# Tropical Discriminants

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# Outline

1. $A$-Discriminants
2. Tropical Geometry
3. Tropical $A$-Discriminants
4. The Newton Polytope of $\Delta_A$
5. Regular Subdivisions and $\Delta$-Equivalence of Triangulations

joint work/project with Alicia Dickenstein and Bernd Sturmfels

# 1. $A$-Discriminants

*[Gelfand, Kapranov, Zelevinsky 1992]*

$A = (a_1 \cdots a_n) \in \mathbb{Z}^{d\times n}$, $\operatorname{rk} A = d$, $(1,\ldots,1) \in$ row span $A$

$Q_A = \operatorname{conv}\{a_1,\ldots,a_n\}$ polytope in $\mathbb{R}^d$, $\dim Q_A = d-1$

$$X_A = \mathcal{V}\left(\langle x^u - x^v \,|\, u,v \in \mathbb{N}^n \text{ with } Au = Av\rangle\right)$$

projective toric variety

$$X_A^* = \text{cl}\ \{\xi \in (\mathbb{CP}^{n-1})^* \,|\, H_\xi \text{ tangent to } X_A \text{ at a regular point}\}$$

dual variety

If $\operatorname{codim} X_A^* = 1$,

$$X_A^* = \mathcal{V}(\Delta_A),$$

where $\Delta_A$ is a unique irreducible polynomial, the $A$-discriminant.

# $A$-Discriminants: Classical Examples

1. Discriminant of a quadratic polynomial in 1 variable

$$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$$

$f$ has a double root $\iff$ $\Delta_f = a_1^2 - 4a_2a_0 = 0$

$$\Delta_f = \Delta_A \in \mathbb{Z}[a_0, a_1, a_2] \quad \text{for} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

2. Discriminant of a degree $n$ polynomial in 1 variable

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad a_n \neq 0$$

$f$ has a double root $\iff$ $\Delta_f = 0$

$$\Delta_f = \Delta_A \in \mathbb{Z}[a_0, \ldots, a_n] \quad \text{for} \quad A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & n \end{pmatrix}$$

# $A$-Discriminants: Classical Examples

2. Resultant of two polynomials in 1 variable

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad a_n \neq 0, \qquad\qquad g(x) = \sum_{i=0}^{m} b_i x^i, \quad b_m \neq 0,$$

$f$ and $g$ have a common root $\Longleftrightarrow$ $\mathrm{Res}(f,g) = 0$

$\mathrm{Res}(f,g) = \Delta_A \in \mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_m]$ for

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & n & 0 & 1 & \ldots & m \end{pmatrix}$$

$\mathrm{Res}(f,g) =$ determinant of the Sylvester matrix

# $A$-Discriminants: Classical Examples

3. Discriminant of a deg $2$ homogeneous polynomial in $3$ variables

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} \qquad \Delta_A = \det \begin{pmatrix} 2a_1 & a_2 & a_3 \\ a_2 & 2a_4 & a_5 \\ a_3 & a_5 & 2a_6 \end{pmatrix}$$

4. Discriminant of a deg $3$ homogeneous polynomial in $3$ variables

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$\deg \Delta_A = 12$, $\quad 2040$ terms

# $A$-Discriminants

Call $A$ defective if $\operatorname{codim} X_A^* > 1$.

The dual variety $X_A^*$ is also of interest in the defective case.

Goal: Derive information on $\Delta_A$, resp. $X_A^*$, for instance

- degree and extreme monomials of $\Delta_A$
- dimension, degree and Chow form of $X_A^*$

directly from $A$, without any reference to defining equations.

Ansatz: Study the tropicalization of $X_A^*$ !

# 2. Tropical Geometry

$(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$, $\quad x \oplus y := \min\{x, y\}$, $\quad x \otimes y := x + y$

tropical semi-ring

complex projective varieties $\xrightarrow{\tau}$ polyhedral fans

$Y \subseteq \mathbb{C}\mathbb{P}^{n-1}$ irreducible variety, $\dim Y = r$
$I_Y \subseteq \mathbb{C}[x_1, \ldots, x_n]$ defining prime ideal

$$\tau(Y) = \{ w \in \mathbb{R}^n \mid \mathrm{in}_w(I_Y) \text{ does not contain a monomial} \}$$

tropicalization of $Y$

$\tau(Y)$ is a pure $r$-dimensional polyhedral fan in $\mathbb{R}^n$,
respectively $\mathbb{T}\mathbb{P}^{n-1} = \mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1)$.

# Examples of Tropicalized Varieties

## 1. $Y$ hypersurface in $\mathbb{CP}^{n-1}$

$f \in \mathbb{C}[x_1,\ldots,x_n]$ irreducible polynomial defining Y
$\mathrm{New}(f)$ Newton polytope, $\mathcal{N}_{\mathrm{New}(f)}$ its normal fan

$$\tau(Y) \;=\; \text{codim } 1\text{-skeleton of } \mathcal{N}_{\mathrm{New}(f)}$$

*Proof:*

$$\begin{aligned}
\tau(Y) &= \{\, w \in \mathbb{R}^n \,|\, \mathrm{in}_w(f) \text{ is not a monomial}\,\} \\
&= \{\, w \in \mathbb{R}^n \,|\, \dim\big(\mathrm{New}(\mathrm{in}_w(f))\big) > 0\} \\
&= \{\, w \in \mathbb{R}^n \,|\, \dim\big(w\text{-maximal face of } \mathrm{New}(f)\big) > 0\} \\
&= \bigcup_{\substack{\sigma \in \mathcal{N}_{\mathrm{New}(f)} \\ \mathrm{codim}\,\sigma > 0}} \sigma
\end{aligned}$$

# Examples of Tropicalized Varieties

2. $Y = X_A$ toric variety, $A \in \mathbb{Z}^{d \times n}$

$$\tau(Y) \;=\; \text{row span } A$$

*Proof:*

$$\begin{aligned} I_{X_A} &= \langle\, x^u - x^v \,|\, u, v \in \mathbb{N}^n \text{ with } Au = Av \rangle \\ \tau(Y) &= \{\, w \in \mathbb{R}^n \,|\, \mathrm{in}_w(f) \text{ is not a monomial for any } f \in I_{X_A} \} \\ &= \{\, w \in \mathbb{R}^n \,|\, wu = wv \text{ whenever } Au = Av \} \\ &= \text{row span } A \end{aligned}$$

3. $Y = V$ linear, resp. projective subspace

$$\tau(Y) \;=\; \mathcal{B}(M(V))$$

Bergman fan of the matroid associated with $V$

# Digression: Bergman Fans of Matroids

$M$ connected matroid on $\{1,\ldots,n\}$, $\operatorname{rk} M = r$

$M_w \;=\; \{\sigma \in M \,|\, \sigma$ basis with maximal $w$-cost $\}$ for $w \in \mathbb{R}^n$

$$\mathcal{B}(M) \;=\; \{w \in \mathbb{R}^n \,|\, M_w \text{ is loop-free } \}$$

Bergman fan

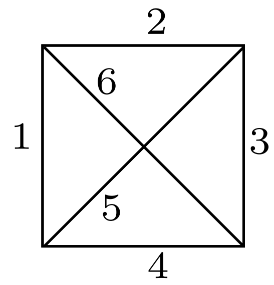
$$B(M) \;=\; \mathcal{B}(M) \cap \left\{ w \in \mathbb{R}^n \,\middle|\, \sum w_i = 0, \sum w_i^2 = 1 \right\}$$

Bergman complex

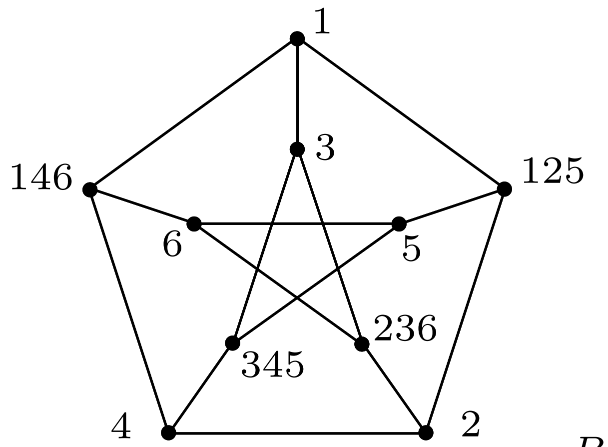
$\mathcal{B}(M)$ is a $(\operatorname{rk} M - 1)$-dimensional subfan of $\mathcal{N}_{P(M)}$, where $P(M)$ is the matroid polytope of $M$.

# Examples of Bergman Fans

$M = M(K_4)$
$r = 3,\ n = 6$

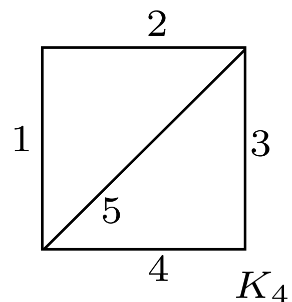



$K_4$

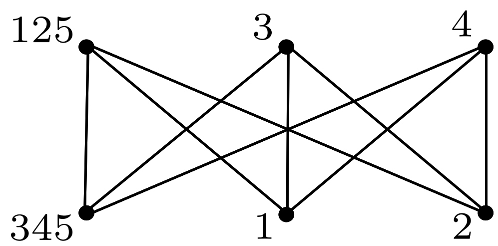



$B(M(K_4))$

$M = M(K_4 \backslash e)$
$r = 3,\ n = 5$




$K_4 \backslash e$




$B(M(K_4 \backslash e))$

# Digression: Nested Set Fans of Matroids

$M$ connected matroid on $\{1, \ldots, n\}$, $\mathrm{rk}\, M = r$
$\mathcal{L}_M$ lattice of flats

$\mathcal{G} \subseteq (\mathcal{L}_M)_{>\hat{0}}$ building set if for any $X \in \mathcal{L}_M$ and $\max \mathcal{G}_{\leq X} = \{G_1, \ldots, G_k\}$, there exists an isomorphism

$$\phi_X : \quad \prod_{i=1}^{k} [\hat{0}, G_i] \quad \longrightarrow \quad [\hat{0}, X]\,.$$

$\mathcal{G}_{\min}$: irreducibles, dense edges, connected flats
$\mathcal{G}_{\max}$: $\mathcal{L}_M \setminus \{\hat{0}\}$

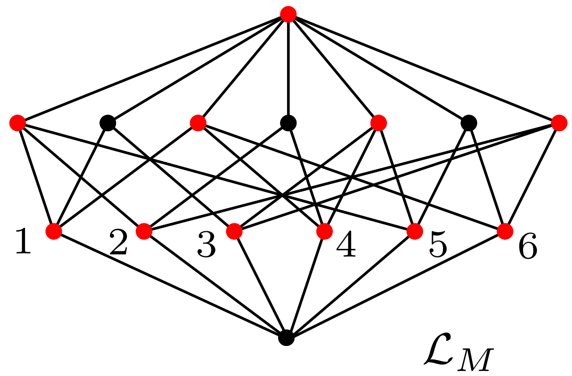
$\mathcal{S} \subseteq \mathcal{G}$ nested set if for any pairwise incomparable $X_1, \ldots, X_t \in \mathcal{S}$, $t \geq 2$, $\bigvee X_i \notin \mathcal{G}$.

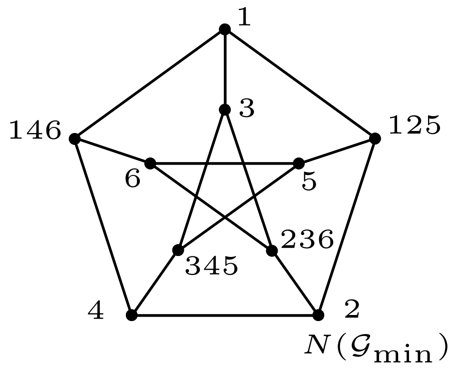
$N(\mathcal{G})$ abstract simplicial complex of nested sets
$\mathcal{N}(\mathcal{G})$ realization as a simplicial fan in $\mathbb{R}^n$

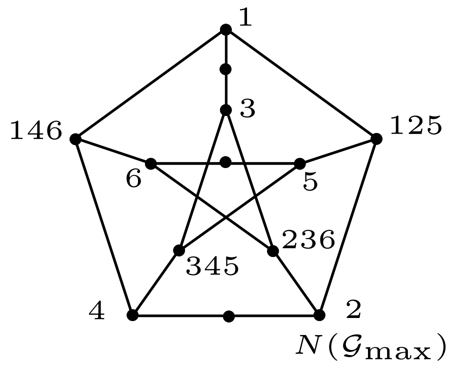
# Examples of Nested Set Fans

$M = M(K_4)$

$r = 3,\ n = 6$

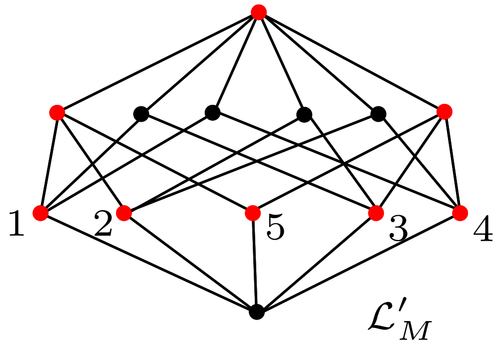


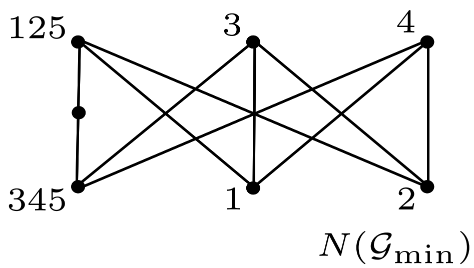



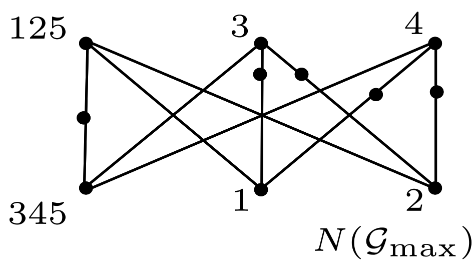




$M = M(K_4 \setminus e)$

$r = 3,\ n = 5$



# Bergman Fans versus Nested Set Fans

Proposition: *[F. & Sturmfels ’04; F. & Müller ’03]*

$\mathcal{B}(M)$ is subdivided by $\mathcal{N}(\mathcal{G})$ for any building set $\mathcal{G}$ in $\mathcal{L}_M$.

$\mathcal{N}(\mathcal{G})$ is subdivided by $\mathcal{N}(\mathcal{G}')$ for any building sets $\mathcal{G} \subseteq \mathcal{G}'$ in $\mathcal{L}_M$.

Back to tropical geometry:

$V$ a linear subspace in $\mathbb{C}^n$, $M$ the associated matroid,
$\mathcal{G}$ any building set in $\mathcal{L}_M$

$$\tau(V) \;=\; \text{supp}\ \mathcal{B}(M) \;=\; \text{supp}\ \mathcal{N}(\mathcal{G})$$

# 3. Tropical $A$-Discriminants

$$A = (a_1 \cdots a_n) \in \mathbb{Z}^{d\times n}, \ \operatorname{rk} A = d, \quad (1,\dots,1) \in \text{row span } A$$

Horn uniformization of $A$-discriminants: *[Kapranov '91]*

The dual variety $X_A^*$ is the closure of the image of the morphism

$$\begin{array}{rccc} \varphi_A : & \mathbb{P}(\ker A) \times (\mathbb{C}^*)^d/\mathbb{C}^* & \longrightarrow & (\mathbb{CP}^{n-1})^* \\ & (u,t) & \longmapsto & (u_1 t^{a_1} : u_2 t^{a_2} : \cdots : u_n t^{a_n})\,. \end{array}$$

Tropical Horn uniformization:

$$\begin{array}{rccc} \tau(\varphi_A) : & \mathcal{B}(\ker A) \times \mathbb{R}^d & \longrightarrow & \mathbb{TP}^{n-1} \\ & (w,v) & \longmapsto & w + vA \end{array}$$

$$\operatorname{im} \tau(\varphi_A) = \mathcal{B}(\ker A) + \text{row span } A$$ Horn fan

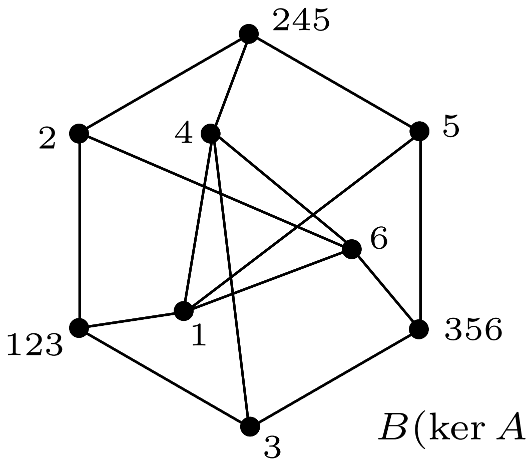
# Tropical $A$-Discriminants

Theorem: *[DFS '05]*

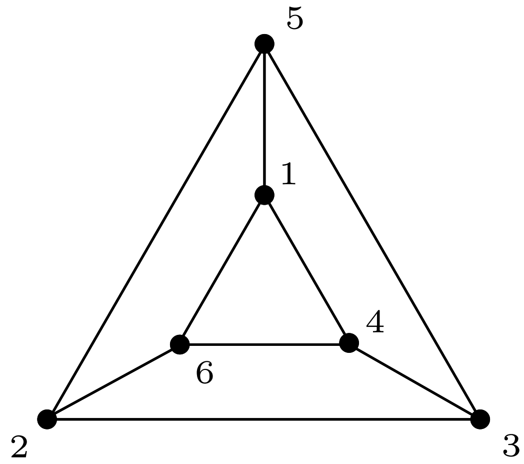
$$\tau(X_A^*) \;=\; \mathcal{B}(\ker A) + \text{row span } A$$

Example:

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$




$B(\ker A)$




$\tau(X_A^*)$

# Tropicalizing Monomials in Linear Forms

$f : \mathbb{C}^m \xrightarrow{U} \mathbb{C}^r \xrightarrow{V} \mathbb{C}^s$

$U \in \mathbb{C}^{r\times m}$ linear map, $V \in \mathbb{Z}^{s\times r}$ monomial map

$$f_i(x_1,\ldots,x_m) \quad = \quad \prod_{k=1}^{r} (u_{k1}x_1 + \cdots + u_{km}x_m)^{v_{ik}}$$

$Y_{UV}$ := closure of $\operatorname{im} f$

Examples:

$r = s$, $V = I_r$: $\quad Y_{UV} = \operatorname{im} U \quad$ linear space

$m = r$, $U = I_m$: $\quad Y_{UV} = X_{V^T} \quad$ toric variety

Theorem: *[DFS '05]*

$$\tau(Y_{UV}) \;=\; V \circ \tau(\operatorname{im} U) \;=\; V \circ \mathcal{B}(M(\operatorname{im} U))$$

# Tropicalizing Monomials in Linear Forms

## Retrieving the tropical discriminant:

Set $m = r$, $r = n + d$, $s = d$, $B$ a Gale dual of $A$,

$$U = \begin{pmatrix} B & 0 \\ 0 & I_d \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} I_n & A^T \end{pmatrix}.$$

Then, $Y^{\mathbb{P}}_{UV} = X^*_A$ , and

$$\tau(X^*_A) = \mathcal{B}(\ker A) + \text{row span } A\,.$$

# 4. The Newton Polytope of $\Delta_A$

$A \in \mathbb{Z}^{d\times n}$ non-defective, $\mathcal{L}$ lattice of flats of $M(\ker A)$,
$\mathcal{N}$ any nested set fan of $\mathcal{L}$,

$\sigma \in \mathcal{N} \;\longleftrightarrow\; \sigma_1,\ldots,\sigma_{n-d-1} \;\in\; \{0,1\}^n$

$w \in \mathbb{R}^n$ generic, $i \in \{1,\ldots,n\}$,
$\mathcal{N}_{i,w} \;=\; \{\sigma \in \mathcal{N} \,|\, \text{row span } A \cap \mathbb{R}_{>0}\{\sigma_1,\ldots,\sigma_{n-d-1},-w,-e_i\} \neq \emptyset\,\}$.

Theorem: *[DFS '05]*

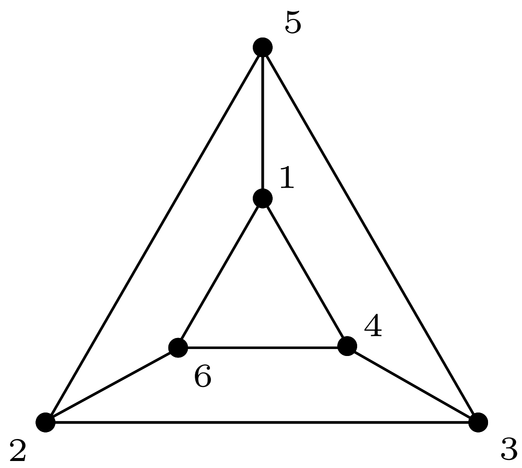
$$\deg_{x_i}(\mathrm{in}_w(\Delta_A)) \;=\; \sum_{\sigma\in\mathcal{N}_{i,w}} \Big|\det\big(A^T,\sigma_1,\ldots,\sigma_{n-d-1},e_i\big)\Big|,$$

in fact, $\deg_{x_i}(\mathrm{in}_w(\Delta_A))$ is the number of intersection points of $w+\mathbb{R}_{>0}e_i$ with $\tau(X_A^*)$, counted with multiplicity.

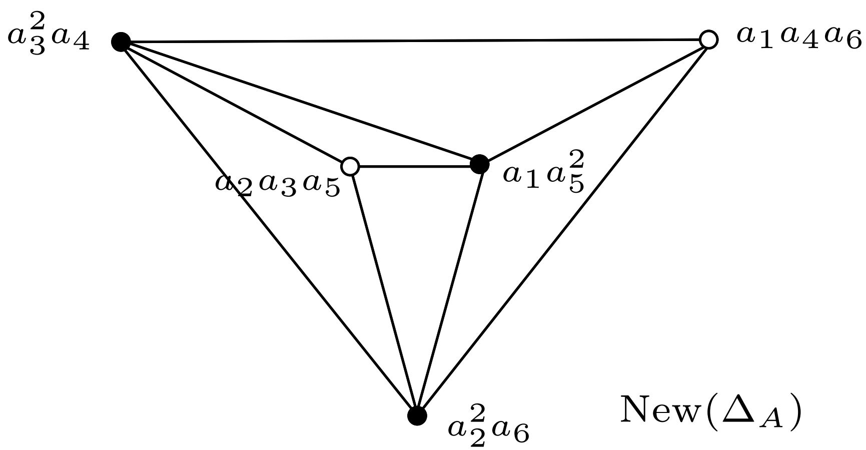
# The Newton Polytope of $\Delta_A$

Example:

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$




$\tau(X_A^*)$




$\mathrm{New}(\Delta_A)$

# 5. Regular Subdivisions

$A = (a_1 \cdots a_n) \in \mathbb{Z}^{d\times n}$, point configuration $a_1, \dots, a_n$ in $\mathbb{Z}^d$

For $w \in \mathbb{R}^n$,

$$\Pi_w := \begin{cases} \text{regular subdivision of } A \text{ given by lower facets of} \\ \operatorname{conv}\{(a_1, w_1), \dots, (a_n, w_n)\} \subseteq \mathbb{R}^d \times \mathbb{R}\,. \end{cases}$$

For $\sigma$ a maximal cell in $\Pi_w$, $M(\sigma)$ the associated $\operatorname{rk} d$ matroid, define the affine linear function $\psi_\sigma : \mathbb{R}^d \to \mathbb{R}$ by

$$\psi_\sigma(a) = \begin{cases} 1\,, & \text{if } A \text{ is a co-loop in } M(\sigma)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

and set $\sigma^* := \sigma \cup \{a \in A \,|\, \psi_\sigma(a) < 1\} \subseteq A$.
Call $a \in \sigma$ a strong co-loop if it is a co-loop in $M(\sigma^*)$.

# Regular Subdivisions

Theorem: *[DFS '05]*

The tropical $A$-discriminant $\tau(X_A^*)$ equals

$$\{\ w \in \mathbb{R}^n \mid \Pi_w \text{ has a maximal cell } \sigma \text{ with no strong co-loops }\}.$$

Corollary:

$\tau(X_A^*)$ is a subfan of the secondary fan $\Sigma(A)$.

# $\Delta$-Equivalence of Triangulations

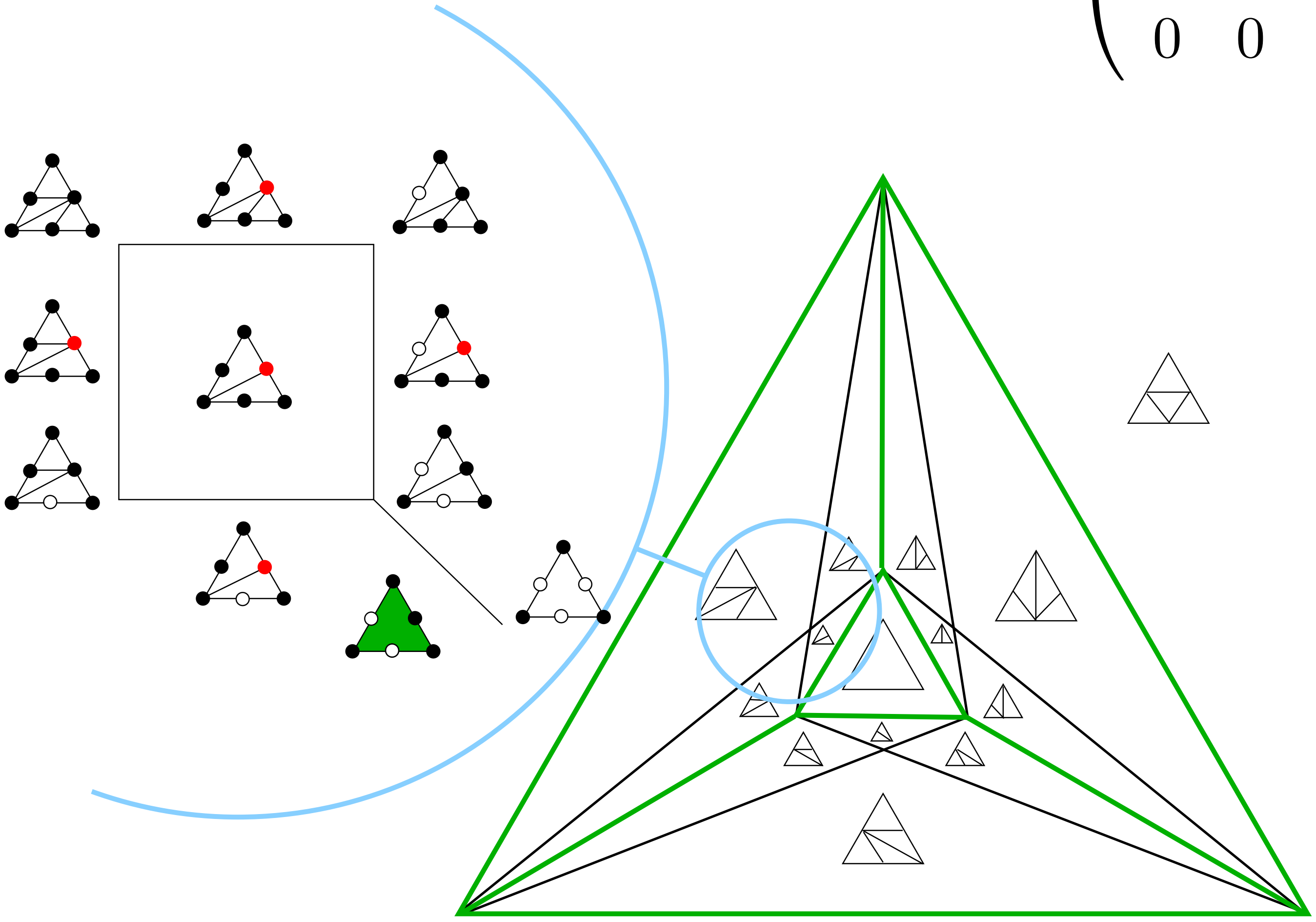
For a non-defective point configuration $A$, two regular triangulations $\Pi_w$ and $\Pi_{w'}$ are called $\Delta$-equivalent if

$$\mathrm{in}_w(\Delta_A) \; = \; \mathrm{in}_{w'}(\Delta_A)\,.$$

Corollary: Two neighboring triangulations $\Pi_w$ and $\Pi_{w'}$ are $\Delta$-equivalent if and only if every cell in their coarsening has a strong co-loop.

# Example

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$



$$\tau(X_A^*) \subseteq \Sigma(A)$$