MINKOWSKI'S SUCCESSIVE MINIMA AND THE LATTICE POINT ENUMERATOR

MARTIN HENK

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We give a brief survey on classical and new inequalities involving Minkowski's successive minima $\lambda_i(K)$ of a 0-symmetric convex body $K \subset \mathbb{R}^n$ and their relation to the lattice point enumerator of K. The i-th successive minimum, $i \in \{1, \ldots, n\}$, is defined by

$$\lambda_i(K) = \min\{\lambda > 0 : \dim(\lambda\, K \cap \mathbb{Z}^n) \geq i\},$$

i.e., $\lambda_i(K)$ is the smallest positive number λ such that $\lambda\, K$ contains at least i linearly independent lattice points. Minkowski proved two fundamental theorems for the successive minima and the volume of a 0-symmetric convex body which can be written in the following way.

Theorem (Minkowski, 1896):

$$\mathrm{vol}(K) \leq \left(\frac{2}{\lambda_1(K)}\right)^n,$$

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{vol}(K) \leq \prod_{i=1}^{n}\frac{2}{\lambda_i(K)}.$$

All these bounds are best possible and the upper bound in the second inequality is a generalisation of the first inequality. In a joint paper with Ulrich Betke and Jörg M. Wills, 1993, we studied similar relations when the volume is replaced by the lattice point enumerator $\mathrm{G}(K) = \#(K \cap \mathbb{Z}^n)$ and proved the following.

Theorem (Betke, H., Wills, 1993):

$$\mathrm{G}(K) \leq \left(\frac{2}{\lambda_1(K)} + 1\right)^n,$$

$$\left(1 - \frac{\lambda_1(K)}{2}\right)\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{G}(K) \leq \prod_{i=1}^{n}\left(\frac{2}{\lambda_i(K)} + 1\right),$$

where we could verify the upper bound in the second relation *only* in the planar case $n = 2$. Observe that all these inequalities imply Minkowski's bounds for the volume. We strongly believe that the above upper bound holds true in all dimensions and so we have the following conjecture.

Conjecture:

$$\mathrm{G}(K) \leq \prod_{i=1}^{n}\left(\frac{2}{\lambda_i(K)} + 1\right).$$

Denote the right hand side by $\mathrm{L}(K)$. Then $\mathrm{L}(\rho\, K) = \prod_{i=1}^{n}\left(\rho\,\frac{2}{\lambda_i(K)}+1\right)$ for $\rho \in \mathbb{R}_{\geq 0}$, and hence we may regard it as a polynomial of degree n for any complex number $s \in \mathbb{C}$. Thus we set

$$\mathrm{L}(s, K) = \prod_{i=1}^{n}\left(s\,\frac{2}{\lambda_i(K)}+1\right).$$

On the other hand we know that for a 0-symmetric lattice polytope P the lattice point enumerator $\mathrm{G}(mP)$, $m \in \mathbb{N}$, is a polynomial (the so called Ehrhart polynomial) in m, i.e., there exist functionals $\mathrm{G}_i(P)$ depending only on P such that

$$\mathrm{G}(mP) = \sum_{i=0}^{n} \mathrm{G}_i(P)\, m^i.$$

Again we may regard the right hand side as a polynomial for any complex number $s \in \mathbb{C}$ and so let

$$\mathrm{G}(s, P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\, s^i = \prod_{i=1}^{n}\left(\frac{s}{\gamma_i(P)}+1\right),$$

where $-\gamma_i(P)$ are the roots of the Ehrhart polynomial. In a joint work with Achill Schürmann and Jörg M. Wills we are interested in relations between the zeros $-\gamma_i(P)$ of $\mathrm{G}(s, P)$ and the zeros $-\lambda_i(P)/2$ of $\mathrm{L}(s, P)$. In terms of the roots of the Ehrhart polynomial Minkowski's inequalities for the successive minima can be rewritten as

$$\left(\prod_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)^{1/n} \leq \left(\prod_{i=1}^{n}\gamma_i(P)\right)^{1/n} \leq n!^{1/n}\left(\prod_{i=1}^{n}\frac{\lambda_i(P)}{2}\right)^{1/n}.$$

Our main result so far gives an analogous relation for the arithmetic mean.

Theorem (H., Schürmann, Wills, 2005):

$$\frac{1}{n}\sum_{i=1}^{n}\gamma_i(P) \leq \frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_i(P)}{2}.$$

This inequality is best possible and equivalent to the more geometric statement

$$\frac{\mathrm{G}_{n-1}(P)}{\mathrm{vol}(P)} \leq \sum_{i=1}^{n}\frac{\lambda_i(P)}{2}$$

which in particular shows the following.

Corollary:

$$\mathrm{G}_{n-1}(P) \leq \sum_{j=1}^{n}\prod_{i\neq j}\frac{2}{\lambda_i(P)}.$$

Martin Henk, Universität Magdeburg, Institut für Algebra und Geometrie, Universitätsplatz 2, D-39106 Magdeburg, Germany
E-mail address: henk@math.uni-magdeburg.de