Ehrhart Quasi-polymomials of Clebsch–Gordan Coefficients

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Talk Outline

1. Lie algebras and their Clebsch–Gordan coefficients
2. Polytopes for Clebsch–Gordan coefficients
3. . . . and their computational applications.
4. A conjecture

Lie Algebras

Definition. *A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot\,,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t.*

- $[X, Y] = -[Y, X],$ *for all* $X, Y \in \mathfrak{g}$,
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$ *for all* $X, Y, Z \in \mathfrak{g}$.

Examples.

- $\mathrm{End}(V)$ — endomorphisms on vector space V, with

$$[X, Y] = XY - YX.$$

- $\mathfrak{sl}_r(\mathbb{C})$ — traceless $r \times r$ matrices over $\mathbb{C}$.

Simple Lie Algebras and Their Representations

Definition. *A simple Lie Algebra contains no nontrivial ideal.*

Fact. Simple Lie algebras come in only four infinite types (the so-called classical types)

$$A_r, \quad B_r, \quad C_r, \quad D_r, \qquad r = 1, 2, \ldots,$$

plus five sporadic cases: G_2, F_4, E_6, E_7, E_8.

E.g., Type A_{r-1} consists of the Lie algebras isomorphic to $\mathfrak{sl}_r(\mathbb{C})$, $r = 1, 2, \ldots$.

Definition. *A representation of a Lie algebra is a linear map* $\rho : \mathfrak{g} \to \mathrm{End}(V)$ *s.t.*

$$\begin{aligned} \rho([X, Y]) &= [\rho(X), \rho(Y)] \\ &= \rho(X)\rho(Y) - \rho(Y)\rho(X), \quad \textit{for all } X, Y \in \mathfrak{g}. \end{aligned}$$

An irreducible representation (irrep) is one in which no nontrivial subspace is fixed under $\rho(\mathfrak{g})$.

Decomposing Representations

Let $\mathfrak{g}$ be a simple Lie algebra.

Then the irreps of $\mathfrak{g}$ are indexed by elements of a semigroup $S_{\mathfrak{g}} \hookrightarrow \mathbb{Z}^r$ of highest weights:

$$V_\lambda \text{ — the irrep of } \mathfrak{g} \text{ with highest weight } \lambda \in S_{\mathfrak{g}}.$$

The dimension r of $S_{\mathfrak{g}}$ is the rank of $\mathfrak{g}$.

Any representation decomposes into irreps:

$$W = \bigoplus_{\nu \in S_{\mathfrak{g}}} C_W^\nu V_\nu.$$

Clebsch–Gordan Coefficients

Definition. *Given highest weights λ and μ, we can write*

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \in S_{\mathfrak{g}}} C^{\nu}_{\lambda\mu} V_\nu .$$

*The values $C^{\nu}_{\lambda\mu}$ are the **Clebsch–Gordan** coefficients for $\mathfrak{g}$.*

When $\mathfrak{g}$ is of type A_r ($\mathfrak{g} \cong \mathfrak{sl}_{r+1}(\mathbb{C})$), $C^{\nu}_{\lambda\mu}$ is called a **Littlewood–Richardson** coefficient.

Enter Polyhedra

Kostka numbers and Clebsch–Gordan coefficients are clearly nonnegative integers. This suggests a combinatorial interpretation . . .

Idea: **Encode as lattice points in Polyhedra** (Johnson (1979), Berenstein–Zelevinsky (1988), Knutson–Tao (1999), Berenstein–Zelevinsky (2001), Pak–Vallejo (2004), Baldoni-Silva–Beck–Cochet–Vergne (2005))

Theorem. [Berenstein & Zelevinsky, 2001] *There exist* **explicitly described linear inequalities** *for families of polytopes that contain a number of lattice points equal to the Clebsch–Gordan coefficients.*

In type A_{r-1} ($\mathfrak{g} \cong \mathfrak{sl}_r(\mathbb{C})$), these polyhedra have particularly nice descriptions.

Polytopes for Clebsch–Gordan Coefficients

Definition. *A **hive pattern** is a triangular array of real numbers*

$$\begin{array}{ccccccccc}
 & & & & h_{00} & & & & \\
 & & & h_{10} & & h_{01} & & & \\
 & & h_{20} & & h_{11} & & h_{02} & & \\
 & \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \\
h_{r0} & & h_{r-1,1} & & \cdots & & h_{1,r-1} & & h_{rr}
\end{array}$$

*satisfying the **Rhombus Inequalities**: in every "little rhombus" of entries,*

$$\begin{array}{ccccccccccccc}
 & & & & & & c & & & & & & \\
c & & b & & & & & & & & a & & c \\
 & & & & \text{or} & a & & b & \text{or} & & & & \\
 & a & & d & & & & & & d & & b & \\
 & & & & & & d & & & & & &
\end{array}$$

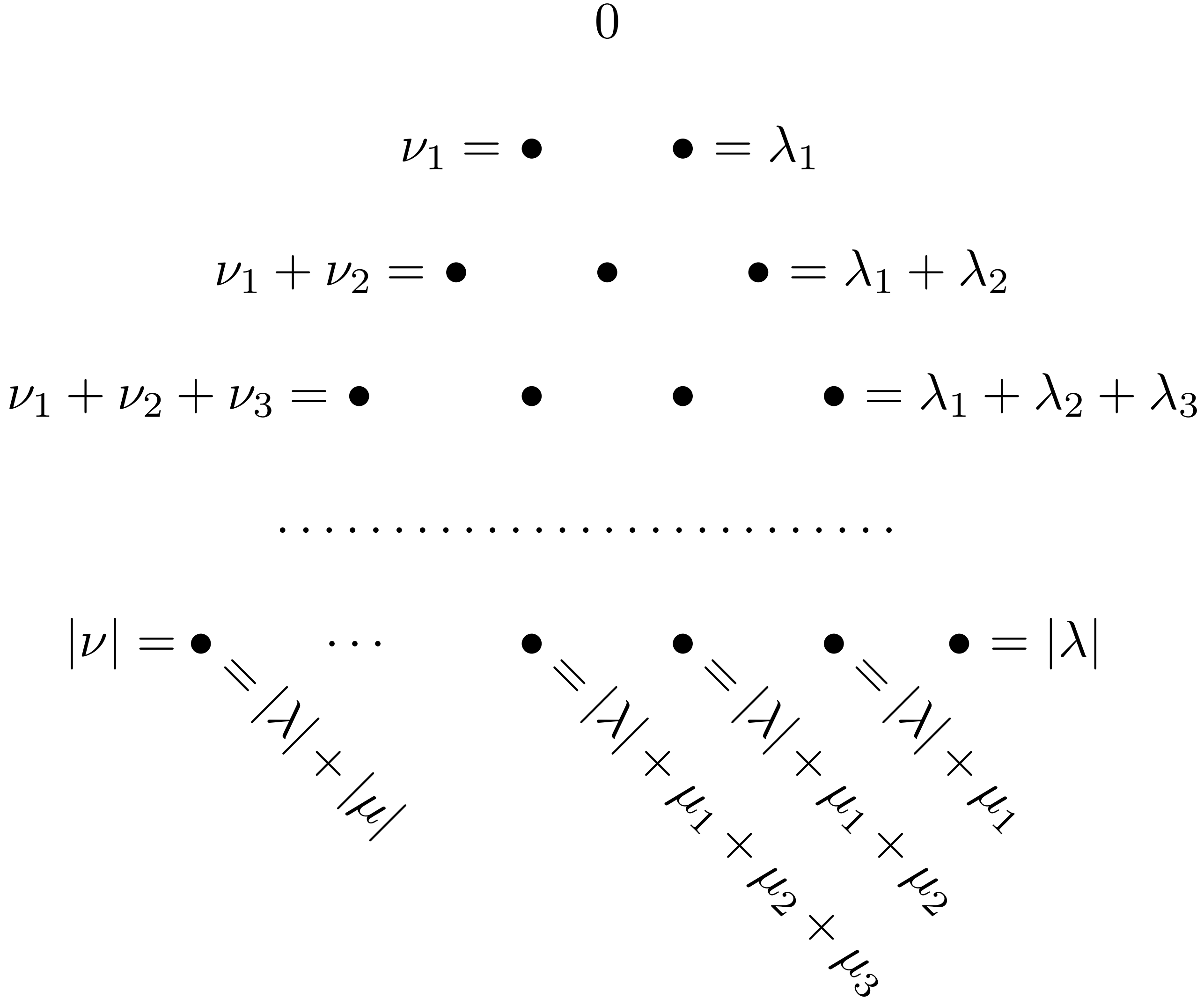
we have $a + b \geq c + d$. (Sum at obtuse angles is $\geq$ sum at acute angles).

Example. A Hive pattern:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 8 & & 5 & & & \\
 & & 13 & & 12 & & 8 & & \\
 & 18 & & 17 & & 15 & & 11 & \\
20 & & 20 & & 18 & & 16 & & 12
\end{array}$$

Hive Polytopes

Definition. *Given integral vectors* $\lambda, \mu, \nu \in \mathbb{Z}^r$*, the* $\boldsymbol{hive\ polytope}$ $H_{\lambda\mu}^{\nu}$ *is those hive patterns with boundary*



Knutson and Tao introduced the Hive polytopes (1999) and proved that

Theorem. [Knutson & Tao] *Given highest weights λ, μ, ν for type A_r, the number of integer lattice points in $H_{\lambda\mu}^{\nu}$ is the Clebsch–Gordan coefficient $C_{\lambda\mu}^{\nu}$ for type A_r:*

$$C_{\lambda\mu}^{\nu} = |H_{\lambda\mu}^{\nu} \cap \mathbb{Z}^d|.$$

More generally, Berenstein and Zelevinsky introduced polytopes $BZ_{\lambda\mu}^{\nu}$ for *any* simple Lie algebra such that

Theorem. [Berenstein & Zelevinsky] *Given highest weights λ, μ, ν for a simple Lie algebra $\mathfrak{g}$, the number of integer lattice points in $BZ_{\lambda\mu}^{\nu}$ is the Clebsch–Gordan coefficient $C_{\lambda\mu}^{\nu}$ for $\mathfrak{g}$:*

$$C_{\lambda\mu}^{\nu} = |BZ_{\lambda\mu}^{\nu} \cap \mathbb{Z}^d|.$$

This means we can use polytopes to compute Clebsch–Gordan coefficients *effectively*.

Lattice Point Enumeration

Theorem. [Barvinok] *Integer lattice points in polyhedra can be enumerated in polynomial time when the dimension is fixed.*

Corollary. [DeLoera & M.] *For fixed simple Lie algebra $\mathfrak{g}$, there is an algorithm to compute the Clebsch–Gordan coefficient $C_{\lambda\mu}^{\nu}$ in polynomial time.*

This algorithm has been implemented for the classical types $A_r, B_r, C_r, D_r,$ and compares favorably to the standard techniques for computing Clebsch–Gordan coefficients.

Deciding whether $C_{\lambda\mu}^{\nu} > 0$

A further application of polyhedral algorithms

Theorem. [Khachian's ellipsoid algorithm] *Deciding whether a polytope nonempty can be done in polynomial time (no need to fix dimension).*

Theorem. [Knutson & Tao] *Every nonempty hive polytope contains an integral point—indeed, an integral* vertex*.*

Corollary. [DeLoera & M.] *In type A, for arbitrary rank, deciding whether $C_{\lambda\mu}^{\nu} > 0$ can be done in polynomial time.*

λ, μ, ν	$C_{\lambda\mu}^{\nu}$	`LattE` runtime	`LiE` runtime
(18, 11, 9, 4, 2) (20, 17, 9, 4, 0) (26, 25, 19, 16, 8)	453	0m03.86s	0m00.12s
(30, 24, 17, 10, 2) (27, 23, 13, 8, 2) (47, 36, 33, 29, 11)	5231	0m05.21s	0m02.71s
(38, 27, 14, 4, 2) (35, 26, 16, 11, 2) (58, 49, 29, 26, 13)	16784	0m06.33s	0m25.31s
(47, 44, 25, 12, 10) (40, 34, 25, 15, 8) (77, 68, 55, 31, 29)	5449	0m04.35s	1m55.83s
(60, 35, 19, 12, 10) (60, 54, 27, 25, 3) (96, 83, 61, 42, 23)	13637	0m04.32s	23m32.10s
(73, 58, 41, 21, 4) (77, 61, 46, 27, 1) (124, 117, 71, 52, 45)	557744	0m07.02s	$>$ 24 hours

`LattE` vs. `LiE`

λ, μ, ν	$C_{\lambda\mu}^{\nu}$	`LattE`
(935, 639, 283, 75, 48) (921, 683, 386, 136, 21) (1529, 1142, 743, 488, 225)	1303088213330	0m08s
(6797, 5843, 4136, 2770, 707) (6071, 5175, 4035, 1169, 135) (10527, 9398, 8040, 5803, 3070)	4590729012405243338	0m10s
(859647, 444276, 283294, 33686, 24714) (482907, 437967, 280801, 79229, 26997) (1120207, 699019, 624861, 351784, 157647)	11711220003870071391294871475	0m08s

Computing large weights with `LattE`, type A.

Stretched Clebsch–Gordan Coefficients

The number of lattice points in dilations $n\,BZ^{\nu}_{\lambda\mu}$, $n = 1, 2, \ldots$, is the **stretched Clebsch–Gordan coefficient**

$$n \mapsto C^{n\nu}_{n\lambda,n\mu}, \quad n = 1, 2, \ldots .$$

Hence, $C^{n\nu}_{n\lambda,n\mu}$ is a quasi-polynomial function of n.

λ, μ, ν	$C^{n\nu}_{n\lambda, n\mu}$
(0, 15, 5) (6, 15, 6) (12, 15, 3)	$\begin{cases} \frac{68339}{64}\,n^5 + \frac{407513}{384}\,n^4 + \frac{13405}{32}\,n^3 + \frac{9499}{96}\,n^2 + \frac{107}{8}\,n + 1, \ n \text{ even} \\ \frac{68339}{64}\,n^5 + \frac{407513}{384}\,n^4 + \frac{13405}{32}\,n^3 + \frac{16355}{192}\,n^2 + \frac{659}{64}\,n + \frac{75}{128}, \ n \text{ odd} \end{cases}$
(8, 1, 3) (8, 6, 14) (11, 13, 3)	$\begin{cases} \frac{121}{576}\,n^6 + \frac{1129}{640}\,n^5 + \frac{6809}{1152}\,n^4 + \frac{163}{16}\,n^3 + \frac{2771}{288}\,n^2 + \frac{191}{40}\,n + 1, \ n \text{ even} \\ \frac{121}{576}\,n^6 + \frac{1129}{640}\,n^5 + \frac{6809}{1152}\,n^4 + \frac{1933}{192}\,n^3 + \frac{659}{72}\,n^2 + \frac{8003}{1920}\,n + \frac{93}{128}, \ n \text{ odd} \end{cases}$
(10, 5, 6) (0, 7, 12) (5, 4, 10)	$\begin{cases} \frac{669989}{960}\,n^5 + \frac{286355}{384}\,n^4 + \frac{10803}{32}\,n^3 + \frac{7993}{96}\,n^2 + \frac{1427}{120}\,n + 1, \ n \text{ even} \\ \frac{669989}{960}\,n^5 + \frac{286355}{384}\,n^4 + \frac{10803}{32}\,n^3 + \frac{15509}{192}\,n^2 + \frac{10081}{960}\,n + \frac{65}{128}, \ n \text{ odd} \end{cases}$

Stretched Clebsch–Gordan coefficients for B_3.

λ, μ, ν	$C^{n\nu}_{n\lambda, n\mu}$
(1, 13, 6) (5, 11, 7) (14, 15, 5)	$\begin{cases} \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{85397}{720}\,n^2 + \frac{883}{60}\,n + 1, \ n \text{ even} \\ \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{657931}{5760}\,n^2 + \frac{3097}{240}\,n + 3/4, \ n \text{ odd} \end{cases}$
(9, 0, 8) (7, 7, 3) (8, 12, 9)	$1/30\,n^5 + 3/8\,n^4 + \frac{19}{12}\,n^3 + \frac{25}{8}\,n^2 + \frac{173}{60}\,n + 1$
(10, 10, 15) (10, 7, 15) (11, 3, 15)	$\begin{cases} \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{43039}{240}\,n^2 + \frac{357}{20}\,n + 1, \ n \text{ even} \\ \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{144751}{960}\,n^2 + \frac{883}{80}\,n + \frac{25}{64}, \ n \text{ odd} \end{cases}$

Stretched Clebsch–Gordan coefficients for C_3.

Observe that the quasi-polynomials all have period 2. This is a general result for classical Lie algebras:

Theorem. *If λ, μ, ν are highest weights for a classical Lie algebra, then there are two polynomials $f_0(n), f_1(n)$ (not necessarily distinct) s.t.*

$$C^{n\nu}_{n\lambda,n\mu} = \begin{cases} f_0(n), & n \textit{ even}, \\ f_1(n), & n \textit{ odd}. \end{cases}$$

The quasi-polynomials we've computed for stretched Clebsch–Gordan coefficients motivate the following conjecture:

Conjecture. If $\mathfrak{g}$ is a classical Lie algebra, the coefficients of stretched Clebsch–Gordan coefficients for $\mathfrak{g}$ are always nonnegative.

On the computation of Clebsch–Gordan coefficients and the dilation effect (with Jesús A. De Loera), to appear in *Experiment. Math.*, arXiv:math.RT/0501446.

Software available at `math.ucdavis.edu/~tmcal`.