Complexes of Graph Homomorphisms

D.N. Kozlov: *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*;
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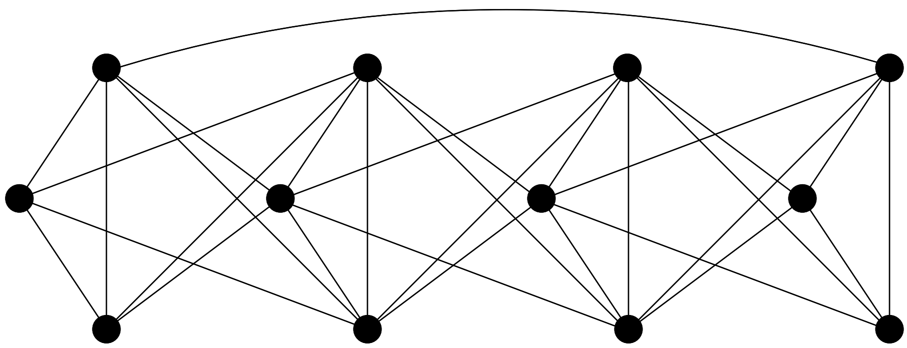
E. Babson, D.N. Kozlov: *Proof of the Lovász Conjecture*;
Annals of Mathematics, to appear.

D.N. Kozlov: *Cohomology of colorings of cycles*;
arXiv:math.AT/0507117

Fragestellung:

How many colors does one need to color the vertices of a given graph G, so that if two vertices are connected by an edge, then they get different colors?



The minimal number of colors is called **chromatic number** and is denoted by $\boldsymbol{\chi(G)}$.

In general, it is very difficult to compute $\chi(G)$ or even to find bounds.

It is even NP-hard to decide whether $\chi(G) = 3$.

Ansatz:

$$\begin{array}{ccc} \text{Graph } G & \longrightarrow & \text{Topological space } X(G) \\ & & \downarrow \\ \begin{array}{c}\text{Combinatorial} \\ \text{properties of } G\end{array} & \longleftarrow & \begin{array}{c}\text{Topological} \\ \text{properties of } X(G)\end{array} \end{array}$$

The topological spaces occurring in this context are finite **regular CW complexes**, where the cells are products of simplices.

At our disposal we have theorems expressing *topological obstructions*, such as the Borsuk-Ulam theorem.

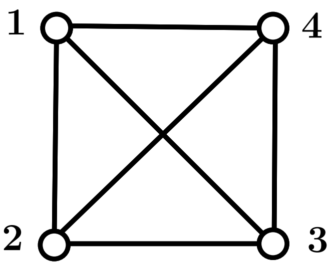
Historic Interlude.

Definition.
Let G be a graph. The **neighborhood complex $\mathcal{N}(G)$** is a simplicial complex defined as follows:

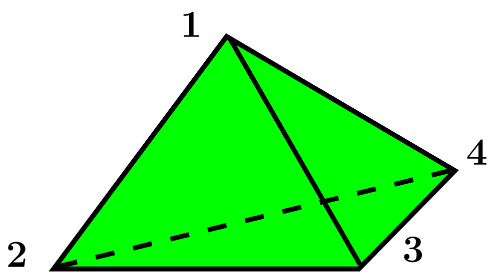
- vertices of $\mathcal{N}(G)$ are the vertices of G;
- a set of vertices A forms a simplex if and only if all vertices in A have a common neighbor.

In particular, every vertex v in G defines a maximal simplex consisting of all neighbors of v.

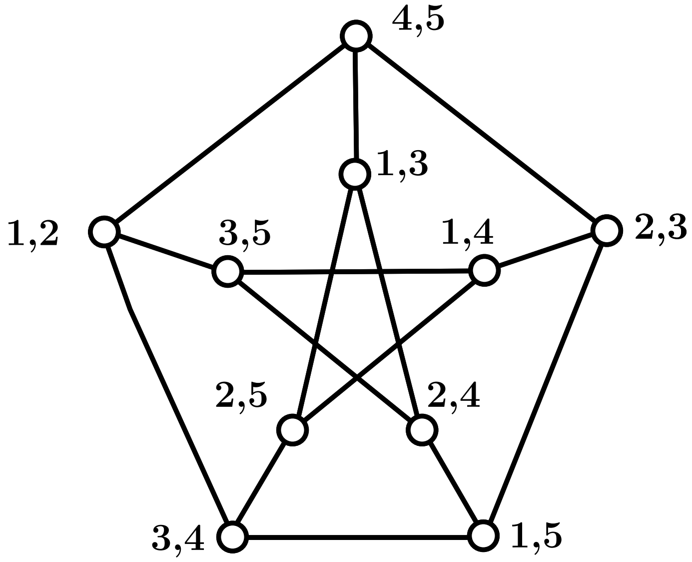
Examples.



K_4



$\mathcal{N}(K_4)$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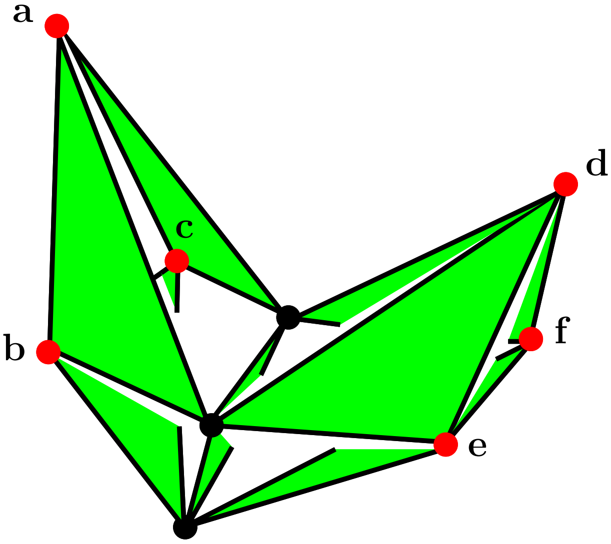
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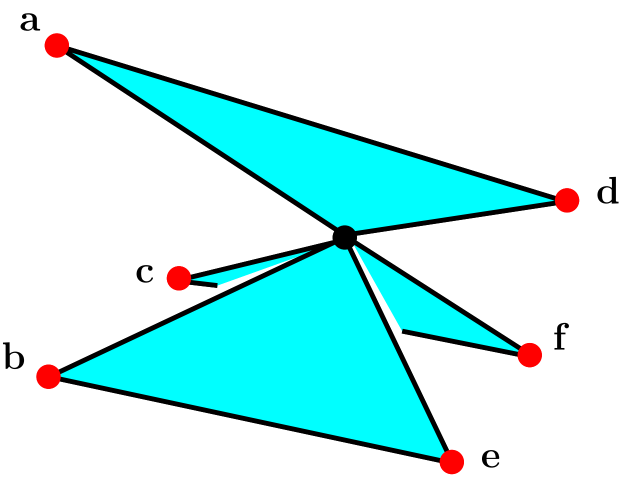
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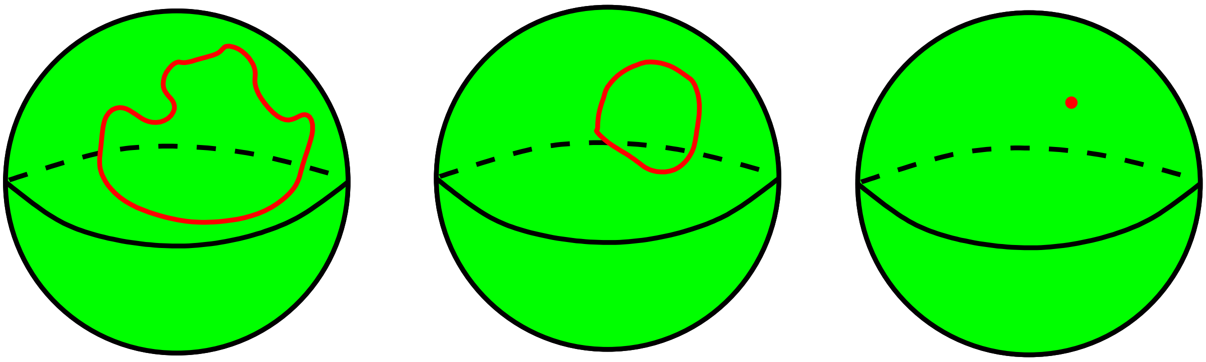
Definition.

A topological space C is called ***k*-connected**, if every continuous map $\phi : S^m \to C$ can be extended to a map $\widetilde{\phi} : B^{m+1} \to C$, for any $-1 \le m \le k$.

Equivalently: **homotopy groups** up to dimension k are trivial.



Theorem (Lovász, 1978).

Let G be a graph and $k \in \mathbb{Z}$, $k \ge -1$.
Then

$$\mathcal{N}(G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \ge k+3 \,.$$

Using this theorem Lovász has proved the Kneser Conjecture.

Equivalently, Lovász' theorem can be formulated as follows:

$$\texttt{Hom}\,(K_2, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+3\,.$$

We shall define $\texttt{Hom}\,(-,-)$ shortly.

Lovász Conjecture.

Let G be a graph and $r, k \in \mathbb{Z}$, $r \geq 1$, $k \geq -1$.
Then

$$\texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+4\,.$$

Theorem (Babson & K., 2003).

(a) The Lovász Conjecture is true.

(b) Let $m, k \in \mathbb{Z}$, $m \geq 1$, $k \geq -1$. Then

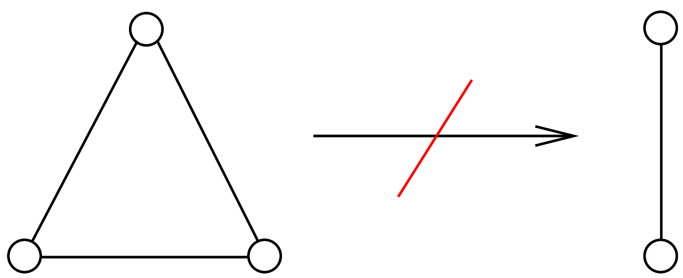
$$\texttt{Hom}\,(K_m, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+m+1\,.$$

Definition.

Let T, G be two graphs. A **graph homomorphism** from T to G is a map $\phi : V(T) \to V(G)$, such that for every edge (x, y) in T the image $(\phi(x), \phi(y))$ is an edge in G.

Observation:

$$G \text{ is } n\text{-colorable} \;\Leftrightarrow\; \exists\, \phi : G \to K_n.$$



A composition of two homomorphisms $\phi_1 : G_1 \to G_2$ and $\phi_2 : G_2 \to G_3$ is again a homomorphism $\phi_2 \circ \phi_1 : G_1 \to G_3$.

This gives **Graphs** - the category which has graphs as objects and graph homomorphisms as morphisms.

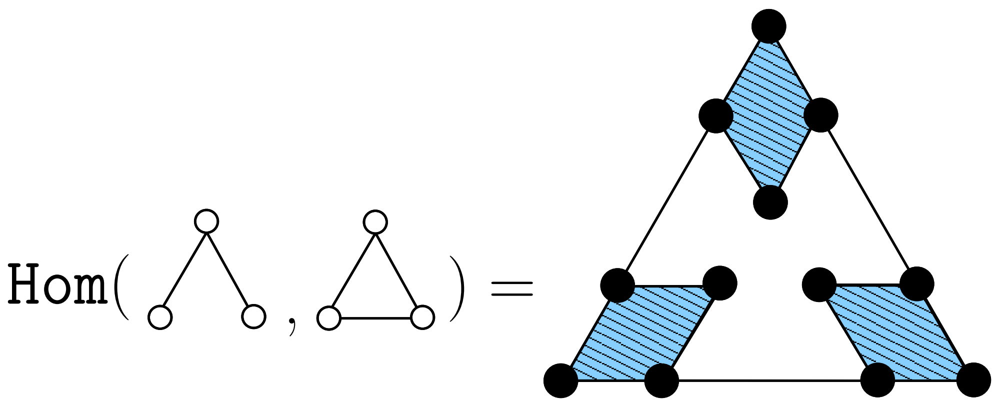
Geometry of graph colorings: idea.

A cell in $\texttt{Hom}\,(T, K_n)$ is a collection of color lists, one for each vertex of T, such that an arbitrary choice of colors, one from each list, is a valid coloring of T.

Now replace K_n with an arbitrary graph G.

- The vertices of G are the colors.
- A homomorphism $T \longrightarrow G$ is thought of as a valid coloring.

Example.



Definition.

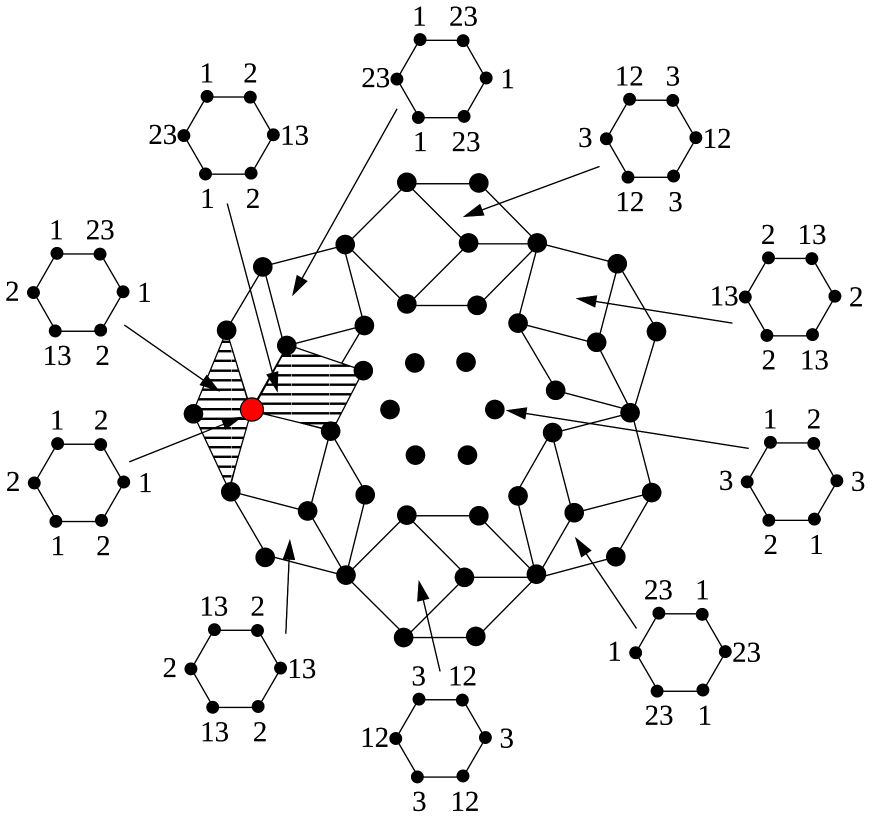
Let T and G be two graphs. The $\mathtt{Hom}$-complex $\mathtt{Hom}\,(\boldsymbol{T},\boldsymbol{G})$ is the subcomplex of $\prod_{x\in V(T)}\Delta^{V(G)}$ defined by the following condition:

the cell $\sigma=\prod_{x\in V(T)}\sigma_x$ is in $\mathtt{Hom}\,(T,G)$ if and only if whenever two vertices $x,y\in V(T)$ are connected by an edge, the pair (σ_x,σ_y) is a complete bipartite subgraph of G.

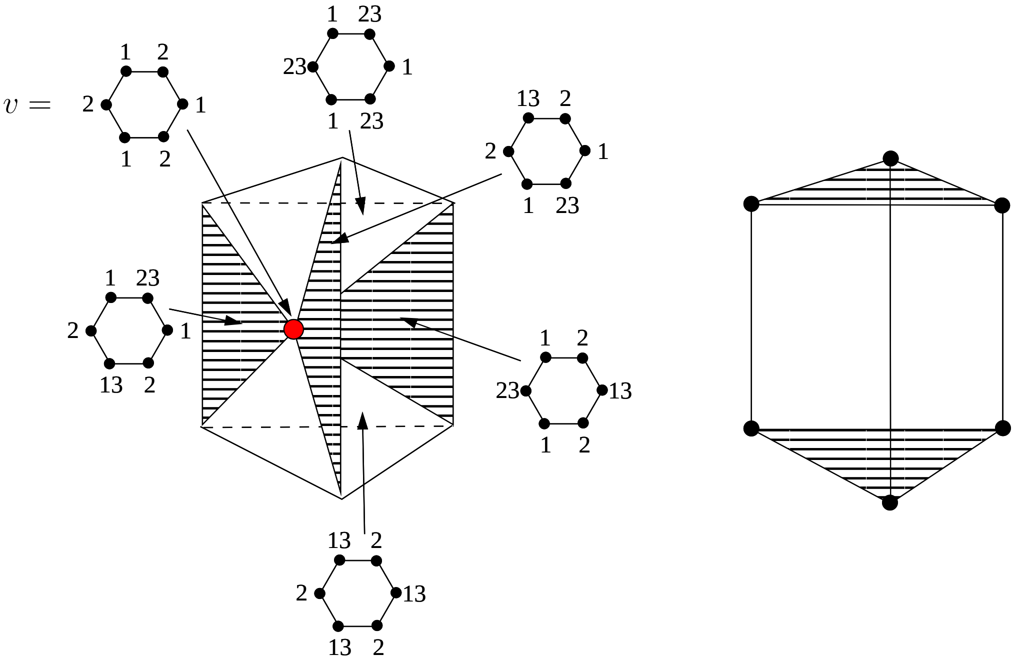
Note that

- the cells of $\mathtt{Hom}\,(T,G)$ are indexed by all set maps $\eta: V(T)\rightarrow 2^{V(G)}\setminus\{\emptyset\}$, such that for any $(x,y)\in E(T)$, and any $\tilde{x}\in\eta(x)$, $\tilde{y}\in\eta(y)$, we have $(\tilde{x},\tilde{y})\in E(G)$;

- the cells in the closure of each cell η are indexed by all maps $\tilde{\eta}: V(T)\rightarrow 2^{V(G)}\setminus\{\emptyset\}$, such that $\tilde{\eta}(v)\subseteq\eta(v)$, for all $v\in V(T)$.



$\mathtt{Hom}\,(C_6, K_3)$



Properties of the Hom -complexes:

(1) Cells in $\texttt{Hom}\,(T,G)$ are products of simplices:

this is a **prodsimplicial** complex.

More precisely, every cell η is a product of $|V(T)|$ simplices of dimension $|\eta(x)|-1$ for $x \in V(T)$.

(2) Bd $\texttt{Hom}\,(K_2,G)$ and $\mathcal{N}(G)$ have the same simple homotopy type.

(3) $\texttt{Hom}\,(T,-)$ is a covariant and $\texttt{Hom}\,(-,G)$ is a contravariant functor from **Graphs** to **Top**.

$\longrightarrow$ If $\phi : G \longrightarrow G'$ is a graph homomorphism, then, for an arbitrary graph H, we shall denote the induced topological maps by
$\phi^H : \texttt{Hom}\,(H,G) \longrightarrow \texttt{Hom}\,(H,G')$ and
$\phi_H : \texttt{Hom}\,(G',H) \longrightarrow \texttt{Hom}\,(G,H)$.

$\longrightarrow$ As a consequence of (3), the group $\mathcal{A}ut(T) \times \mathcal{A}ut(G)$ acts on $\texttt{Hom}\,(T,G)$.

$\longrightarrow$ When G has no loops and $\phi \in \mathcal{A}ut(T)$ flips an edge, the induced map $\phi_G \circlearrowright \texttt{Hom}\,(T,G)$ has no fixed points.

Definition.

A CW complex X is called **$\mathbb{Z}_2$-space**, if $\mathbb{Z}_2$ acts freely on X.

In this case, there exists a continuous map $\psi : X \to S^\infty$. The induced quotient map $\phi : X/\mathbb{Z}_2 \to \mathbb{RP}^\infty$ is up to homotopy independent of the choice of ψ.

It induces an algebra map

$$\phi^* : H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2),$$

which is independent of the choice of ψ.

Let z be the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Then

$$\boldsymbol{w_1(X)} := \phi^*(z)$$

is called the **Stiefel-Whitney class.**

Stiefel-Whitney classes are functorial:

if $\psi : X \to Y$ is a $\mathbb{Z}_2$-map, and $\widetilde{\psi} : X/\mathbb{Z}_2 \to Y/\mathbb{Z}_2$ is the induced map between the quotient spaces, then

$$\widetilde{\psi}^*(w_1(Y)) = w_1(X)\,.$$

An application of Stiefel-Whitney classes:

Borsuk-Ulam Theorem.

There is no continuous map $S^{k+1} \longrightarrow S^k$, which commutes with the antipodal maps on S^{k+1} and S^k.

Lovász Conjecture:

$$\texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+4\,.$$

$$k=-1: \quad \texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad \text{nonempty} \quad \Longrightarrow \quad \chi(G) \geq 3\,.$$

$$k=0: \quad \texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad \text{connected} \quad \Longrightarrow \quad \chi(G) \geq 4\,.$$

Proof for $k=0$.
Assume $\chi(G) \leq 3$, then there exists a graph homomorphism

$$\phi : G \to K_3.$$

It induces a $\mathbb{Z}_2$-map

$$\phi^{C_{2r+1}} : \texttt{Hom}\,(C_{2r+1}, G) \to \texttt{Hom}\,(C_{2r+1}, K_3).$$

A direct analysis shows that the connected components of the complex $\texttt{Hom}\,(C_{2r+1}, K_3)$ can be indexed with the signed number of times C_{2r+1} winds around K_3. This number α is odd:

$$\alpha = \pm 1, \pm 3, \ldots, \pm(2s+1),$$

and $s \geq 0$. The $\mathbb{Z}_2$-action on $\texttt{Hom}\,(C_{2r+1}, K_3)$ swaps the connected components by changing the sign of the winding number.
This contradicts the fact that $\texttt{Hom}\,(C_{2r+1}, G)$ is connected.

Scheme of the Proof (Babson & K., 2003):

• Since $\texttt{Hom}\,(C_{2r+1}, G)$ is $(n-1)$-connected there exists a $\mathbb{Z}_2$-map

$$f : S_a^n \to \texttt{Hom}\,(C_{2r+1}, G).$$

• Assume $\chi(G) \leq n+2$, then $\exists\, \phi : G \to K_{n+2}$.
This induces a $\mathbb{Z}_2$-map

$$\phi^{C_{2r+1}} : \texttt{Hom}\,(C_{2r+1}, G) \to \texttt{Hom}\,(C_{2r+1}, K_{n+2}).$$

• Since $\phi^{C_{2r+1}} \circ f : S_a^n \to \texttt{Hom}\,(C_{2r+1}, K_{n+2})$ is a $\mathbb{Z}_2$-map and $w_1^n(S_a^n) \neq 0$, we conclude that

$$w_1^n(\texttt{Hom}\,(C_{2r+1}, K_{n+2})) \neq 0.$$

• On the other hand, for odd n, spectral sequence computations yield $w_1^n(\texttt{Hom}\,(C_{2r+1}, K_{n+2})) = 0 \Longrightarrow$ a contradiction.

• For even n the obstructions are found simply by computing $H^*(\texttt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$.

• Further work on Lóvasz Conjecture and our (BK) conjecture that $w_1^n(\texttt{Hom}\,(C_{2r+1}, K_{n+2})) = 0$ also for even n by Živaljević, Schultz.

Facts about Hom's I.

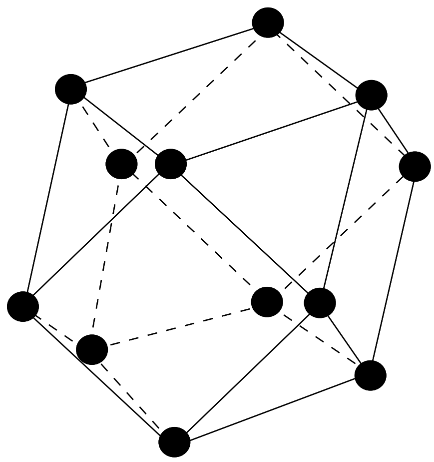
Theorem. (Babson & K., 2003)

Let $m, n \in \mathbb{Z}$, $n \geq m \geq 2$. Then $\texttt{Hom}\,(K_m, K_n)$ is homotopy equivalent to a wedge of $(n-m)$ - dimensional spheres.

Note that $\texttt{Hom}\,(K_2, K_n)$ can be realized as a boundary of an $(n-2)$ - dimensional polytope, in particular it is homeomorphic to S^{n-2}.

Example.

$$\texttt{Hom}\,(K_2, K_4) =$$



Facts about Hom's II.

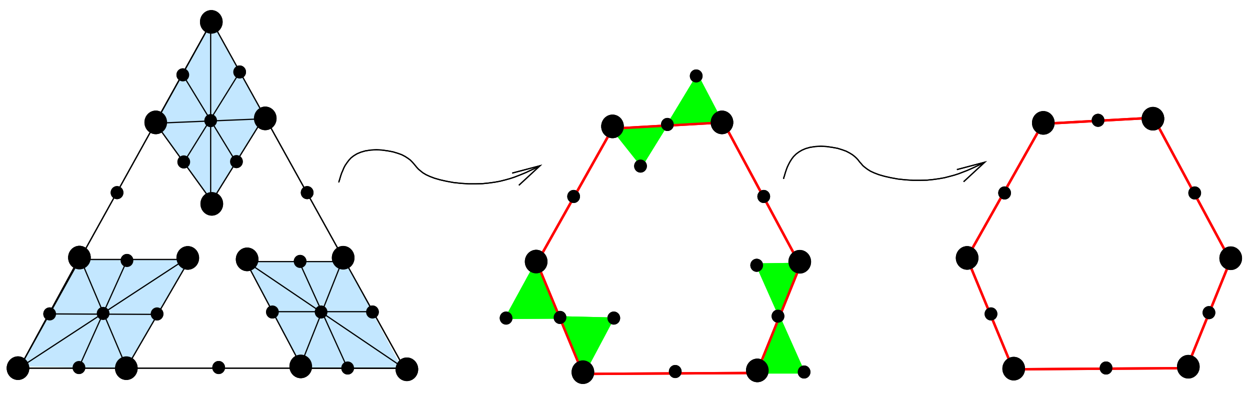
Replacing G with $G - v$ is called a *fold*.
Some special cases:

- A tree folds onto any of its edges.
- Let F be a forest, then $\bar{F}$ folds onto K_m, where m is the maximal cardinality of an independent set in F.

Theorem. (K., 2004)

Let G and H be two graphs, and let u, v be vertices of T, such that $N(v) \subseteq N(u)$.

Then $\operatorname{Bd} \mathtt{Hom}\,(G, H)$ collapses onto $\operatorname{Bd} \mathtt{Hom}\,(G - v, H)$, whereas $\mathtt{Hom}\,(H, G)$ collapses onto $\mathtt{Hom}\,(H, G - v)$.



$\mathtt{Hom}\,(L_3, K_3)$ $\qquad$ X $\qquad$ $\mathtt{Hom}\,(K_2, K_3)$

Facts about `Hom`'s III.

Theorem. (Čukić & K., 2004)

Let G be a graph of maximal valency d, then the complex $\mathtt{Hom}\,(G, K_n)$ is at least $(n - d - 2)$-connected.

This was conjectured by Babson & K., short proof: Engström, 2005.

Theorem. (Čukić & K., 2004)

Every connected component of $\mathtt{Hom}\,(C_m, C_n)$ is either a point or is homotopicaly equivalent to S^1.

Facts about $\texttt{Hom}$'s VI.

Universality Theorem. (Csorba, Živaljević, 2004)

For each finite, free $\mathbb{Z}_2$-complex X, there exists a graph G, such that $\texttt{Hom}\,(K_2, G)$ is $\mathbb{Z}_2$-homotopy equivalent to X.

Conjecture. (Csorba, 2004)

$\texttt{Hom}\,(C_5, K_n)$ is homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^{n-1})$, for all $n \geq 1$.

For example, $\texttt{Hom}\,(C_5, K_4) \cong \mathbb{RP}^3$.

Cohomology of complexes of cycles I.

$\texttt{Hom}_+$-construction.

Similar to $\texttt{Hom}$, empty coloring lists are allowed.

- $\texttt{Hom}_+(T, G)$ is a simplicial complex.

- $\texttt{Hom}_+(T, K_n)$ is isomorphic to the n-fold join of the independence complex of T.

- There exists a simplicial map

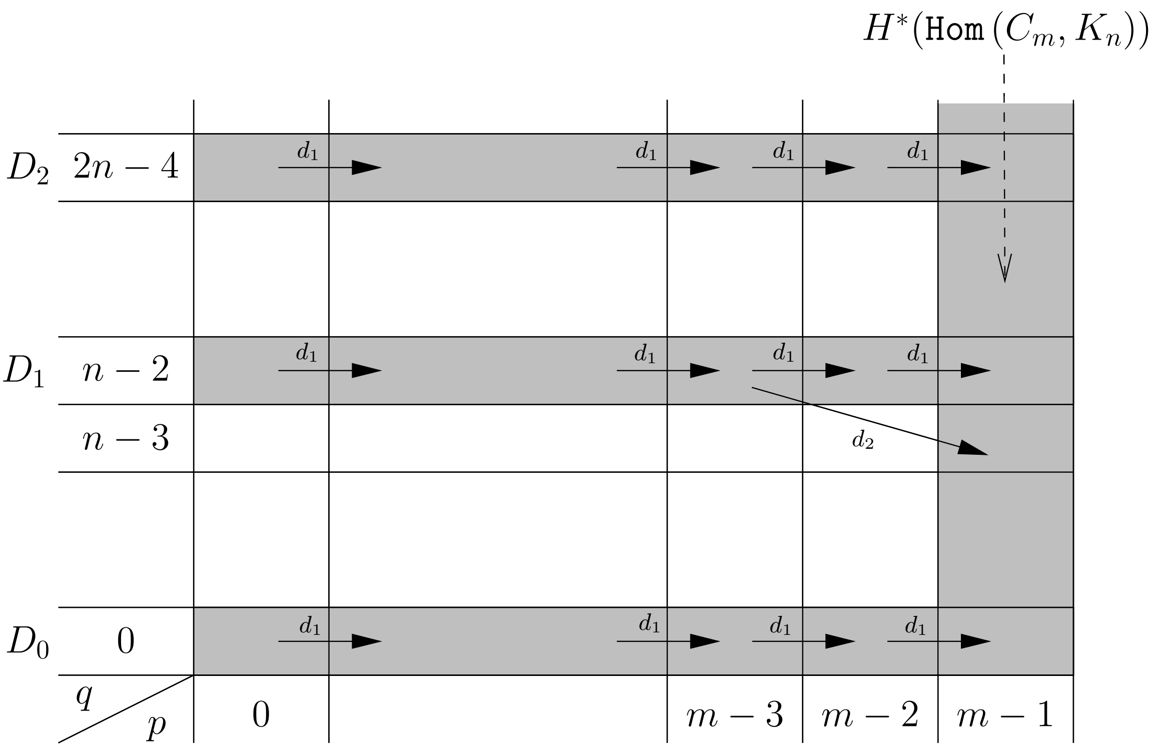
$$\text{supp} : \texttt{Hom}_+(T, G) \to \Delta_{|V(T)|} .$$

- $\texttt{Hom}\,(T, G) = \text{supp}^{-1}(\rho_0)$, where ρ_0 is the barycenter of the simplex $\Delta_{|V(T)|}$.

Cohomology of complexes of cycles II.

- The spectral sequence: the filtration on $\texttt{Hom}_+(T,G)$ is given by $\operatorname{supp}^{-1}(\Delta^i_{|V(T)|})$, where $\Delta^i_{|V(T)|}$ is the i-skeleton of $\Delta_{|V(T)|}$.

- The first tableau looks like this:

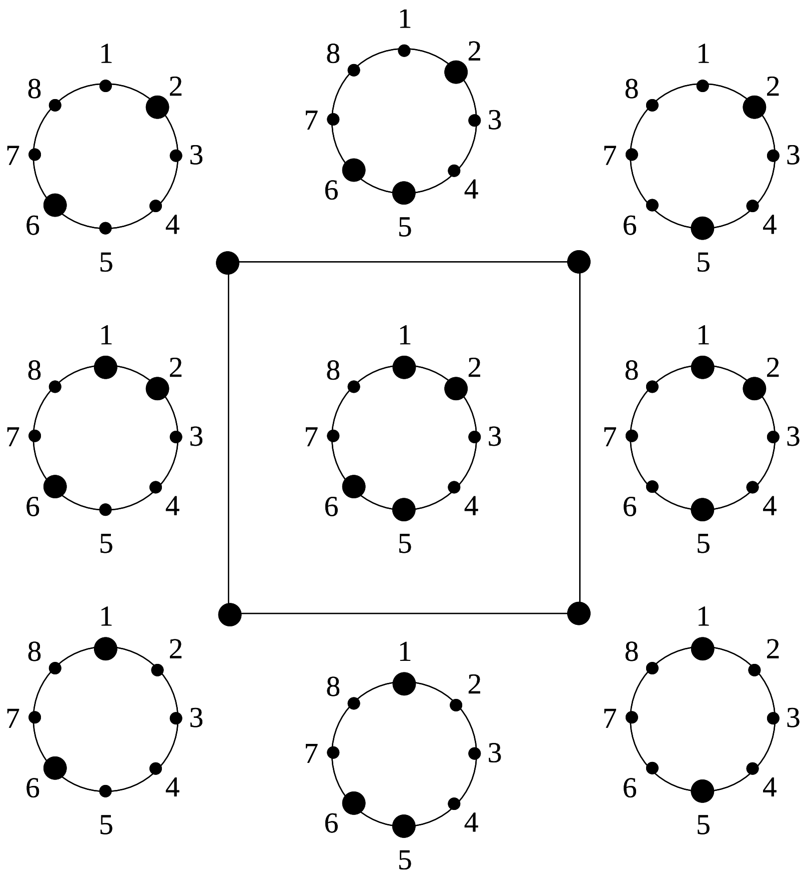


Cohomology of complexes of cycles III.

Definition.

Let $m, n, g \in \mathbb{N}$, $\mathbf{\Phi_{m,n,g}}$ is the cubical complex whose cells are indexed by all possible collections of m sets, where each set either contains a single element of the circular n-set, or consists of a pair of two elements at distance 1; the sets are requested to be at minimal distance g, where the distance between two sets is defined as the minimum of the distances between their elements.

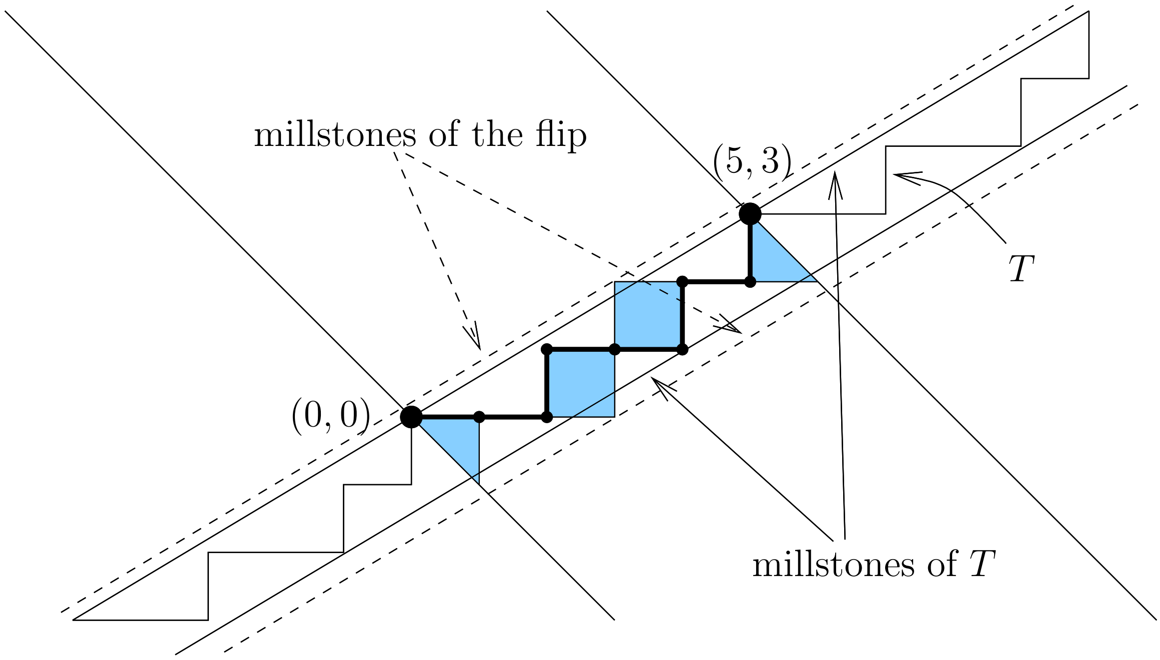
Example. Cell indexing in $\Phi_{2,8,2}$:



Cohomology of complexes of cycles IV.

Definition. Let m, n, and g be natural numbers. The cubical complex $TF_{m,n,g}$ is defined as follows:
vertices of $TF_{m,n,g}$ are indexed by all possible (m, n)-torus fronts, whose horizontal legs have length at least g, here the length of the leg is taken to be the number of vertices it contains;
the higher-dimensional cubes of $TF_{m,n,g}$ are indexed by all possible flips of (m, n)-torus fronts, whose members are vertices of $TF_{m,n,g}$.

The rows in the first tableau can be reinterpreted as computing homology of $\Phi_{m,n,g} = TF_{m,n-m,g}$.



One can grind torus front complexes until only thin fronts are left.

Cohomology of complexes of cycles V.

Theorem. (K., 2005)

For any integers m, n, such that $m \geq 5$, $n \geq 4$, we have

$$\widetilde{H}^*(\mathtt{Hom}\,(C_m, K_n); \mathbb{Z}) = \left(\bigoplus_{t=1}^{\lfloor (m-2)/3 \rfloor} A_{t,m,n} \right) \oplus B_{m,n},$$

where

$$A_{t,m,n} = \begin{cases} \mathbb{Z}(tn-3t) \oplus \mathbb{Z}(tn-3t+1), & \text{if } n \text{ is odd or } m+t \text{ is odd,} \\ \mathbb{Z}_2(tn-3t+1), & \text{if } n \text{ is even and } m+t \text{ is even,} \end{cases}$$

and

$$B_{m,n} = \begin{cases} \mathbb{Z}^{2^n-3}(nk-m), & \text{if } m = 3k, \\ \mathbb{Z}(nk-m+2), & \text{if } m = 3k+1, \\ \mathbb{Z}(nk-m), & \text{if } m = 3k-1. \end{cases}$$

Examples:

- $\widetilde{H}^*(\mathtt{Hom}\,(C_6, K_4); \mathbb{Z}) = A_{1,6,4} \oplus B_{6,4} = \mathbb{Z}(1) \oplus \mathbb{Z}(2) \oplus \mathbb{Z}^{13}(2) = \mathbb{Z}(1) \oplus \mathbb{Z}^{14}(2)$;
- $\widetilde{H}^*(\mathtt{Hom}\,(C_8, K_6); \mathbb{Z}) = A_{1,8,6} \oplus A_{2,8,6} \oplus B_{8,6} = \mathbb{Z}(3) \oplus \mathbb{Z}(4) \oplus \mathbb{Z}_2(7) \oplus \mathbb{Z}(10)$.

Summary and Outlook

The fact that the Lovász Conjecture is true implies that $\mathtt{Hom}\,(-,-)$ construction is interesting and produces nontrivial bounds for the chromatic number.

It is known that the following more general conjecture is false:

$$\chi(G) \geq \chi(T) + \operatorname{conn} \mathtt{Hom}\,(T, G) + 1.$$

Is truth somewhere in the middle?

How much information about the graph colorings is contained in the algebro-topological invariants of the $\mathtt{Hom}$-complexes?

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