Polyhedra and Polytopes:
Algebra and Combinatorics

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These three lectures are designed to cover the theory and background to the proof of the necessity of McMullen's conditions in the g-theorem, which characterizes the f-vectors of simple polytopes.

The algebra of polyhedra

At the heart of much of the metrical theory of convex bodies and related topics, as well as some of the combinatorial theory of polytopes, lies the concept of a valuation. For polytopes, it turns out to be very useful to develop the initial part of the theory of valuations at the more general level of polyhedra (polyhedral sets). Although valuations imply only additive relations on polyhedra, it is convenient right from the outset to have an underlying ring structure. Let $\mathcal{Q} = \mathcal{Q}(\mathbb{V})$ denote the family of non-empty polyhedra in a finite-dimensional vector space $\mathbb{V}$ over an arbitrary ordered field $\mathbb{F}$ (further conditions on $\mathbb{F}$ will be imposed later); $\mathcal{P} = \mathcal{P}(\mathbb{V})$ denotes the sub-family of polytopes. The *polyhedron ring* $\Gamma = \Gamma(\mathbb{V})$ has a generator $[P]$, called its *class*, for each $P \in \mathcal{Q}$; formally, define $[\emptyset] := 0$. The addition on Γ is induced by the *valuation relation*:

$$[P \cup Q] + [P \cap Q] = [P] + [Q]$$

whenever P, Q are polyhedra such that $P \cup Q$ is convex. The multiplication on Γ is induced by *Minkowski addition*:

$$[P] \cdot [Q] = [P + Q].$$

The unity on Γ is $1 := [o]$ ($[t]$ is shorthand for $[\{t\}]$ when $t \in \mathbb{V}$). There is then a range of structure theorems on Γ, among which one will recognize generalized angle-sum relations. Of course, all these carry over to the polytope ring Π generated by the classes $[P]$ with $P \in \mathcal{P}$, where the implications are perhaps more familiar.

What is less familiar – indeed, very widely unknown – is the co-ring structure on Γ. The laws of a co-ring dualize those of a ring. Thus, for example, multiplication is an additive homomorphism $\mu\colon \Gamma \otimes \Gamma \to \Gamma$, with certain additional properties such as associativity – the distributive laws are taken care of by the formulation – and the existence of a unity. Then co-multiplication is an additive homomorphism $\kappa\colon \Gamma \to \Gamma \otimes \Gamma$, given here by

$$[P]\kappa := \sum_{F \leqslant P} [\operatorname{relint} F] \otimes [C(F, P)],$$

where $\operatorname{relint} F$ is the relative interior of F (its class is well-defined in Γ), and $C(F, P)$ is the angle-cone generated by P at F (by $F \leqslant P$ is meant that F is a non-empty face of P, with P itself specifically allowed). There will be an appropriate definition of co-associativity, and there will exist a corresponding co-unity. Indeed, with this choice of κ (others yield a co-ring structure as well), $\varGamma$ will be a bi-ring, meaning that the ring and co-ring operations are compatible. This may seem to be unnecessarily abstract, but it will be shown that some well-known valuations – and many variants of them – arise naturally through the co-ring structure.

Polytope and weight algebras

While the polyhedron ring $\varGamma$ has a rich algebraic structure (as a co- and bi-ring as well), its quotients by powers T^{k+1} of the translation ideal $T := \langle [t] - 1 \mid t \in \mathbb{V} \rangle$ with $k \geqslant 1$ seem not to play a very important rôle. In contrast, the corresponding quotients $\varPi/T^{k+1}$ are very important. When $k = 0$, the quotient is, in all but one trivial respect, a graded algebra over the base field $\mathbb{F}$. When $k > 0$, only a rational algebra is obtained (modulo the same triviality), but this can be turned into an $\mathbb{F}$-algebra when further factored by what is called the *algebra ideal* A.

These quotient algebras can be identified with algebras of tensor weights on polytopes; at this stage, for convenience, one assumes that $\mathbb{F}$ is square-root-closed. A *tensor weight* a assigns a (symmetric) tensor $a(F)$ to each face F of a polytope P; these weights, for different faces, are linked by the *Green-Minkowski connexions* (*GMC*)

$$\sum_{F \lhd G} a(F)\langle u(F,G), t\rangle = a(G)t,$$

where $u(F,G)$ is the unit outer normal to $G \leqslant P$ at its facet F (the meaning of $F \lhd G$) in the linear space $G_{\|}$ parallel to aff G, and $t \in G_{\|}$; the GMC generalize Green's theorem and the Minkowski relations for facet-areas of polytopes. There is a resulting graded $\mathbb{F}$-algebra $\mathcal{W}(\mathbb{V})$; its restriction $\mathcal{W}(P)$ to a simple d-polytope P is isomorphic to the face-ring of the dual of P, so that it already captures some of the combinatorial properties of P. It will be seen later that the corresponding scalar weight algebras $\varOmega(P)$ are of even greater importance.

Duality considerations also suggest a different multiplication $\boxtimes$ on tensor weights; this is vaguely related to a multiplication on valuations introduced by Alesker. It arises from the fibre polytope construction, and permits brief descriptions of, for example, Schneider's mixed polytope construction.

The g-theorem

As was said earlier, the g-theorem characterizes the face-vectors of simple polytopes. If P is a simple d-polytope, meaning that each vertex of P lies in exactly d facets, with $f_j = f_j(P)$ j-faces for each $j = 0, \ldots, d$ (thus $f_d = 1$), then $(f_0, \ldots, f_d)$ is its f-vector, and

$$f(P,\tau) := \sum_{j=0}^{d} f_j \tau^j = \sum_{F \leqslant P} \tau^{\dim F}$$

is its f-polynomial. The analogous h-vector $(h_0, \ldots, h_d)$ (with $h_r = h_r(P)$) and h-polynomial $\sum_{r=0}^{d} h_r \tau^r$ are defined by

$$h(P, \tau) := f(P, \tau - 1).$$

Thus, $f(P, \tau) = h(P, \tau + 1)$, so that the f_j are linear combinations of the h_r with non-negative integer coefficients. Finally, the g-polynomial is defined by

$$g(P, \tau) = \sum_{r=0}^{d+1} g_r \tau^r := (1 - \tau) h(P, \tau),$$

and the corresponding g-vector is $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ (this contains all the necessary information, because of the Dehn-Sommerville equations $g_r = -g_{d-r+1}$).

McMullen's conditions can be stated succinctly as: $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ *is the g-vector of some simple d-polytope if and only if there is a standard algebra* $A = \bigoplus_{r \geqslant 0} A_r$ *such that* $g_r = \dim A_r$ *for each* $r = 0, \ldots, \lfloor d/2 \rfloor$. (It should, perhaps, be mentioned that the original formulation of the conditions was expressed in terms of the inequalities, due to Macaulay, relating these dimensions.)

Stanley proved the necessity of McMullen's conditions using the hard Lefschetz theorem, applied to the cohomology of the toric variety derived from a *rational* simple polytope P. This proof was widely felt (particularly by the speaker) to employ much heavier machinery than should be needed; more to the point, a proof working entirely within convexity was obviously desirable. Such a proof, which was found by the speaker, initially used the polytope algebra Π/T, but was subsequently refined to use the scalar weight algebra Ω. It is this proof which will be presented in this lecture.

Some interesting consequences of the method of proof will also be mentioned. These include the Brunn-Minkowski and Alexandrov-Fenchel inequalities for mixed volumes of polytopes, which thus permit algebraic rather than analytic proofs.