Enumeration of Flags in Eulerian Posets

Louis Billera
Cornell University, Ithaca, NY 14853

Conference on
Algebraic and Geometric Combinatorics
Anogia, Crete
20-26 August, 2005

Abstract

These three lectures will develop the theory of flag enumeration in polytopes and Eulerian posets, including the connection of this theory, via duality of Hopf algebras, to the theory of quasisymmetric functions. In the end, we will see some applications of this to enumeration in hyperplane arrangements and convex closures and to Kazhdan-Lusztig polynomials of Coxeter groups.

Lecture I: Faces in polytopes and chains in posets

- *f-vectors of convex polytopes and the g-theorem:* We begin by reviewing the problem of counting faces in convex polytopes. In this case of simplicial (and simple) poloytopes, the answer is given by the g-theorem. For general polytopes of dimension larger than three, the answer is unknown.

- *flag f-vectors of graded posets:* The study of flag f-vectors of Eulerian posets grew from a generally unsuccessful attempt to find an extension of McMullen's g-conditions to arbitrary polytopes. The flag f-vector of a polytope Q counts all chains of faces according to their corresponding sets of dimensions. Considering that you are counting chains in its lattice of faces $P = \mathcal{F}(Q)$, this leads to the more general question of understanding flag f-vectors of classes of graded posets.

- *Eulerian posets and the* **cd**-*index:* In the case of polytopes (and spherical decompositions), the resulting face lattice is Eulerian, so particular attention has been focused on Eulerian posets. It is known that their flag f-vectors satisfy a system of equations, analogous to the classical Dehn-Sommerville equations for simplicial polytopes, implying that the number of independent components is given by the Fibonacci numbers. The **cd**-index, an invariant derived from the flag f-vector, exhibits this Fibonacci phenomenon very concisely.

- *Convolutions of flag f-vectors:* In order to derive inequalities that are satisfied by flag f-vectors of polytopes, Kalai gave a method of lifting known inequalities from polytopes to those of higher dimension by means of a convolution operation. This operation also serves to derive *equations* and can be used to derive the generalized Dehn-Sommerville equations for Eulerian posets from the fact that the Euler relation must hold in every interval.

Lecture II: Enumeration algebra, quasisymmetric functions and the peak algebra

- *The enumeration algebra for Eulerian posets:* Thinking of flag f-vectors as functions on graded posets, taking as operations formal addition and convolution, one has the *enumeration algebra*, a noncommutative associative algebra of all such functions over graded posets. Restricting these operators to Eulerian posets, one is led to divide out by the (2-sided) ideal of forms that vanish on these, leading to a quotient algebra $A_{\mathcal{E}}$, the *Eulerian enumeration algebra.*.

- *Quasisymmetric functions and P-partitions:* A quasisymmetric function is a formal power series in $\mathbb{Q}[[x_1, x_2, \ldots]]$ whose coefficients are invariant under shifting the variables without changing order. These were originally defined by Gessel to describe the weight enumerators of P-partitions and shown to form an algebra $\mathcal{Q}$. A natural basis for $\mathcal{Q}$ is associated to the descent sets of certain permutations, and multiplication in this algebra corresponds to shuffling permutations and reading the resulting descent sets.

- *The quasisymmetric function of a graded poset:* To a graded poset P, with rank function $r(\cdot)$, Ehrenborg associated the quasisymmetric function

$$F(P) := \sum_{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_k = \hat{1}} x_1^{r(t_0,t_1)} x_2^{r(t_1,t_2)} \cdots x_k^{r(t_{k-1},t_k)},$$

where the sum is over all multichains in P and $r(x,y) = r(y) - r(x)$. Considering the *monomial basis* $\{M_S\}$ or the *fundamental basis* $\{L_S\}$ for $\mathcal{Q}$, one gets

$$F(P) = \sum_S f_S\, M_S = \sum_S h_S\, L_S,$$

where f_S and h_S represent the flag f-vector and the related flag h-vector of P. Thus we can view the quasisymmetric function $F(P)$ as a coordinate-free way to study flag enumeration.

- *Peak quasisymetric functions and enriched P-partitions:* A signed theory of P-partitions was later developed by Stembridge. In this theory, the algebra spanned by all the weight enumerators of so-called *enriched* P-partitions is a subalgebra $\Pi \subset \mathcal{Q}$, called the *peak* algebra. This name derives from the fact that, analogous to the interpretation of $\mathcal{Q}$ in terms of descents of permutations, Π has an interpretation in terms of their *peaks*. Analogous to the fundamental basis $\{L_S\}$ of $\mathcal{Q}$, there is the *peak basis* $\{\Theta_S\}$ for Π, where S runs over the (Fibonacci many) possible peak subsets of $[n] := \{1, 2, \ldots, n\}$.

- *Connection to the enumeration algebra via Hopf algebra duality:* In a more recent development, Bergeron, Mykytiuk, Sottile and van Willigenburg showed that the two Fibonacci-enumerated algebras $A_{\mathcal{E}}$ and Π are closely related. (They are not isomorphic, since the second is commutative, while the first is not.) They showed that both are Hopf algebras with appropriate notions of comultiplication, and as such, they are *dual* Hopf algebras. That is, they are dual as vector spaces, and the multiplication of one is dual to the comultiplication of the other. (This is already hinted at by the relation $F(P) = \sum f_S\, M_S$.) Thus P is Eulerian if and only if $F(P) \in \Pi$, so it is reasonable to ask for the representation of such $F(P)$ in terms of the peak basis $\{\Theta_S\}$. It turns out that this is given by the **cd**-index!

Lecture III: Applications to arrangements, convex closures and Coxeter groups

- *The Stembridge map $\mathcal{Q} \longrightarrow \Pi$:* To relate ordinary P-partitions to their corresponding enriched P-partitions, Stembridge defined a map $\vartheta : \mathcal{Q} \to \Pi$ via $L_S \mapsto \Theta_{\Lambda(S)}$, where $\Lambda(S)$ denotes the peak set associated to the set S. This map has had unexpected, and as yet unexplained, resonance with the certain questions in geometric enumeration.

- *Geometric lattices and arrangements:* In the 1970's, Zaslavsky showed that one can obtain interesting enumerative invariants of a hyperplane arrangement directly from the underlying lattice of flats L of the arrangement. It was later shown that the entire flag f-vector of the arrangement is determined by L. We now know that the flag f-vector of the arrangement can be obtained from the flag f-vector of L. Perhaps the most elegant expression of this is given in terms of the corresponding quasisymmetric functions by

$$\vartheta\left(F(L \cup \widehat{0})\right) = 2\ F(Z),$$

  where $L \cup \widehat{0}$ is the lattice L augmented by a new minimal element, and Z is the (face lattice of) the zonotope dual to the arrangement.

- *Meet distributive lattices and convex closures:* Geometric lattices are the lattices of closed sets of matroids, which can be defined by closure relations abstracting the properties of linear or affine span. If one instead abstracts convex span, one gets the set of *convex closure* operators; the corresponding lattices of closed sets of these are the *meet-distributive* lattices. These generalize the classical distributive lattices in a natural way. Again we we have a relation

$$\vartheta\left(F(L^* \cup \widehat{0})\right) = 2\ F(\mathcal{L}),$$

  where L is any meet-distributive lattice and $\mathcal{L}$ is the face poset of a CW-sphere constructed from the convex closure corresponding to L.

- *Kazhdan-Lusztig polynomials of Coxeter groups:* Eulerian posets also arise from Coxeter groups in the form of the Bruhat order. Work in progress by Brenti and the speaker has shown a direct and simple connection between the Kazhdan-Lusztig polynomial corresponding to a Bruhat interval and an extension of its **cd**-index. In terms of this new index, one can obtain the Kazhdan-Lusztig polynomial as a linear combination of the classical *ballot polynomials*, which count ballot sequences in terms of the eventual final vote.