Sheet 5 — Cyclic polytopes

1. Prove that $\mathcal{C}(n,d)$ is in general position. (Hint: a set of $d+1$ points forms a Vandermonde matrix. Look for its determinant).

2. Count the number of facets of $C(n,d)$; the number of upper facets and that of lower facets. (Hint: distinguish between d even and odd).

3. Which pair of simplices satisfies (IP) in $C(9,4)$?

1234	5678
1256	2378
2468	1357
1346	1459

4. Which set of labels forms a lower/upper facet of $C(9,4)$?

1234
1346
1469
1256
4569

5. Prove that every circuit of $C(n,d)$ is alternating:

$$C_+ = \{i_1, i_3, i_5, \ldots\}$$
$$C_- = \{i_2, i_4, i_6, \ldots\},$$
$$\text{for } 1 \leq i_1 < i_2 < i_3 < \ldots \leq n.$$

6. Show that flipping up in dimension d is cycle-free.

7. **(Open Problem)** For triangulations T, T' of $C(n,d)$ define:

$$T \leq_2 T' \;\; :\Longleftrightarrow \;\; s_T \text{ is weakly lower than } s_{T'}$$
$$\text{(i.e., } s_T(x)_{d+1} \leq s_{T'}(x)_{d+1}, \forall x \in C(n,d)).$$

Prove or disprove:

$$\mathcal{S}_2(n,d) := (\{\text{Triangulations of } C(n,d)\}, \leq_2) = \mathcal{S}_1(n,d).$$