Sheet 1 — Triangulations of point sets

1. Describe the triangulations (and flips between them) of the following point configurations:

   (a) A 0-dimensional point configuration (n points, all equal to one another!).

   (b) A 1-dimensional point configuration in general position (n distinct points along a line)

   (c) n distinct points in dimension 2, $n-1$ of which lie in the same line.

   (d) (advanced) The product of a d-simplex and a segment. Hint: it has exactly $(d+1)!$-triangulations. How can you make them correspond to the $(d+1)!$ permutations of some set of $d+1$ "something"?

   (e) (advanced) The d-dimensional cross-polytope. That is, the set $\{e_1, \dots, e_n\} \cup \{-e_1, \dots, -e_n\}$ where $e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n$. Hint: there are n triangulations.

2. Construct all the four types ("modulo oriented matroid") of point configurations of dimension 2 with 4 elements, and for each one check that indeed it has two triangulations.

3. Construct all the three types ("modulo oriented matroid") of point configurations of dimension 2 with 5 elements in general position, and for each one check that it has five triangulations.

4. Find a Hamiltonian cycle in the graph of flips of a hexagon.

5. Prove that a regular tetrahedron cannot be triangulated into (more than one) regular sub-tetrahedra, even if you are allowed to add as many vertices to the triangulation as you wish. Hint: look at the dihedral angle of the regular tetrahedron.

6. Let K be a simplicial complex and let F_1 and F_2 be two of its faces. Suppose that $F_1 \cap F_2 = \emptyset$ and that $F_1 \cup F_2$ is a face of K. Prove that $\mathrm{lk}(F_2, \mathrm{lk}(F_1, K)) = \mathrm{lk}(F_1 \cup F_2, K)$.