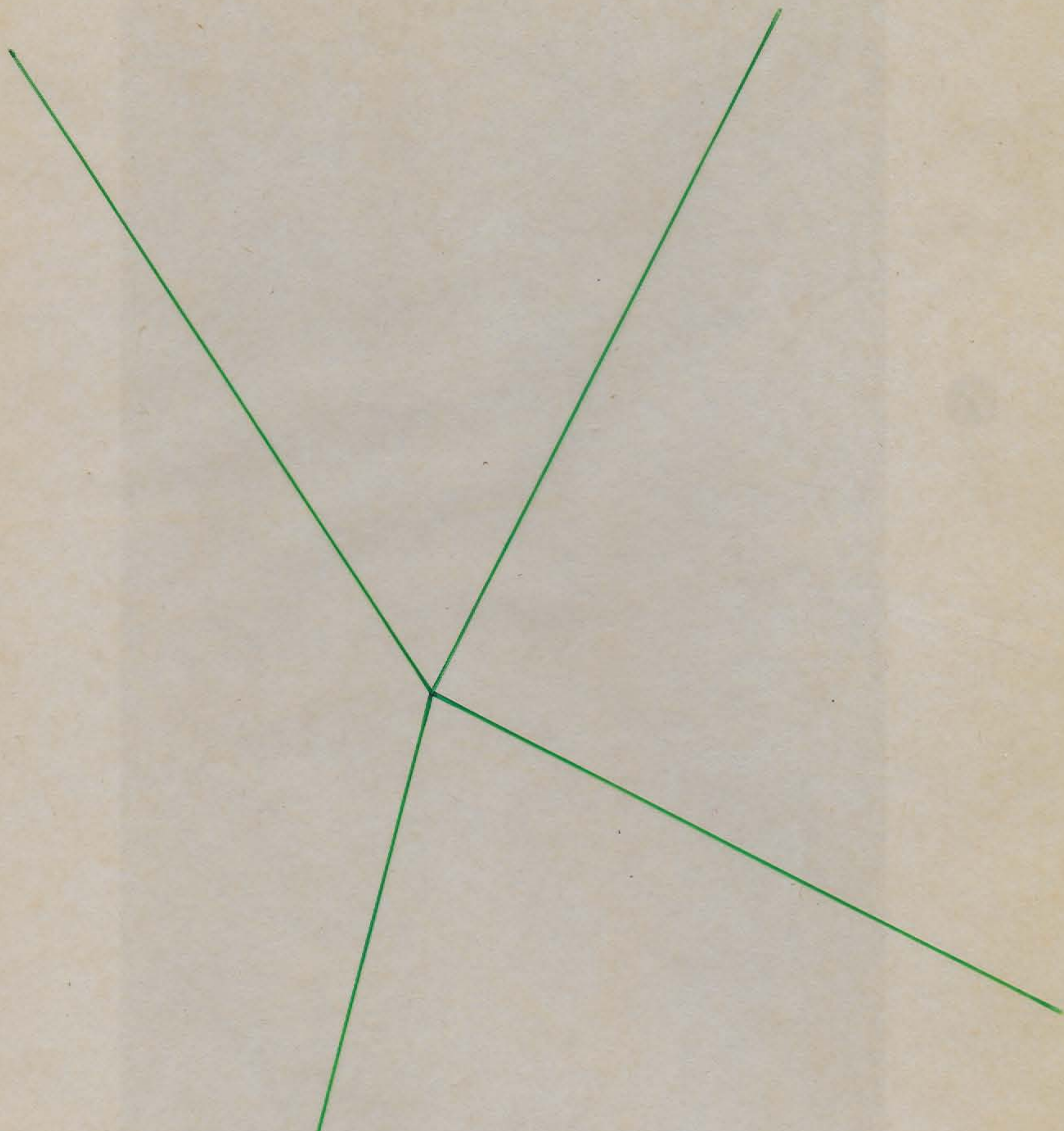
$$\varphi(p_\lambda) = \left(\tfrac{1}{d}\langle p_\lambda, a_\lambda \rangle\right)^{1/d} \geq \left(\tfrac{1}{d}(1-\lambda)\langle p_0, a \rangle + \tfrac{1}{d}\lambda\langle p_1, a \rangle\right)^{1/d}$$

$$\geq (1-\lambda)\varphi(p_0) + \lambda\varphi(p_1).$$

Hence $p = q$ as claimed.

P





Q