Triangulations of polytopes

Francisco Santos

Universidad de Cantabria

http://personales.unican.es/santosf

Outline of the talk

1. Triangulations of polytopes

2. Geometric bistellar flips

3. Regular and non-regular triangulations

4. Geometric bistellar flips in context:

   computational geometry,
   algebraic geometry,
   combinatorial topology,
   topological combinatorics.

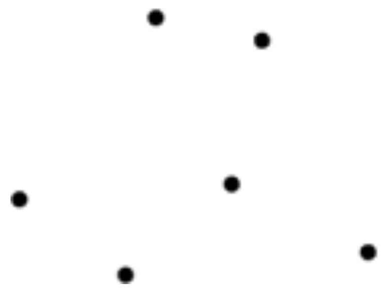
1

Triangulations of polytopes and point sets

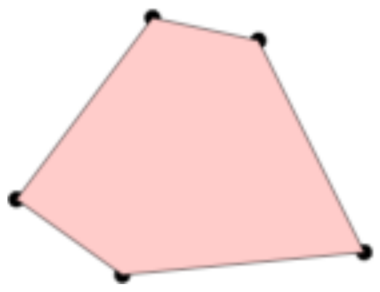
Polytopes

(Convex) polytope := convex hull of finitely many points in $\mathbf{R}^d$.

Equivalently, the bounded intersection of finitely many closed half-spaces.



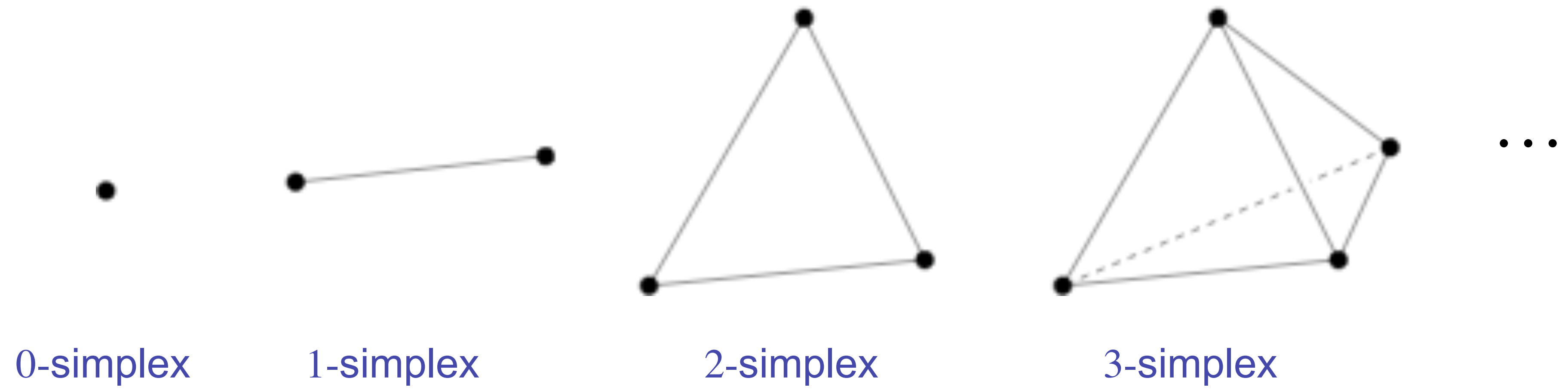
A finite point set in $\mathbf{R}^2$



Its convex hull (a polygon)

Simplices

d-***simplex*** **:=** the simplest of all d-polytopes, with $d+1$ vertices.

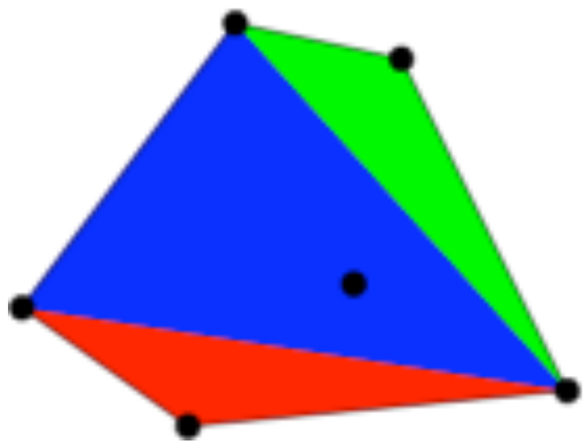


Triangulations of a polytope

A **triangulation** of a polytope P is a partition of it into **simplices** such that:

•The union of all them equals P.
•The intersection of any pair of them is a (possibly empty) common face.

(In other words, a geometric simplicial complex covering P).



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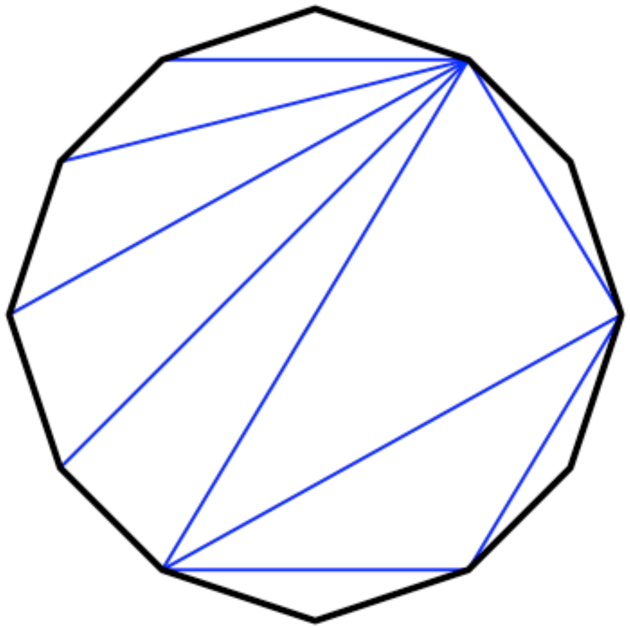
We allow using some interior points as vertices, but normally a fixed set of them. In this sense we speak of "*triangulations of a point set*".

Example: Triangulations of a convex n-gon

To triangulate the n-gon, you just need to insert diagonals, mutually non-crossing, until all regions are triangles:



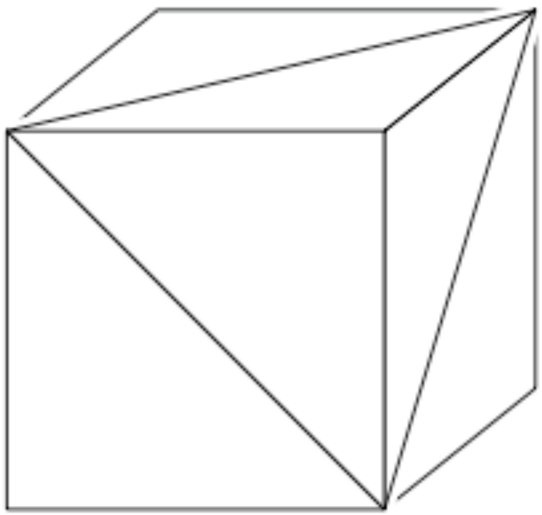
All triangulations of the n-gon have the same number of diagonals $(n - 3)$ and of triangles $(n - 2)$.

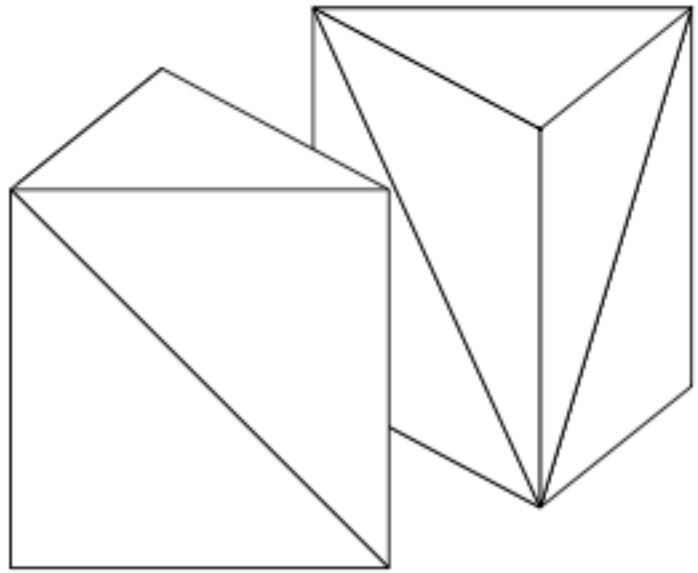
The number of triangulations of an n-gon is the Catalan number

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$$

Example: Triangulations of the 3-cube

There are triangulations of the 3-cube with five tetrahedra (a regular tetrahedron plus four “corners”) and with six tetrahedra (divide the cube into two prisms, then triangulate each with three tetrahedra).





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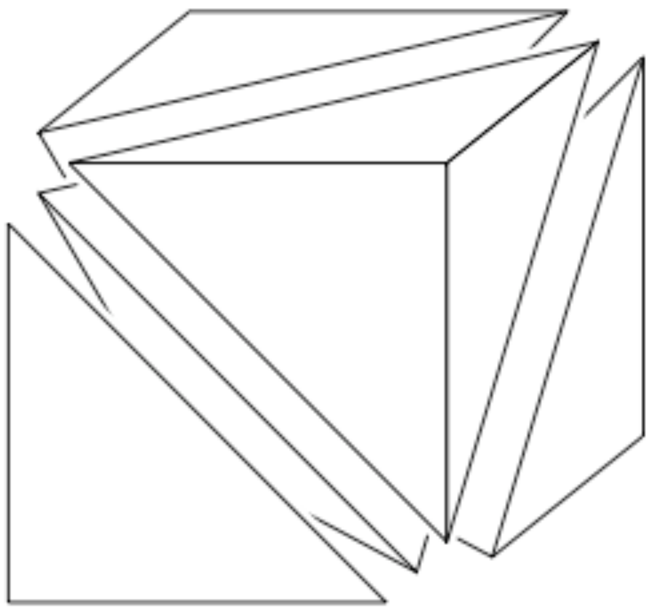

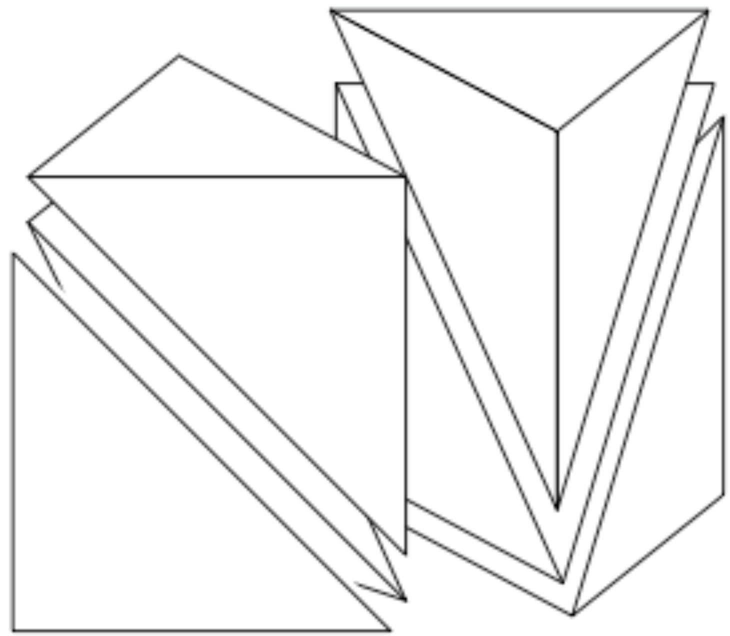

Two versus more dimensions…

In two dimensions:

a) All triangulations of the same point set have the same number of triangles.

b) All non-convex polygons admit a triangulation (without extra vertices).

In three (or more) dimensions:

a) It is *NP-complete* to compute the smallest number of tetrahedra needed to triangulate a convex 3-polytope [Below-de Loera-RichterGebert, 2004]

b) There are non-convex 3-polytopes that cannot be triangulated (without extra vertices) [Schönhardt, 1928]. It is *NP-complete* to decide if a given non-convex 3-polytope can be triangulated [Ruppert-Seidel, 1992]

Example: Triangulations of the n-cube

Triangulations of the n-cube with no extra vertices have *at most* $n!$ n-simplices.

But, how many do they have *at least*? (s_n = "simplexity of the n-cube").

n	1	2	3	4	5	6	7
s_n	1	2	5	16	67	308 (*)	1493 (*)
$r_n := \sqrt[n]{\frac{s_n}{n!}}$	1	1	.941	.904	.890	.868	.840

* [Anderson-Hughes, 1996]

Open question: is $\lim_{n\to\infty} r_n \geq 0$?

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Upper bounds:

[Haiman, 1991]: $\lim_{n\to\infty} r_n \le \inf r_n \le r_7 = 0.840$

[Orden-S., 2003]: $\lim_{n\to\infty} r_n \le 0.816$

Lower bounds: $\lim_{n\to\infty} r_n \sqrt{n} \ge 2$ (Euclidean volume, Hadamard)

$\lim_{n\to\infty} r_n \sqrt{n} \ge \sqrt{6}$ (hiperbolic volume) [Smith, 2000]

Even dim 2 poses hard problems…

Theorem [Mulzer-Rote, 2006] The *Minimum Weight Triangulation* in two dimensions is an NP-hard problem.

The "bible" of NP-completeness [Garey-Johnson, 1979] contains, besides hundreds of NP-complete problems, a list of 14 problems that were "open" (neither known to be polynomial nor NP-complete). One of them was the ***minimum weight triangulation*** problem: given a finite set of points in the plane, compute the triangulation of it with the minimum possible sum of lengths of edges.

2

Geometric bistellar flips

“Spaces of triangulations”

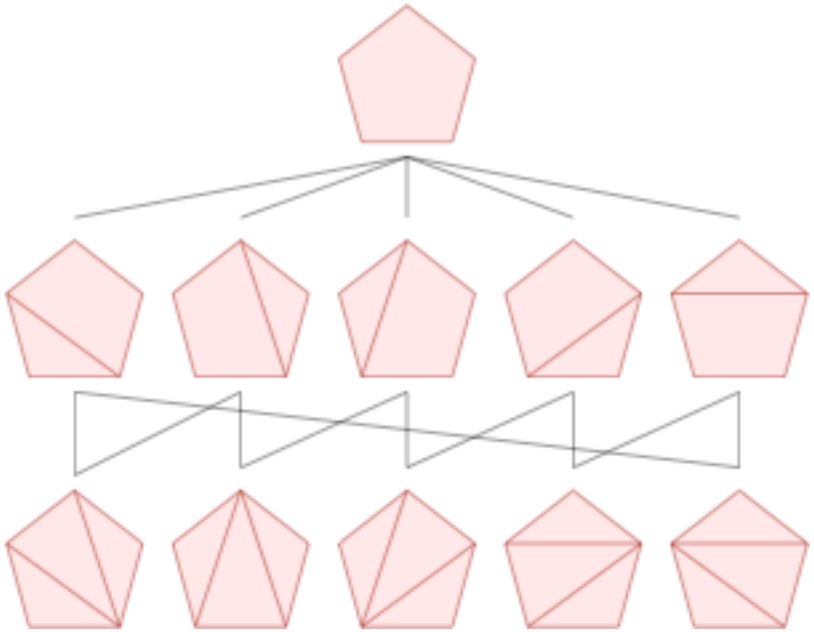
In several contexts (geometric computing / algebraic geometry) it is natural and useful to look at the set of all the possible triangulations of a given point set, and give this set some “structure”.

There are two natural ways to do this:

- The poset of subdivisions
- Geometric bistellar flips

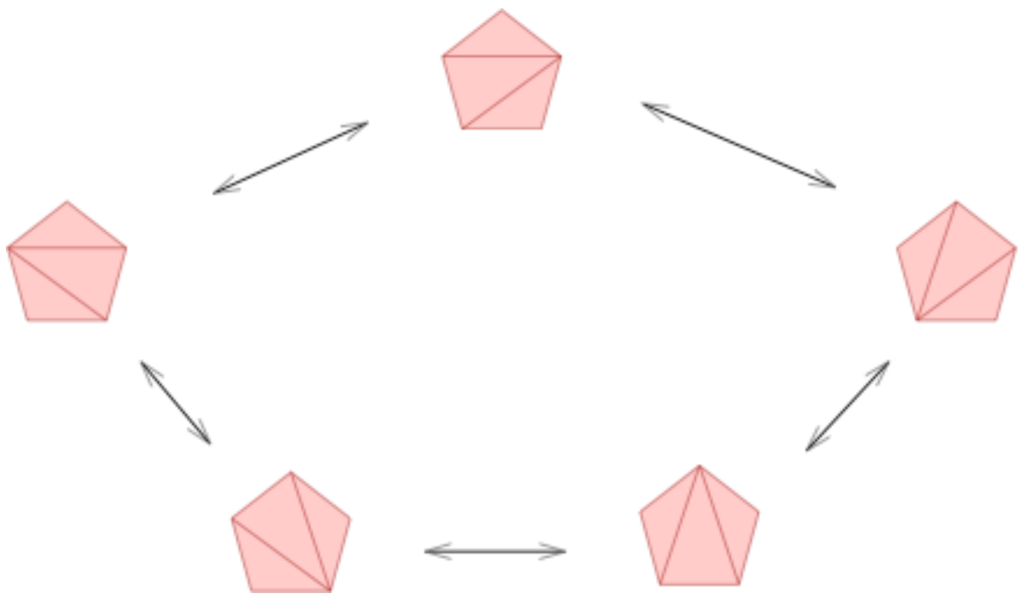
The poset of subdivisions

Polyhedral subdivisions of a polytope P are the decompositions of it into subpolytopes (instead of only simplices) that "cover P" *and* "intersect properly". Polyhedral subdivisions form a *poset* (:=**P**artially **O**rdered **SET**) with the refinement relation:



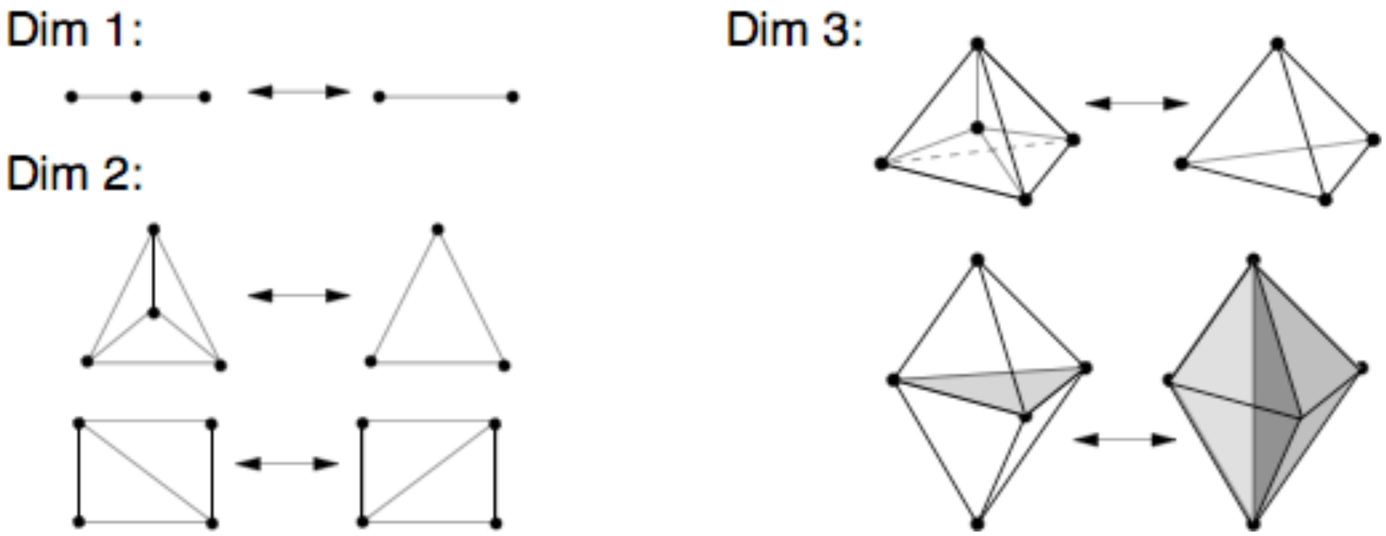
Geometric bistellar flips

(Geometric bistellar) flips are operations that transform a triangulation of a point set into another one, with the minimum possible change ("elementary moves").



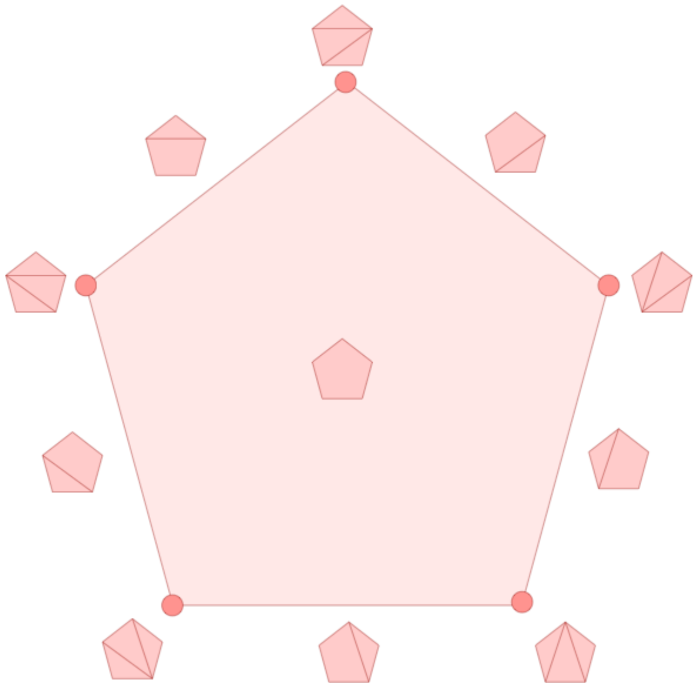
Geometric bistellar flips

They are defined (essentially) as the exchange between the two possible triangulations in a subset of $d+2$ points:



The poset and the graph

Both things are related: two triangulations differ by a flip if and only if there is a polyhedral subdivision that is only refined by them.



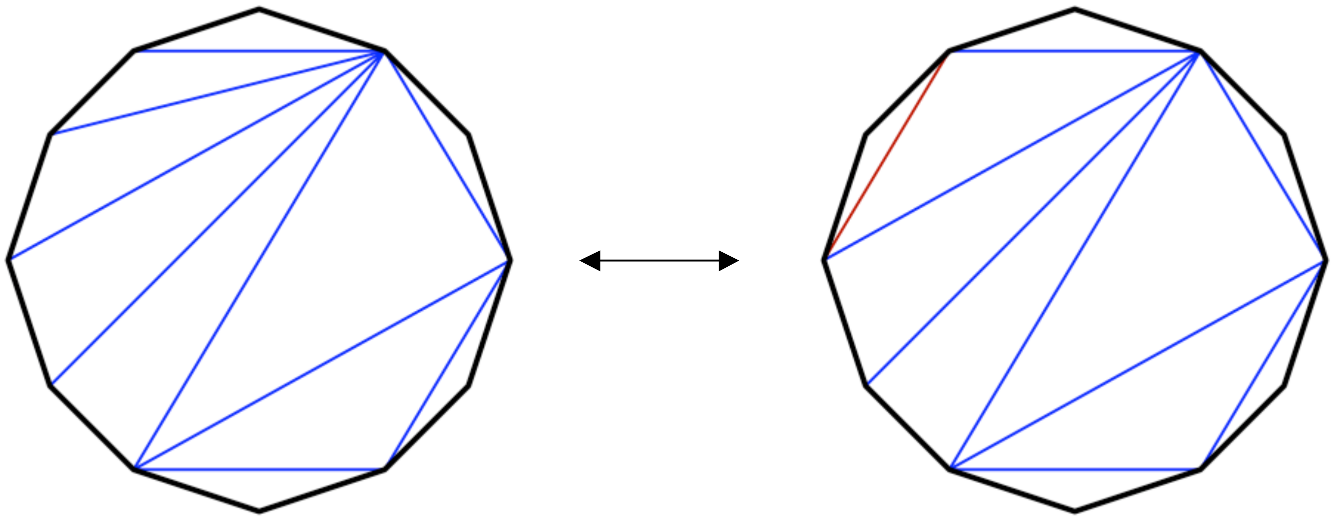
That is:

Triangulations are the **minimal** elements in the poset of subdivisions of a point set.

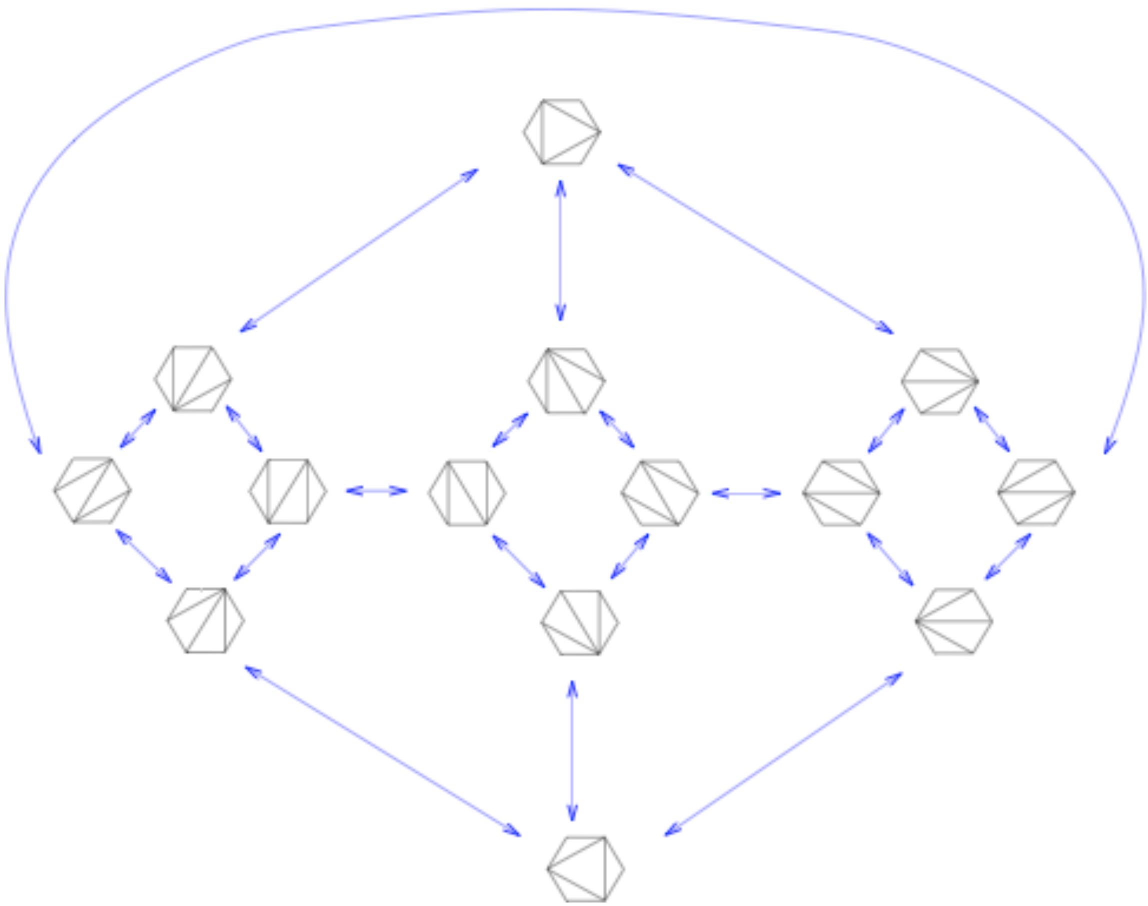
Geometric bistellar flips are the "**next-to-minimal**" elements.

The graph of triangulations of an *n*-gon

The only geometric bistellar flips in this case are “edge-flips”:



The graph of triangulations of an *n*-gon



The graph of flips berween triangulations of a hexagon

The graph of triangulations of an *n*-gon

Some obvious properties of the graph of flips in an n-gon:

- It has dihedral symmetry.

- It is *regular* of degree n-3 (every triangulation has *exactly* n-3 neighbors, because every edge can be flipped).

- It is a connected graph.

Some non-obvious properties (1):

Its *diameter* is at most $2n$-10 for every n (easy), and ***exactly*** $2n$-10 for n sufficiently large (not-so-easy, proof via hyperbolic geometry), [Sleator-Tarjan-Thurston, 1988].

Open question: how large needs n to be in the last sentence? (conjecture: $n \geq 13$).

[**Related conjecture:** for every $n \geq 13$ there is a 3-dimensional polytope with n vertices that cannot be triangulated with less than $2n$-10 tetrahedra]

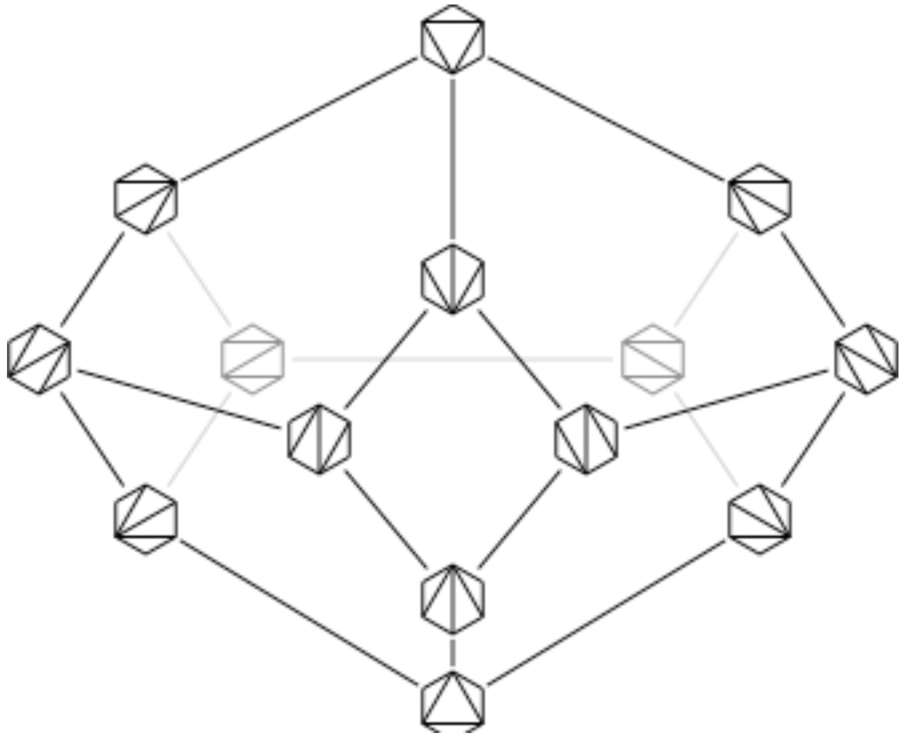
Some non-obvious properties (2):

It is the graph of a polytope of dimension n-3, called **the associahedron**

[Stasheff 1963,

Haiman 1984, Lee 1989,

Fomin-Zelevinskii 2003].

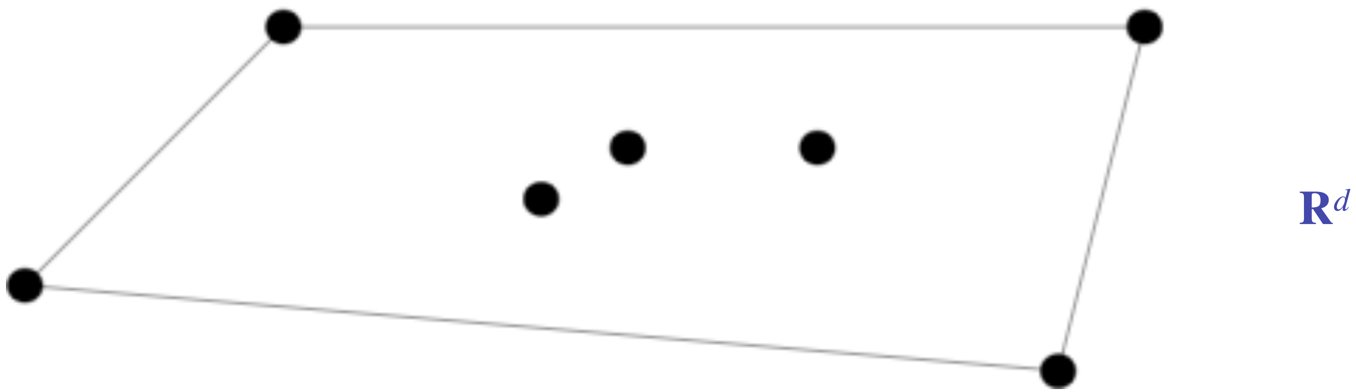


3

Regular triangulations, secondary polytopes

Regular triangulations

(A simple way to triangulate)



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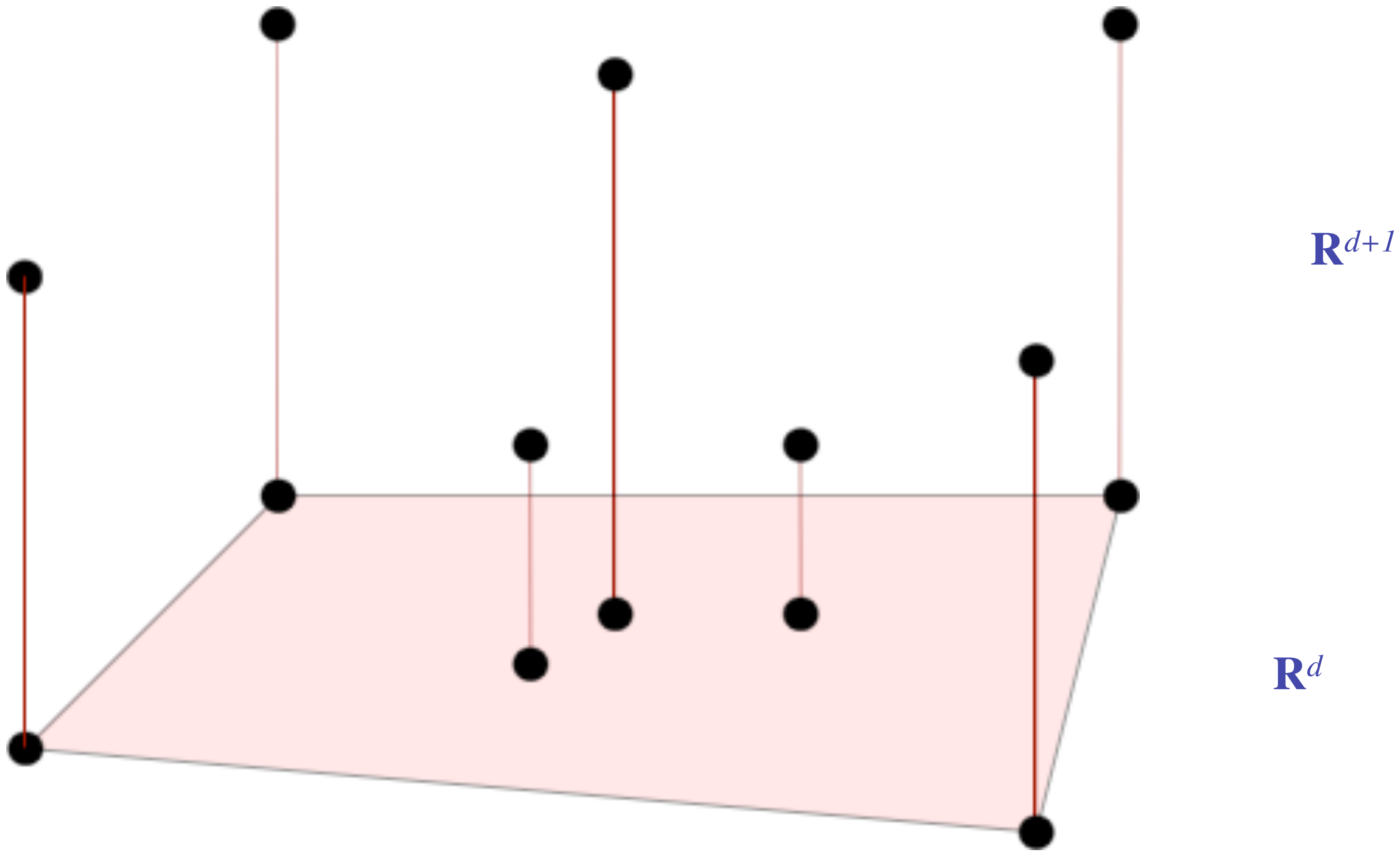

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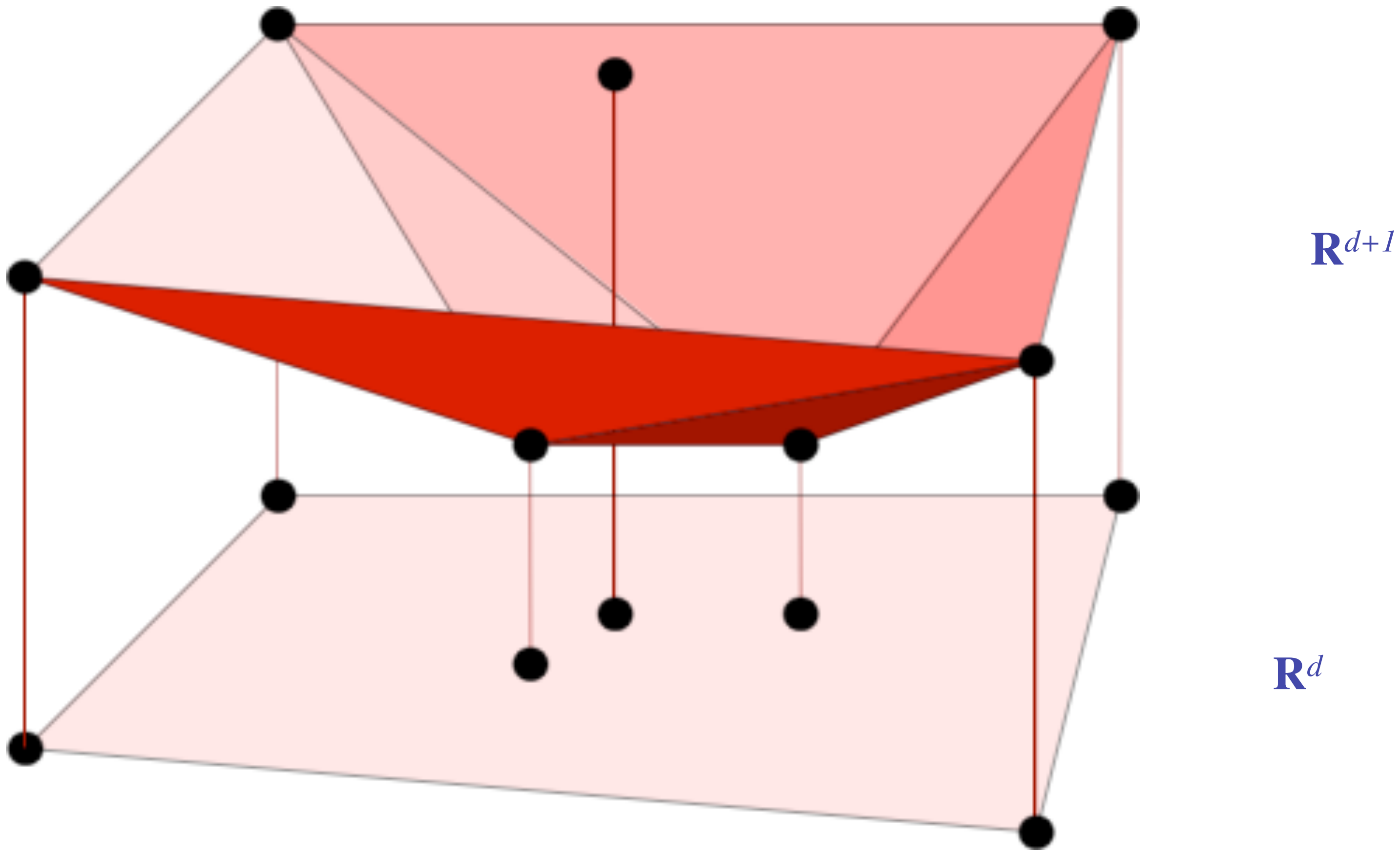

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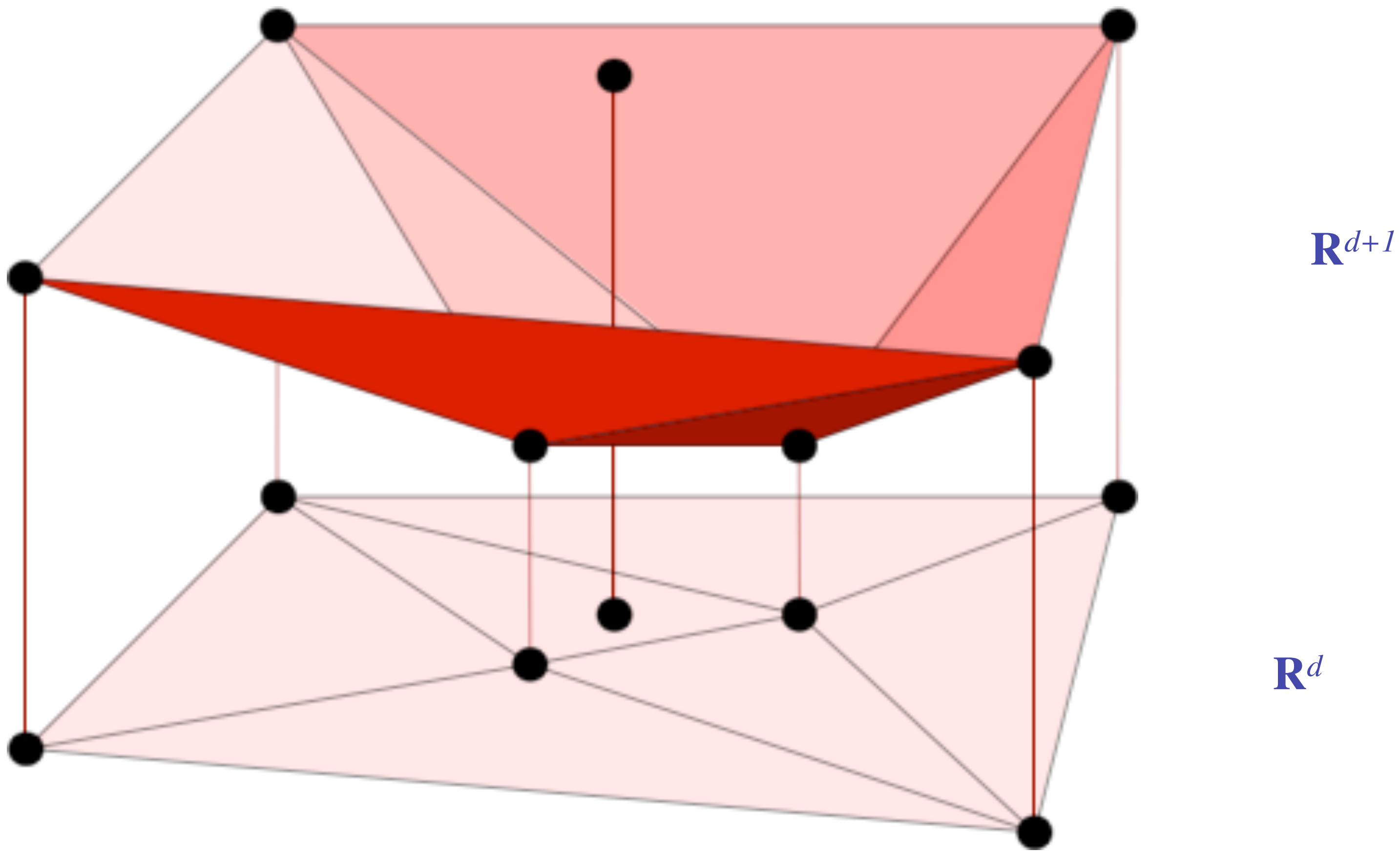
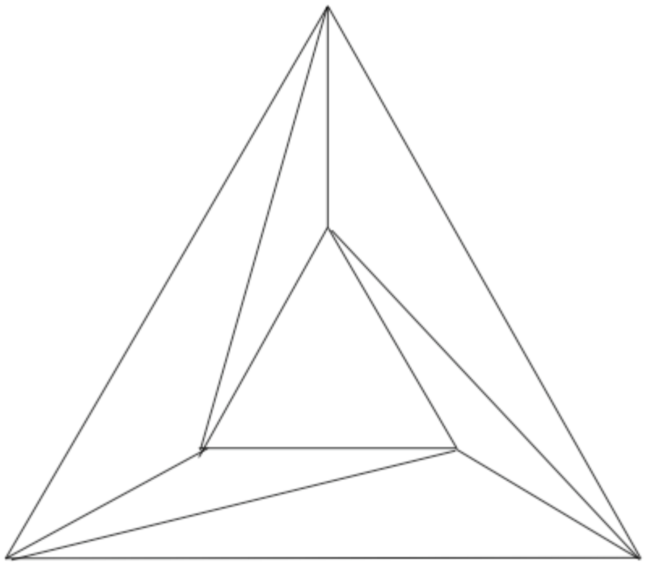


Non-regular triangulations

Obviously, different choices of *lifting* may produce different triangulations. For example, the choice $h(a)=\|a\|^2$ (lift to a paraboloid) produces the **Delaunay triangulation.**

More interestingly, *some* triangulations *cannot* be obtained in this way:



The smallest **non-regular** triangulation

The secondary polytope

Theorem [Gelfand-Kapranov-Zelevinskii, 1990] For every set A of n points in $\mathbf{R}^{d}$, there is a polytope $\Sigma(A)$ of dimension n-d-1 with the following correspondences:

regular triangulations of A	$\longleftrightarrow$	vertices of $\Sigma(A)$
flips between them	$\longleftrightarrow$	edges of $\Sigma(A)$
poset of reg. Subdivisions of A	$\longleftrightarrow$	poset of faces of $\Sigma(A)$

This is called the **secondary polytope** of A.

Corollary: The graph of geometric bistellar flips among ***regular*** triangulations is connected (in fact, $(n$-d-$1)$ - connected), and every triangulation has at least $(n$-d-$1)$ geometric bistellar neighbors.

Question: Does this hold for **non-regular** triangulations too?

Connectivity of the graph of flips. State of the art.

dim 2:

For every point set A in dimension 2:

•The graph of *all* triangulations is connected. In fact, you can always *monotonically* flip from any triangulation of A to one of several canonical triangulations (Delaunay triangulation, pushing, placing, pulling triangulations) [Lawson 1977]

•Every triangulation has at least the "expected" n-3 flips [de Loera-S.-Urrutia, 1998]

•The diameter of the graph is at most $4n$ (although you may need a quadratic number of flips for *monotone* flipping).

Connectivity of the graph of flips. State of the art.

dim 3 and 4:

•There are *highly flip-deficient* triangulations [S., 2000]:

dim 3: $O(\mathrm{sqrt}(n))$ flips,

dim 4: $O(1)$ flips

•*Monotone* flipping to the Delaunay triangulation (or to pulling/pushing triangulations) can fail (*e*ven in dimension three and for points in general and convex position [Joe 1989]).

•It is **open** whether the graph of flips is **connected** (*even in dimension three for point sets in general and convex position*!)

Connectivity of the graph of flips. State of the art.

dim 5 and higher:

There are point sets with disconnected graphs of triangulations. Three constructions so far:

•[S. 2000] There is a triangulation in dimension six with 324 vertices and with **no geometric bistellar flip** at all.

•[S. 2005] There are point sets in **dimension five** (with 26 points) and with a disconnected graph of triangulations. Moreover, they have integer coordinates and possess *unimodular* triangulations in several components of the graph.

•[S. 2006] There is a point set in dimension six (with 17 points) and **in general position** with a disconnected graph of triangulations.

4

Geometric bistellar flips

in context

a) Computational Geometry
(inc. CAGD, meshing, finite element methods,…)

Triangulating a region (of the plane, of 3-space, of a higher dimensional configuration space), is a natural thing to do in *terrain modelling*, *computer graphics*, *numerical analysis*, etc.

In many contexts, one has a fixed set of points (test points where we know our functions exactly) and want to interpolate a *good* triangulation on them. Special relevance in this context has the *Delaunay triangulation* (which is a regular triangulation).

Geometric bistellar flips are used to compute easily and efficiently the Delaunay triangulation, via the so-called *randomized incremental algorithm.*

a) Computational Geometry

- a history of flips -

Lawson [1972-1977-1986]: shows that in 2D, the graph of flips is connected (and you can actually flip *monotonically* from any triangulation to the **Delaunay triangulation**). He also defines flips in arbitrary dimension.

Joe [1989]: you cannot (always) flip *monotonically* to the Delaunay triangulation in 3D (there are “locally optimal” triangulations for the corresponding optimization criterion).

Clarkson-Shor [1989], Joe [1991]: still, you can use bistellar flips to compute the Delaunay triangulation in optimal time (in any dimension), if you use an *incremental* and *randomized* algorithm.

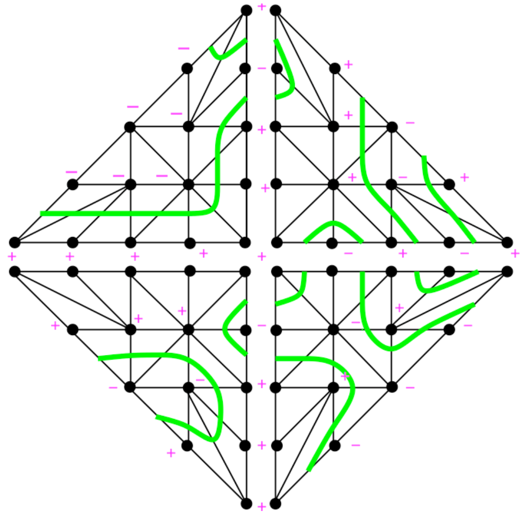
Shewchuk [2003]: adapts the “incremental randomized” algorithm to compute *constrained Delaunay triangulations* (and shows how to deal with point sets in non-general position as well).

b) Algebraic geometry

Toric varieties are complex algebraic varieties that can be defined via (the normal fans of) integer polytopes or via (the binomial ideals associated to affine dependences in) integer point configurations, among other ways.

There is a very precise “dictionary” between **combinatorial properties** of the polytope/point configuration and **algebro-geometric properties** of the variety.

b) Algebraic geometry



1) as a nice "test set" for conjectures and a source of examples and constructions, e.g.:

   Viro's patchworking Theorem (1984): one can construct real projective algebraic curves with prescribed (and complicated) topology starting with regular triangulations of an integer triangle. Used to advance in *Hilbert's XVI problem.*

2) as intermediate steps in the proof of more general theorems ("torification"), e.g.:

   Bernstein-Khovanskii-Koushnirenko Theorem (~1975): The number of solutions in $(\mathbf{C}^*)^n$ of a zero-dimensional polynomial system is at most (and, generically, equal to) the *mixed volume* of the Minkowski sum of the Newton polytopes of the system. (Incidentally, the mixed volume of a Minkowski sum can be computed using regular triangulations [Sturmfels, 1994]).

b) Algebraic geometry

Bistellar flips and factorization of birational morphisms

The following algebraic statement…

Theorem (*weak Oda conjecture; weak factorization for toric varieties*) [Morelli 1996, Wlodarczyk 1997]:
Every proper and equivariant birational map between two nonsingular **toric** varieties is a composite of blowings up and blowings down with smooth centers.

…is essentially equivalent to (and proved as)…

Theorem: Every two triagulations of the same point set can be joined by a sequence of geometric bistellar flips, *if one allows for the **insertion** of arbitrarily many additional points in the intermediate steps.*

Via torification it gives:

Theorem (*weak factorization in characteristic 0*) [Abramovich-Karu-Matsuki-Wlodarczyk 1999]: every birational map between complete nonsingular varieties over an algebraically closed field of characteristic 0 is a composite of blowings up and blowings down with smooth centers.

b) Algebraic geometry

Bistellar flips and toric Hilbert schemes

To every integer point set A we associate its *toric ideal* I_A and define the **toric Hilbert scheme** of A as the "set" (with an appropriate scheme structure) of all binomial ideals with the same multigraded Hilbert function as I_A [Sturmfels 1994, Peeva-Stillman 2002]. It was open whether toric Hilbert schemes are always connected (as usual Hilbert schemes)

Theorem [Maclagan-Thomas 2002]: if A has **unimodular** triangulations not connected to one another by geometric bistellar flips, then the toric Hilbert scheme of A is not connected.

Theorem [S. 2005]: there are integer point sets in $\mathbf{R}^5$ which possess unimodular triangulations in different components of the flip-graph. **In particular, there are non-connected toric Hilbert schemes.**

Other similar schemes are shown to be non-connected with similar techniques: Alexeev's "moduli space of stable toric pairs"; Kapranov-Sturmfels-Zelevinskii's "inverse limit of toric GIT quotients of $\mathbf{CP}^n$"

c) Combinatorial topology

(Topological) bistellar flips between triangulations of a compact manifold have been proposed as a tool for: *(a)* algorithmic *manifold recognition* or *manifold simplification* [Lickorish 1999, Björner-Lutz, 2000] and (*b)* for studying the "space of (combinatorial) triangulations of a manifold" [Nabutovskii 1996, Ambjorn et al. 1997], (with applications to "quantum gravity") Two results in this context are:

Theorem [Pachner, 1991]: any two *PL-homeomorphic* triangulations of a manifold can be related to one another via bistellar flips, *if arbitrarily many vertices are allowed to be added in ntermediate steps*.

Theorem [Dougherty-Faber-Murphy, 2004]: the same is false, even for the 3-sphere, if no additional vertices are allowed. There is a triangulation of the 3-sphere with 15 vertices and no topologiacl bistellar flips, other than "insertion flips".

d) Topological combinatorics

The order complex of a poset: A standard technique in topological combinatorics is to associate to any poset a simplicial complex, whose vertices are the poset elements and whose simplices are the *chains* (totally ordered subsets) in the poset.

Example: if the original poset is the face poset of a cell complex (e.g., faces of a polytope) the order complex equals the *barycentric subdivision* of it.

Order complexes are a natural way to turn *posets* into *topological objects*. In particular, great attention is devoted to the *homotopy type* of a poset.

d) Topological combinatorics

Based on “small examples” and the existence of secondary polytopes, there was the following:

Conjecture [Billera-Kapranov-Sturmfels, 1994]: the poset of *all* polyhedral subdivisions of a point set A of dimension d and with n elements is homotopy equivalent to the $(n\text{-}d\text{-}2)$-dimensional topological sphere.

This conjecture is a special case (the “**simplicial case**”) of a more general conjecture, the *generalized Baues conjecture*, inspired by work of Baues [1980] and disproved in ts full generality by Rambau-Ziegler [1996]. Other cases of the conjecture include:

•The poset of “cellular strings” on a polytope (**proved** [Billera-Kapranov-Sturmfels, 1994]).

•The case of *zonotopal tilings on a zonotope*, equivalent to the *extension space conjecture* in oriented matroid theory (**open**).

d) Topological combinatorics

The existence of disconnected graphs of flips [S. 2000] was "strong evidence" against the simplicial case of the GBC. But it did not directly disproved it. However, the new example in general position [S. 2006] does:

Lemma: if a point set *A* **in general position** has a disconnected graph of triangulations, then the poset of non-trivial subdivisions of either *A* or some subset of it is not connected either.

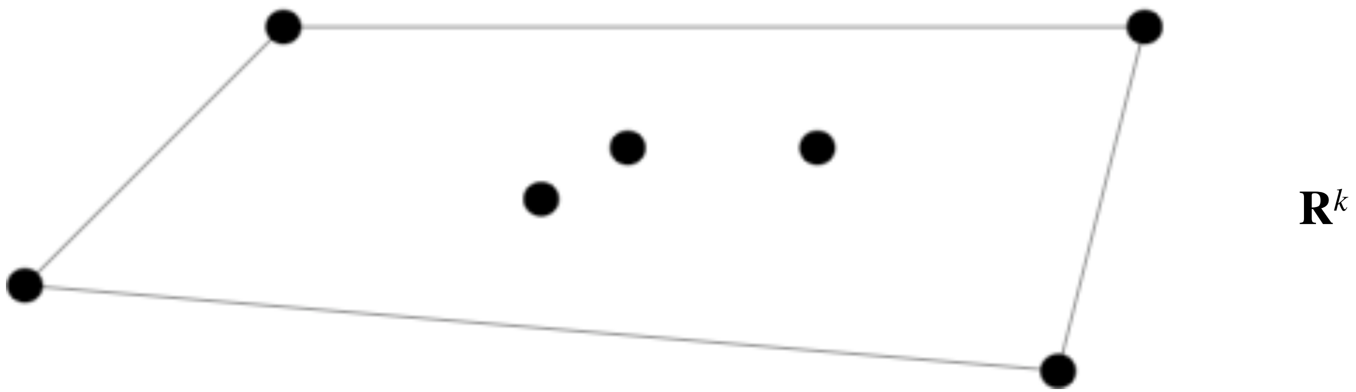
Corollary [S. 2006]: There is a point set **in general position** of dimension 6 and with at most 17 elements whose poset of non-trivial subdivisions is not connected. In particular, this disproves the simplicial case of the GBC.

-- epilogue --

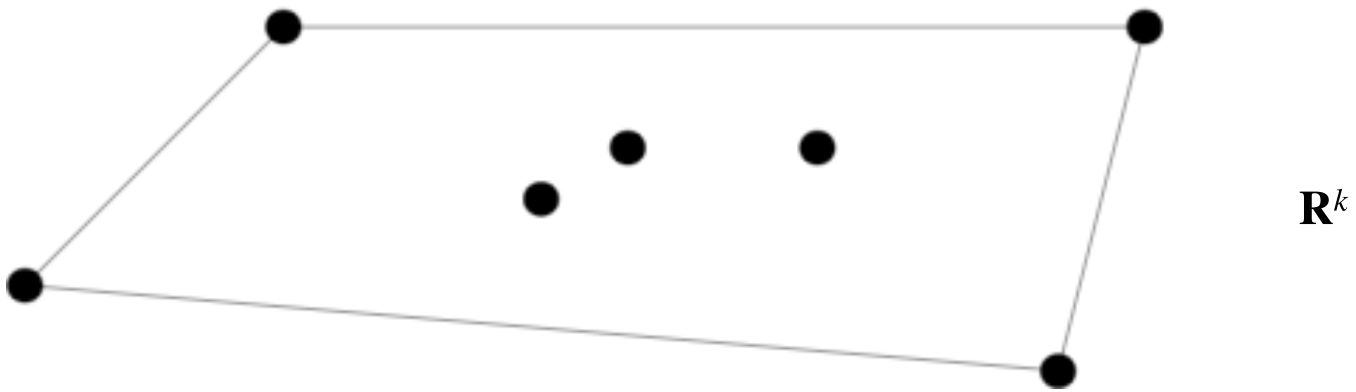
Regular subdivisions

(reprise)

A



$A \quad n, \ k$



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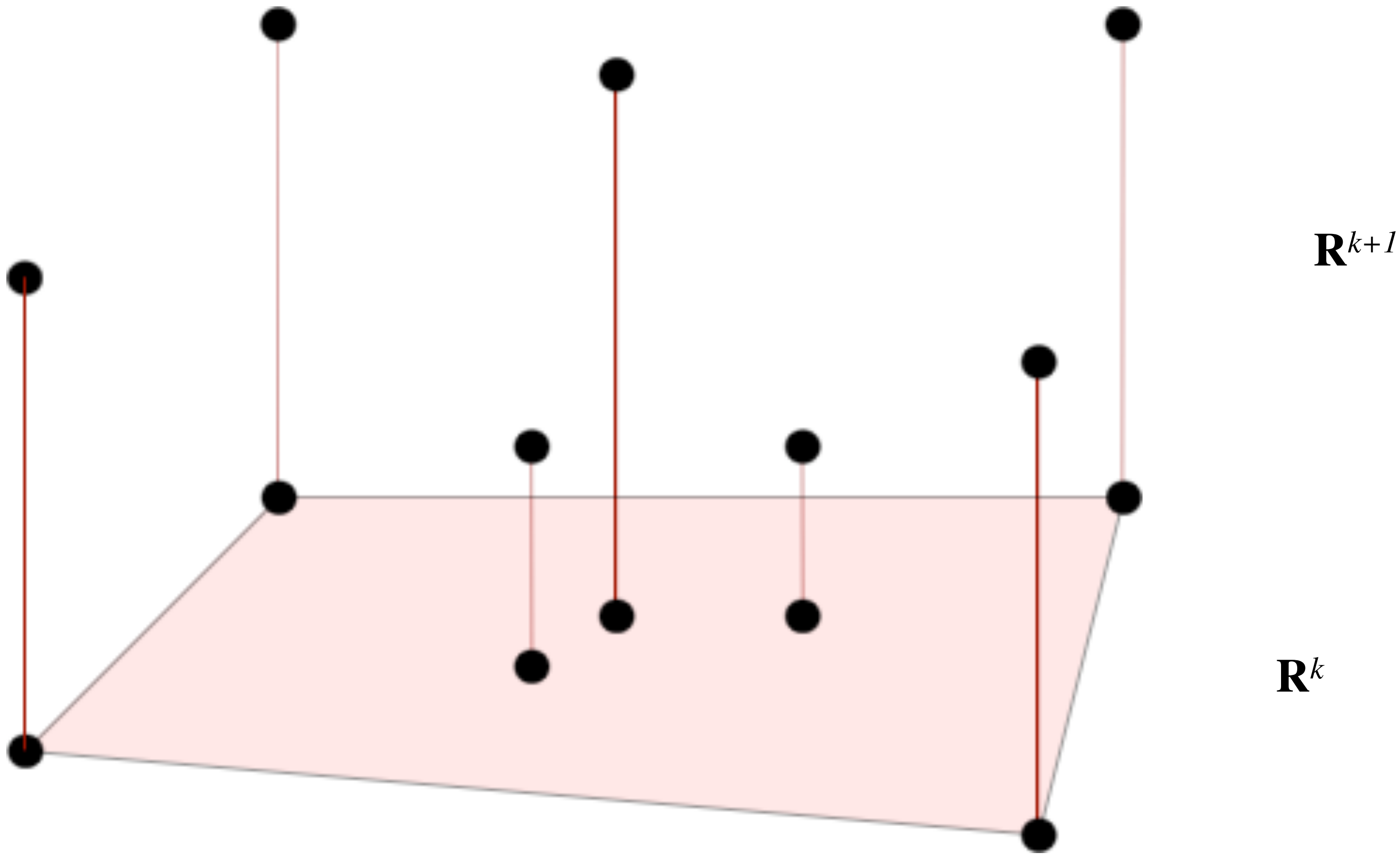

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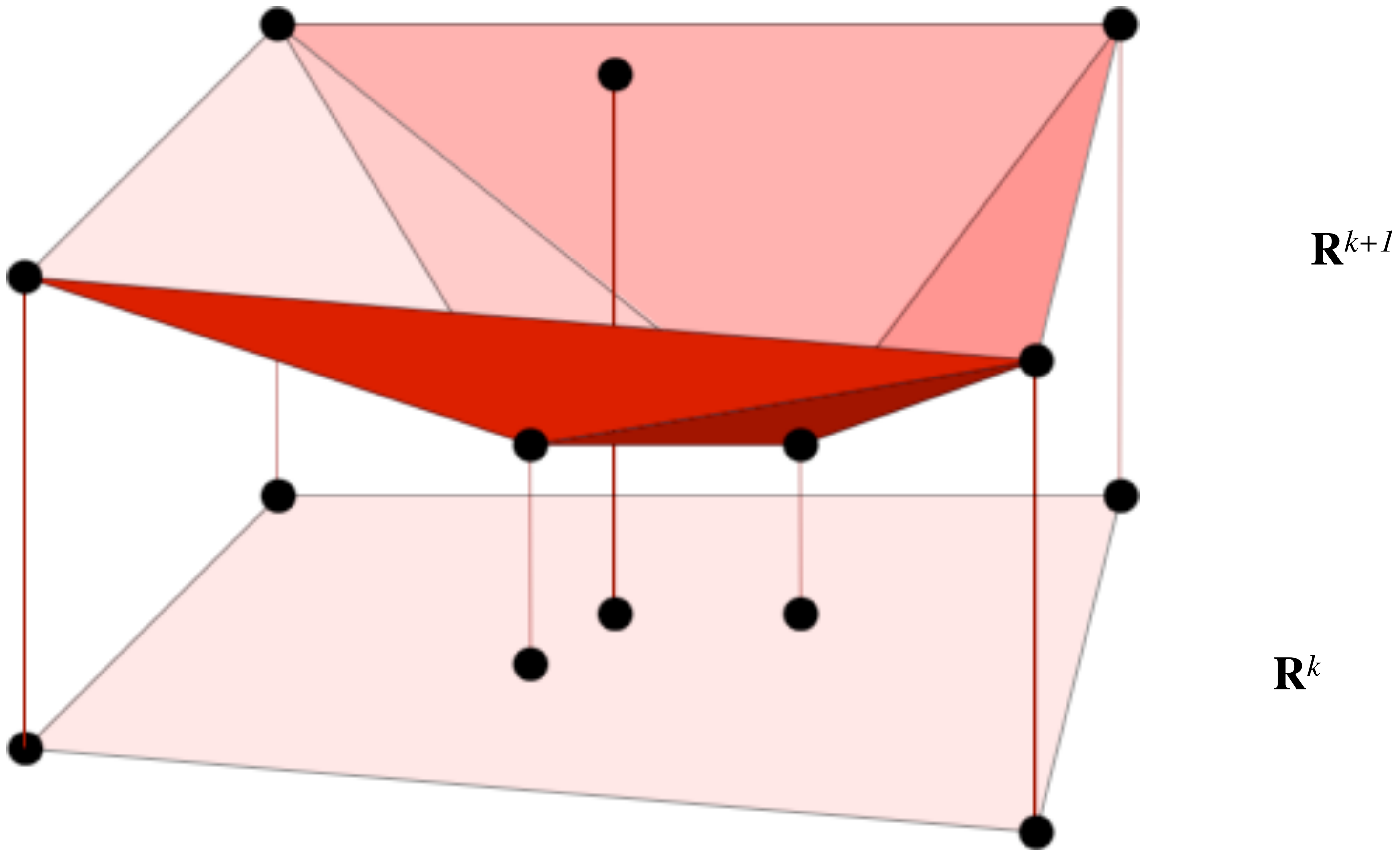

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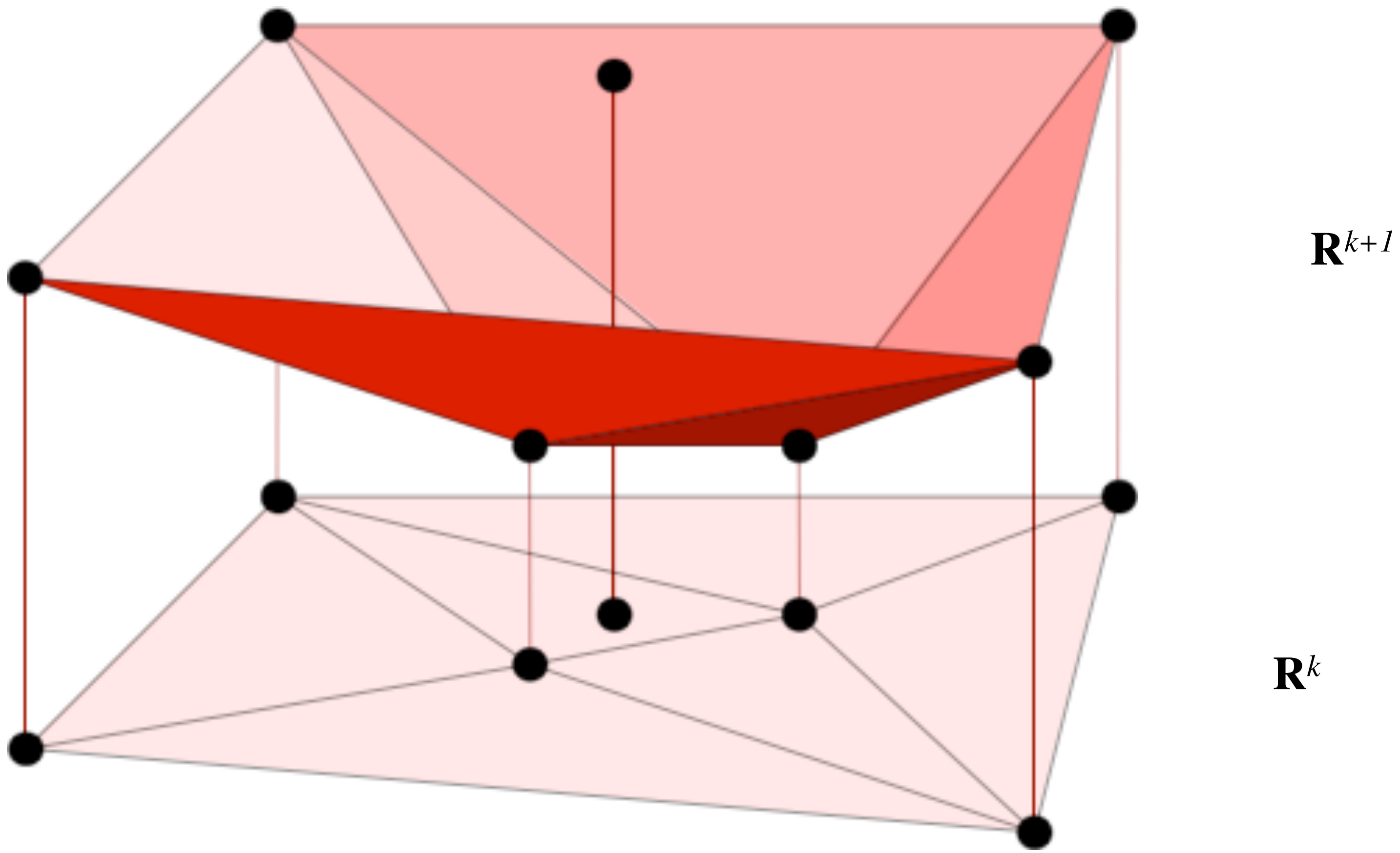
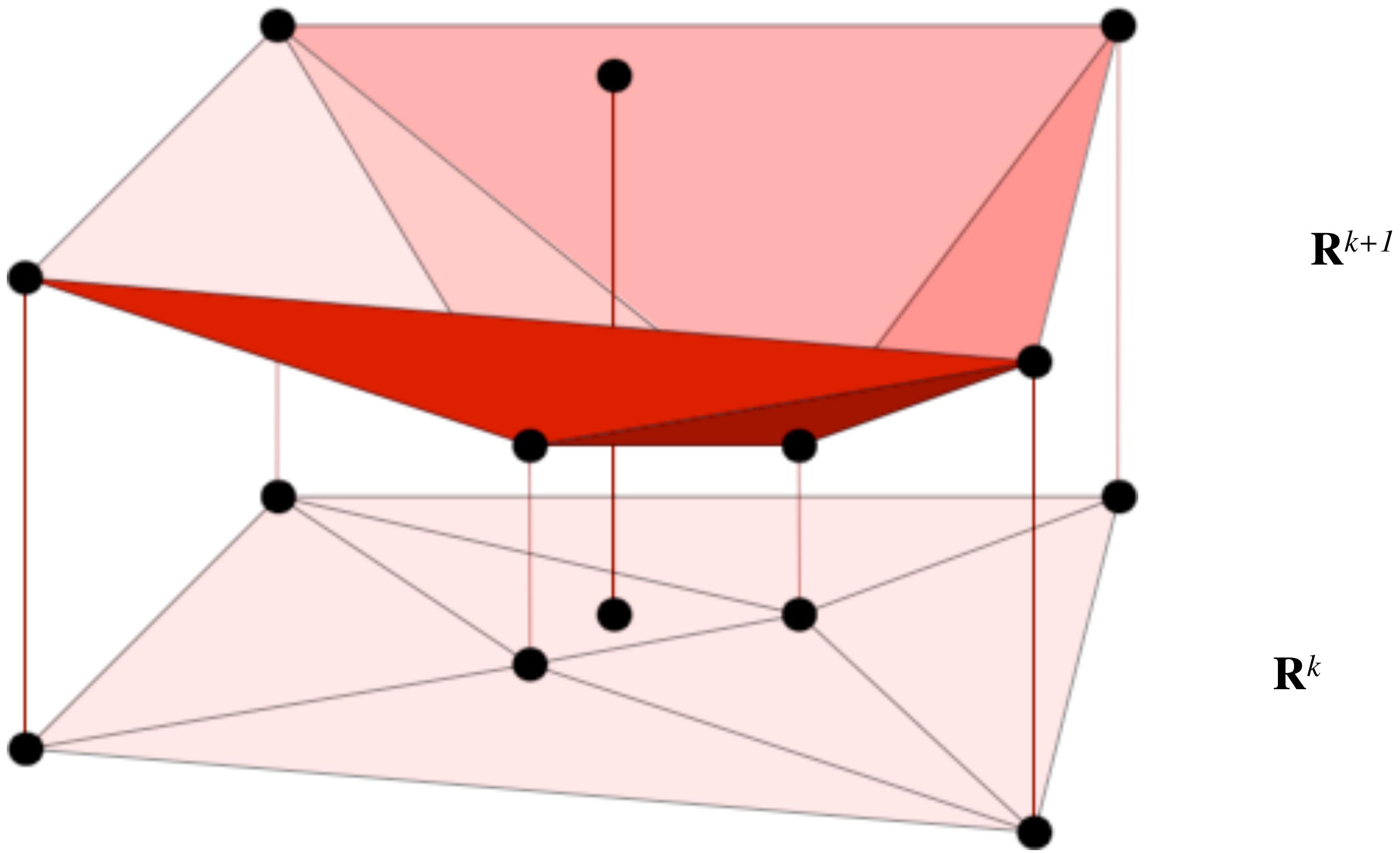


T h A n, k S



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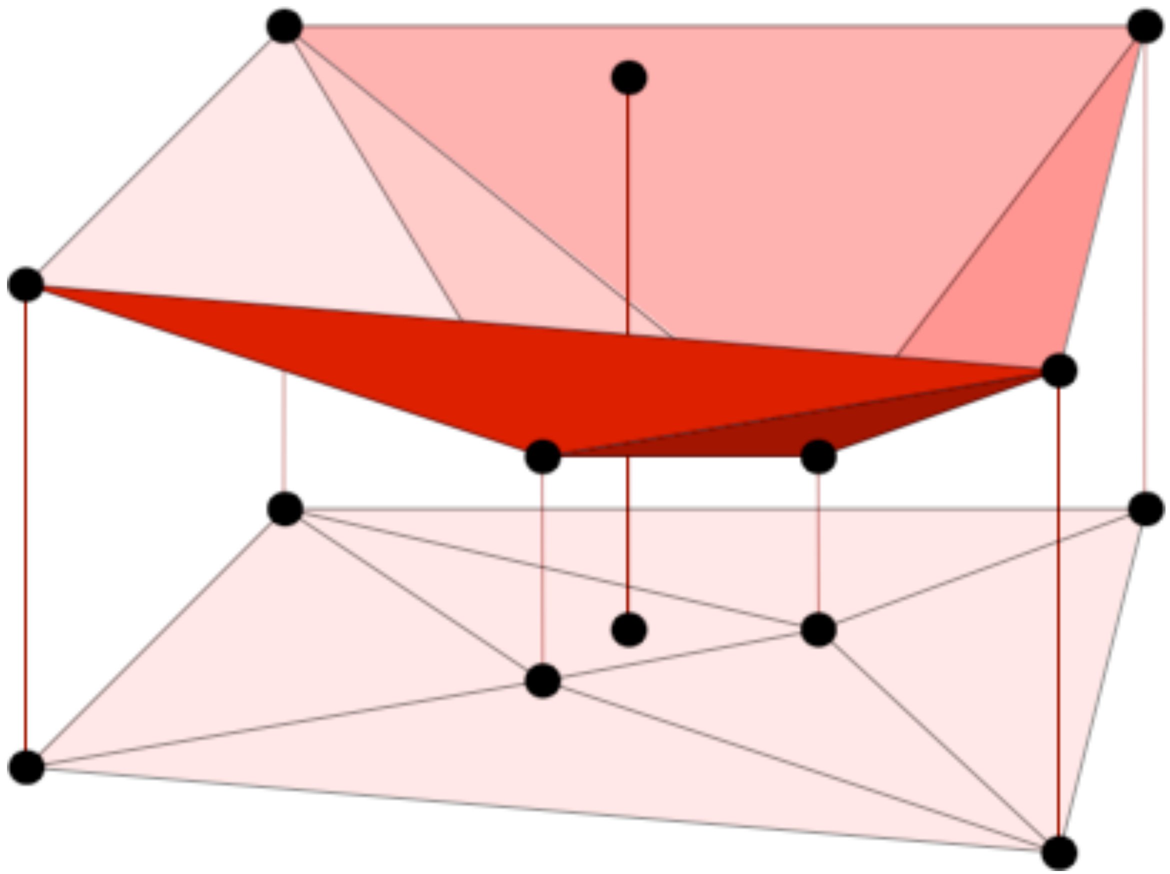
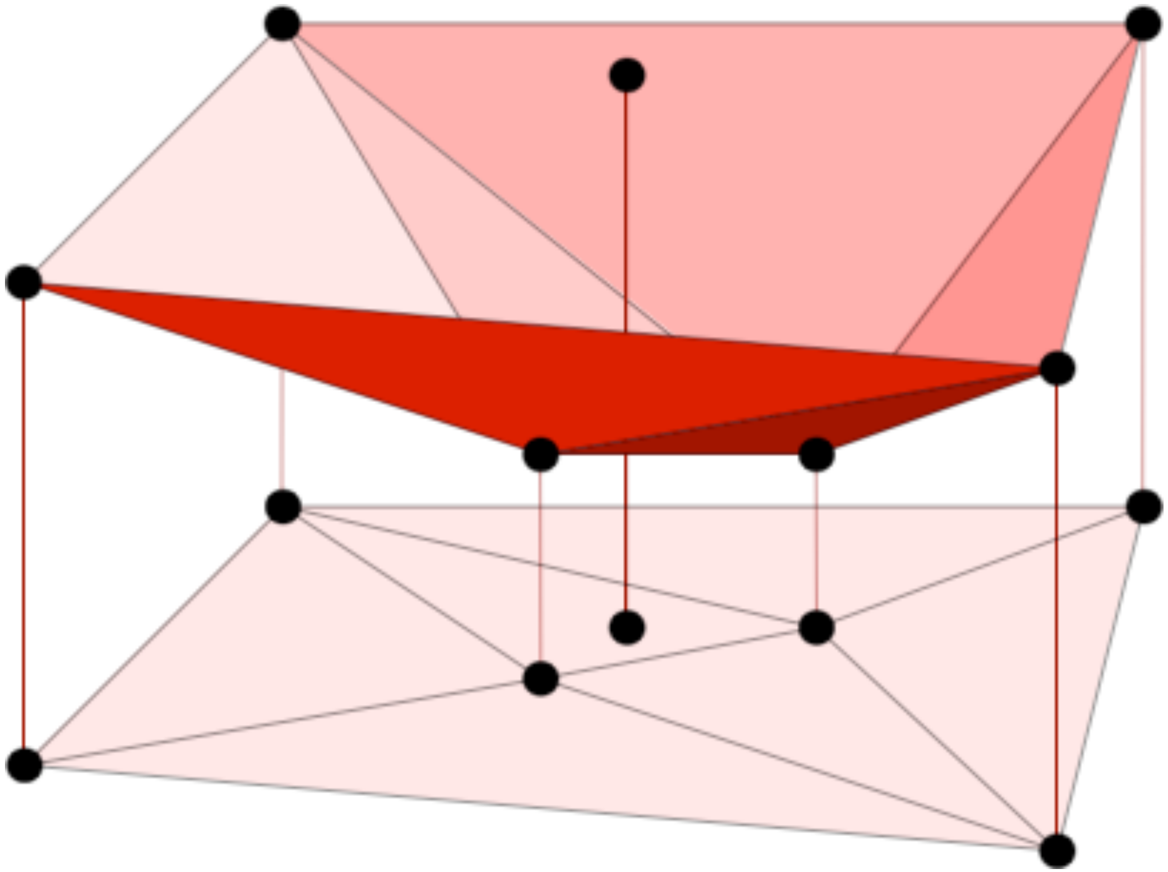


T h A n, k S



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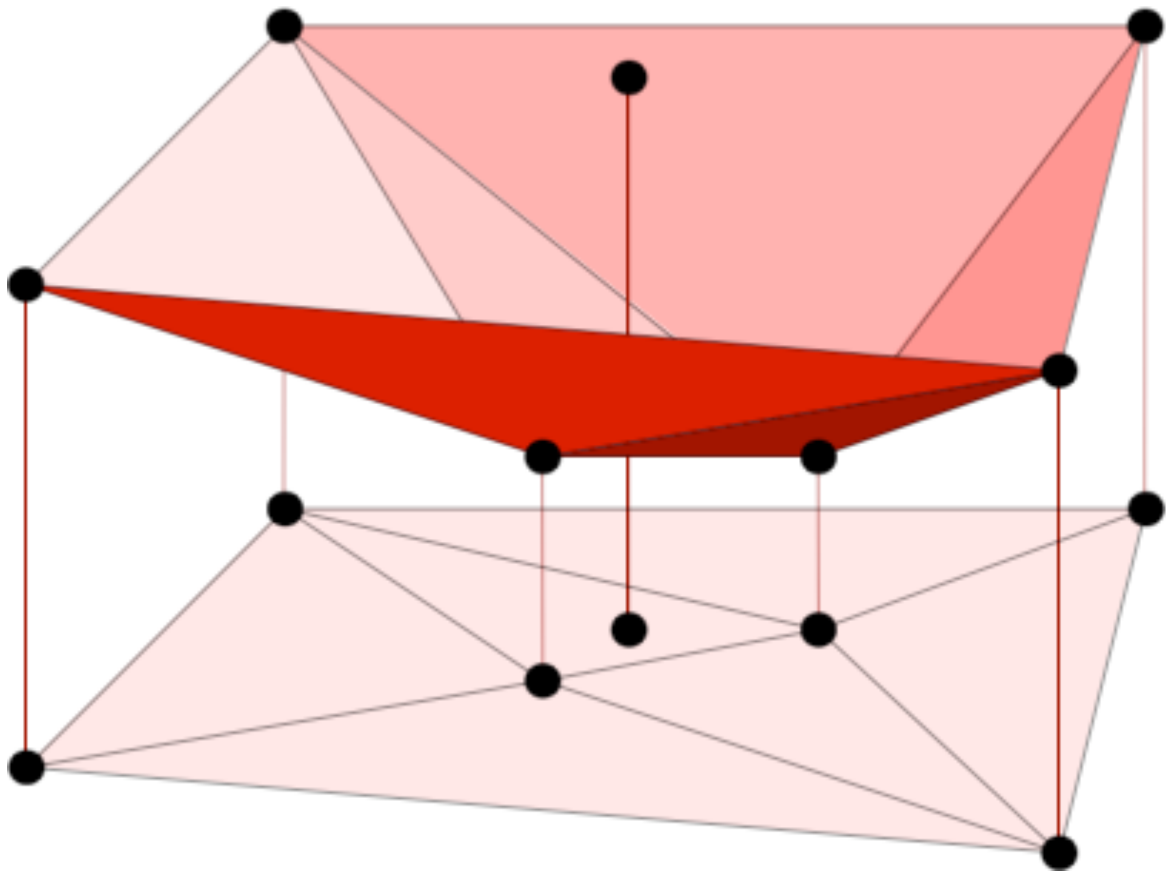

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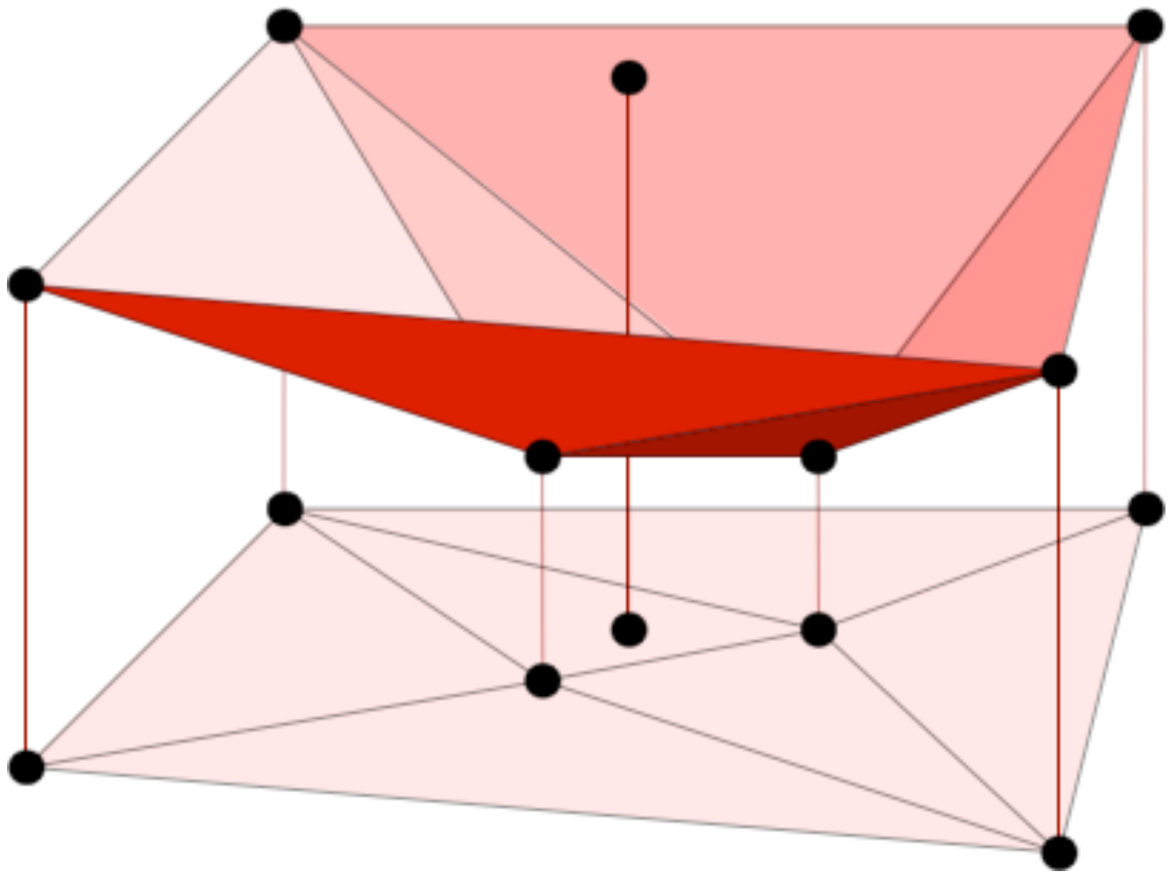

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