GRAVER BASES, MATCHINGS IN SIMPLICIAL COMPLEXES AND TORIC VARIETIES

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Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a finite set. An *abstract simplicial complex* $\mathcal{D}$ on the vertex set $\mathcal{V}$ is a collection of subsets of $\mathcal{V}$ satisfying:

(i) $\{v_i\} \in \mathcal{D}$ for every $i = 1, \ldots, n$,
(ii) if $T \in \mathcal{D}$ and $G \subset T$, then $G \in \mathcal{D}$.

A set $T \subset \mathcal{D}$ of cardinality $m + 1$ has *dimension* $m \geq -1$ and is called an m-*simplex* of $\mathcal{D}$. The 0-simplices of $\mathcal{D}$ are called *vertices*, while the 1-simplices are called *edges*. The *dimension* $dim(\mathcal{D})$ of $\mathcal{D}$ is the maximum of the dimensions of its simplices.

Let $\mathcal{D}$ be an abstract simplicial complex on the vertex set $\mathcal{V}$ and J be a subset of $\Omega := \{0, 1, \ldots, dim(\mathcal{D})\}$.
A set $\mathcal{M} = \{T_1, \ldots, T_s\}$ of simplices of $\mathcal{D}$ is called a *J-matching* in $\mathcal{D}$ if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and dim(T_k)$\in J$ for every $1 \leq k \leq s$.

Set $supp(\mathcal{M}) = \cup_{i=1}^{s} T_i \subset \mathcal{V}$.
A J-matching $\mathcal{M}$ in $\mathcal{D}$ is called a *maximal* J-matching if $supp(\mathcal{M})$ has the maximum possible cardinality among all J-matchings.

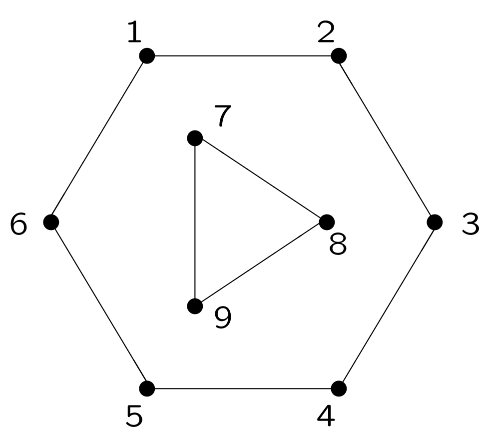
By $\delta(\mathcal{D})_J$ we denote the minimum $card(\mathcal{M})$ among all maximal J-matchings $\mathcal{M}$ in $\mathcal{D}$.

When $\mathcal{D}$ is a simple graph, i.e dim$(\mathcal{D}) \leq 1$, the notion of $\{1\}$-matching in $\mathcal{D}$ coincides with the notion of matching. Also maximal $\{1\}$-matching coincides with the notion of maximal matching in $\mathcal{D}$.

Finally $\delta(\mathcal{D})_{\{1\}}$ equals the matching number of $\mathcal{D}$.
Recall that a subset M of the edges of $\mathcal{D}$ is called a *matching* in $\mathcal{D}$ if there are no two edges which are incident with a common vertex.

M is a *maximal matching* if it has the maximum possible cardinality among all matchings. The cardinality of a maximal matching in $\mathcal{D}$ is commonly known as its *matching number*.

Example. Consider the simplicial complex $\mathcal{D}$ drawn in the figure. The set $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9\}\}$ is a $\{0, 1\}$-matching in $\mathcal{D}$, while the set $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8, v_9\}\}$ is a $\{0, 1, 2\}$-matching in $\mathcal{D}$. Both of them are maximal, since they cover all the vertices of $\mathcal{D}$.



Let $\Bbbk$ be an algebraically closed field and $\Bbbk[x_1, \ldots, x_m]$ be the polynomial ring in the variables $x_1, \ldots, x_m$.

A *binomial* in $\Bbbk[x_1, \ldots, x_m]$ is a difference of monomials. Given a lattice L on $\mathbb{Z}^m$, the ideal

$$I_L := (\{\mathbf{x}^{\alpha_+} - \mathbf{x}^{\alpha_-} | \alpha = \alpha_+ - \alpha_- \in L\})$$

in $\Bbbk[x_1, \ldots, x_m]$ is called *lattice ideal*. Where $\alpha_+ \in \mathbb{N}^m$ and $\alpha_- \in \mathbb{N}^m$ denote the positive and negative part of α, respectively, and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_m^{b_m}$ for $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{N}^m$.

If $rank(L) = k$, then there exists a matrix $\mathbf{M} \in \mathbb{Z}^{(m-k)\times m}$ of rank $m-k$ such that $L \subset ker_{\mathbb{Z}}(\mathbf{M})$.

When $L = ker_{\mathbb{Z}}(\mathbf{M})$, the ideal I_L is prime and called *toric ideal*. The variety $\mathbb{V}(I_L)$ is called *toric variety*.

Let $A = \{\mathbf{a}_i \mid 1 \leq i \leq m\}$ be the set of columns of $\mathbf{M}$, we associate to I_L the rational polyhedral cone
$\sigma = pos_{\mathbb{Q}}(A) := \{d_1\mathbf{a}_1 + \cdots + d_m\mathbf{a}_m \mid d_i \in \mathbb{Q}_{\geq 0}\}$.
We assume that σ is *strongly convex*, i.e. $\{\mathbf{0}\}$ is a face of σ.

With respect to the grading $deg_A(x_i) = \mathbf{a}_i$ of the polynomial ring $\mathbb{k}[x_1, \ldots, x_m]$ the ideal I_L is A-homogeneous.

The *binomial arithmetical rank* bar(I_L) of I_L is the smallest integer s for which there exist binomials $F_1, \ldots, F_s$ in I_L such that $\sqrt{I_L} = \sqrt{F_1, \ldots, F_s}$.

Hence the binomial arithmetical rank is an upper bound for the *arithmetical rank* ara(I_L) of I_L, which is the smallest integer s for which there exists polynomials $F_1, \ldots, F_s$ in I_L such that $\sqrt{I_L} = \sqrt{F_1, \ldots, F_s}$.

When all the polynomials $F_1, \ldots, F_s$ are A-homogeneous, the smallest integer s is called *A-homogeneous arithmetical rank* $\text{ara}_A(I_L)$ of I_L. For a lattice ideal I_L the following inequality holds:

$\text{ht}(I_L) \leq \text{ara}(I_L) \leq \text{ara}_A(I_L) \leq \text{bar}(I_L) \leq \mu(I_L)$.

Problem. Find lower bounds for the minimal number $\mu(I_L)$ of generators, the binomial arithmetical rank and the A-homogeneous arithmetical rank of a lattice ideal.

Let $\sigma = pos_{\mathbb{Q}}(\mathbf{r}_1, \ldots, \mathbf{r}_t) \subset \mathbb{Q}^n$ be a strongly convex rational polyhedral cone. Where $\{\mathbf{r}_1, \ldots, \mathbf{r}_t\}$ is a set of integer vectors, one for each extreme ray of σ.

For a subset E of $\{1, \ldots, t\}$ we denote by σ_E the subcone $pos_{\mathbb{Q}}(\mathbf{r}_i \ |i \ \in E)$ of σ.
The *relative interior* $relint_{\mathbb{Q}}(\sigma_E)$ of σ_E is the set of all positive rational linear combinations of $\mathbf{r}_i$, $i \in E$.

Suppose that σ is not a simplex cone, i.e. the extreme vectors $\mathbf{r}_1, \ldots, \mathbf{r}_t$ are not linearly independent.

The set of cones σ_E, which are not faces of the cone σ, is not empty and form a poset ordered by inclusion. Let $\{\sigma_{\mathbb{E}_1}, \ldots, \sigma_{\mathbb{E}_f}\}$ be the minimal elements of this poset, which are called the *minimal non faces* of σ.

To the cone σ we associate a simplicial complex $\mathcal{D}_\sigma$ with vertices the set $\{\sigma_{\mathbb{E}_1}, \ldots, \sigma_{\mathbb{E}_f}\}$ and $T \subset \{\sigma_{\mathbb{E}_1}, \ldots, \sigma_{\mathbb{E}_f}\}$ belongs to $\mathcal{D}_\sigma$ if

$$\cap_{\sigma_{\mathbb{E}_i} \in T} relint_{\mathbb{Q}} \left(\sigma_{\mathbb{E}_i} \right) \neq \emptyset.$$

To the simplicial complex $\mathcal{D}_\sigma$ we can associate the 1-*skeleton* $\mathbb{G}(\mathcal{D}_\sigma)$ of $\mathcal{D}_\sigma$, formed by the vertices and edges of $\mathcal{D}_\sigma$.

The *complement* $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$ of $\mathbb{G}(\mathcal{D}_\sigma)$ is the graph with the same vertices as $\mathbb{G}(\mathcal{D}_\sigma)$, and $\{v_i, v_j\}$ is an edge of $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$ if and only if $\{v_i, v_j\}$ is not an edge of $\mathbb{G}(\mathcal{D}_\sigma)$.

The *chromatic number* $\gamma(\overline{\mathbb{G}(\mathcal{D}_\sigma)})$ of the graph $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$ is the smallest integer k for which there is a function
$c : Vertices(\overline{\mathbb{G}(\mathcal{D}_\sigma)}) \to \{1, \ldots, k\}$ such that $c(v_i) \neq c(v_j)$ if $\{v_i, v_j\}$ is an edge of $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$.

Theorem. For a lattice ideal I_L with associated cone $\sigma = pos_{\mathbb{Q}}(A)$ we have:

(i) $\mu(I_L) \geq \text{bar}(I_L) \geq \delta(\mathcal{D}_\sigma)_{\{0,1\}} = \delta(\mathcal{D}_\sigma)_{\{0\}} - \delta(\mathcal{D}_\sigma)_{\{1\}}$,

(ii) $\text{ara}_A(I_L) \geq \delta(\mathcal{D}_\sigma)_\Omega \geq \gamma(\overline{\mathbb{G}(\mathcal{D}_\sigma)})$,

(iii) If $\sqrt{I_L} = \sqrt{F_1, \ldots, F_s}$, then

(a) the total number of monomials in the nonzero terms of the polynomials $F_1, \ldots, F_s$ is greater than or equal to the number of vertices $\delta(\mathcal{D}_\sigma)_{\{0\}}$ of $\mathcal{D}_\sigma$.
(b) the total number of A-homogeneous components in $F_1, \ldots, F_s$ is greater than or equal to the chromatic number of $\overline{\mathbb{G}(\mathcal{D}_\sigma)}$.

We consider the polynomial ring $\mathbb{k}[y_1, \ldots, y_t]$, by taking one variable for each extreme vector $\mathbf{r}_i$. From the set $\mathbf{R}_\sigma = \{\mathbf{r}_1, \ldots, \mathbf{r}_t\}$ we can construct the toric ideal $I_{\mathbf{R}_\sigma}$, which is the kernel of the $\mathbb{k}$-algebra homomorphism

$$\phi : \mathbb{k}[y_1, \ldots, y_t] \to \mathbb{k}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$$

given by $\phi(y_i) = \mathbf{z}^{\mathbf{r}_i}$.

The toric variety $\mathbb{V}(I_{\mathbf{R}_\sigma})$ is called *extremal toric variety*.

A binomial $F(\mathbf{u}) := \mathbf{y}^{\mathbf{u}_+} - \mathbf{y}^{\mathbf{u}_-}$ in $I_{\mathbf{R}_\sigma}$ is called *primitive* if there exists no other binomial $\mathbf{y}^{\mathbf{v}_+} - \mathbf{y}^{\mathbf{v}_-} \in I_{\mathbf{R}_\sigma}$ such that $\mathbf{y}^{\mathbf{v}_+}$ divides $\mathbf{y}^{\mathbf{u}_+}$ and $\mathbf{y}^{\mathbf{v}_-}$ divides $\mathbf{y}^{\mathbf{u}_-}$. The set of primitive binomials of $I_{\mathbf{R}_\sigma}$ is finite and is called the *Graver basis* $Gr(\mathbf{R}_\sigma)$ of $I_{\mathbf{R}_\sigma}$.

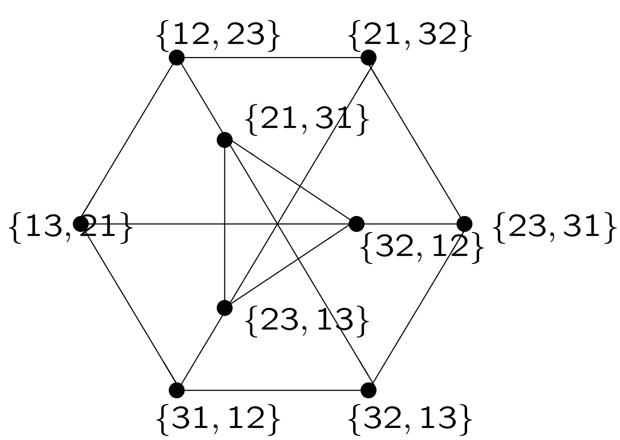
Theorem. Set $\mathcal{E} := \{E \subset \{1, \ldots, t\} \mid \exists\ F(\mathbf{u}) \in Gr(\mathbf{R}_\sigma)$ with $supp(\mathbf{u}_+) = E$ or $supp(\mathbf{u}_-) = E\}$, where $supp(\mathbf{v}) = \{i \in \{1, \ldots, t\} \mid v_i > 0\}$ for $\mathbf{v} = (v_1, \ldots, v_t) \in \mathbb{N}^t$. Then σ_E is a minimal non face of σ if and only if E is a minimal element of $\mathcal{E}$.

Theorem. A set $T = \{\sigma_{\mathbb{E}_i}, \sigma_{\mathbb{E}_j}\}$ is an edge of $\mathcal{D}_\sigma$ if and only if there is a primitive binomial $F(\mathbf{u}) \in I_{\mathbf{R}_\sigma}$ with $supp(\mathbf{u}_+) = \mathbb{E}_i$ and $supp(\mathbf{u}_-) = \mathbb{E}_j$.

Example. Consider the lattice $L = ker_{\mathbb{Z}}(\mathbf{M})$, where $\mathbf{M}$ is the 3×6 matrix with columns the vectors of the set $A_3 = \{\mathbf{r}_{ij} = 2\mathbf{e}_i + \mathbf{e}_j | i, j \in \{1, 2, 3\}, i \neq j\}$ and $\{\mathbf{e}_i | 1 \leq i \leq 3\}$ is the canonical base of $\mathbb{Q}^3$. The cone $\sigma = pos_{\mathbb{Q}}(A_3)$ is associated to the toric ideal $I_L \subset \mathbb{k}[x_{ij}]$. Every vector of A_3 is an extreme vector of σ.

The Graver Base of $I_{A_3} = I_L$ consist of 3 binomials of the form $x_{ij}x_{kj} - x_{ji}x_{ki}$, 3 binomials of the form $x_{ij}^2 x_{ki} - x_{ik}^2 x_{ji}$, 3 binomials of the form $x_{ij}^2 x_{jk} - x_{ji}^2 x_{ik}$, 3 binomials of the form $x_{ji}^3 x_{ki} - x_{ij}^2 x_{jk}^2$ and 3 binomials of the form $x_{ki}^3 x_{ji} - x_{kj}^2 x_{ik}^2$.

The simplicial complex $\mathcal{D}_\sigma$ is drawn in the Figure. We can prove that $\delta(\mathcal{D}_\sigma)_{\{0,1\}} = 5$ and $\delta(\mathcal{D}_\sigma)_{\{0,1,2\}} = \gamma(\overline{\mathbb{G}(\mathcal{D}_\sigma)}) = 4$. In fact $\mathrm{bar}(I_{A_3}) = 5$ and $\mathrm{ara}_{A_3}(I_{A_3}) = 4$. For $\mathrm{ara}(I_{A_3})$ we have that $3 \leq \mathrm{ara}(I_{A_3}) \leq 4$, but it is unknown whether it is 3 or 4.



We can generalize this example letting $A_n = \{\mathbf{r}_{ij} = 2\mathbf{e}_i + \mathbf{e}_j | i, j \in \{1, \ldots, n\}, i \neq j\}$, where $n \geq 3$. We can prove that

$$\text{bar}(I_{A_n}) = 5\binom{n}{3} + 6\binom{n}{4} \text{ and}$$

$$\text{ara}_{A_n}(I_{A_n}) = 4\binom{n}{3} + 6\binom{n}{4}.$$

For $n = 10$ we have that $\text{bar}(I_{A_{10}}) = 1860$ and $\text{ara}_{A_{10}}(I_{A_{10}}) = 1740$, while $80 \leq \text{ara}(I_{A_{10}}) \leq 90$.

In those 80 up to 90 polynomials that generate the radical of $I_{A_{10}}$, there should be totally at least 3600 monomials in at least 1740 A_{10}-homogeneous components.