A counter-example to the Hirsch conjecture

Francisco Santos

Universidad de Cantabria, Spain
http://personales.unican.es/santosf/Hirsch

The mathematics of Klee & Grünbaum — Seattle, July 30, 2010
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Or “Two theorems by Victor Klee and David Walkup”

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Two quotes by Victor Klee:

- A good talk contains no proofs; a great talk contains no theorems.
- Mathematical proofs should only be communicated in private and to consenting adults.
WARNING
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This talk contains material that may not be suited for all audiences.
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This includes, but may not be limited to, mathematical theorems and proofs, pictures of highly dimensional polytopes, and explicit coordinates for them.
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Vertices and edges of a polytope $P$ form a graph (finite, undirected)

The distance $d(a, b)$ between vertices $a$ and $b$ is the length (number of edges) of the shortest path from $a$ to $b$.

For example, $d(a, b) = 2$. 
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Vertices and edges of a polytope \( P \) form a graph (finite, undirected)

The diameter of \( G(P) \) (or of \( P \)) is the maximum distance among its vertices:

\[
\delta(P) = \max\{d(a, b) : a, b \in \text{vert}(P)\}.
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The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)
For every polytope $P$ with $n$ facets and dimension $d$,

$$\delta(P) \leq n - d.$$ 

Theorem (S. 2010+)
There is a 43-dim. polytope with 86 facets and diameter 44.

Corollary
There is an infinite family of non-Hirsch polytopes with diameter

$$\sim (1 + \epsilon)n,$$ even in fixed dimension. (Best so far: $\epsilon = 1/43$).
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Motivation: linear programming

A linear program is the problem of maximization / minimization of a linear functional subject to linear inequality constraints.

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron $P$ with (at most) $n$ facets.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program starting at any feasible vertex and moving along the graph of $P$, in a monotone fashion, until the optimum is attained.
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There are more recent algorithms for linear programming which are proved to be polynomial: (ellipsoid [1979], interior point [1984]). But the simplex method is still one of the most often used, for its simplicity and practical efficiency:
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Finding strongly polynomial algorithms for linear programming is one of the “mathematical problems for the 21st century” according to [Smale 2000]. A polynomial pivot rule would solve this problem in the affirmative.

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- $n - d \leq 6$: [Klee-Walkup, 1967] [Bremner-Schewe, 2008]
- $H(9, 4) = H(10, 4) = 5$ [Klee-Walkup, 1967]
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A “quasi-polynomial” bound

**Theorem** (Kalai-Kleitman 1992): For every $d$-polytope with $n$ facets

$$\delta(P) \leq n^{\log_2 d + 2}.$$  

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Given a linear program with $d$ variables and $n$ restrictions, we consider a random perturbation of the matrix, within a parameter $\epsilon$ (normal distribution).

Theorem [Spielman-Teng 2004] [Vershynin 2006]

The expected diameter of the perturbed polyhedron is polynomial in $d$ and $\epsilon^{-1}$, and polylogarithmic in $n$. 
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**Theorem 1**: The $d$-step Theorem

Klee and Walkup, 1967
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It is possible to go from $a$ to $b$ so that at each step we enter a new facet, one that we had not visited before.

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The d-step Theorem

**Theorem 1 (Klee-Walkup 1967)**

Hirsch $\leftrightarrow$ d-step $\leftrightarrow$ non-revisiting path.

**Proof:** Let $H(n, d) = \max \{ \delta(P) : P \text{ is a } d\text{-polytope with } n \text{ facets} \}$. The key step in the proof is to show that for any $k$:

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\cdots \leq H(2k - 1, k - 1) \leq H(2k, k) = H(2k + 1, k + 1) = \cdots
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That is to say:

1) $H(n, d) \leq H(n + 1, d + 1)$, for all $n$ and $d$.

2) $H(n - 1, d - 1) \geq H(n, d)$, when $n < 2d$. 
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2) $H(n - 1, d - 1) \geq H(n, d)$, when $n < 2d$:

Since $n < 2d$, every pair of vertices $a$ and $b$ lie in a common facet $F$, which is a polytope with one less dimension and (at least) one less facet. Hence, $d_P(a, b) \leq d_F(a, b) \leq H(n - 1, d - 1)$. 
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\forall a, b \in \text{vert}(P), \quad \exists a', b' \in \text{vert}(P'), \quad d_{P'}(a', b') \geq d_P(a, b).
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Choose an arbitrary facet $F$ of $P$. Let $P'$ be the wedge of $P$ over $F$. Then:

\[ \forall a, b \in \text{vert}(P), \quad \exists a', b' \in \text{vert}(P'), \quad d_{P'}(a', b') \geq d_{P}(a, b). \]
The $d$-step Theorem

**Theorem 1** (Klee-Walkup 1967)

Hirsch $\iff$ $d$-step $\iff$ non-revisiting path.

**Proof:** Let $H(n, d) = \max\{\delta(P) : P$ is a $d$-polytope with $n$ facets\}. The key step in the proof is to show that for any $k$:

$$\cdots \leq H(2k - 1, k - 1) \leq H(2k, k) = H(2k + 1, k + 1) = \cdots$$

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\[ d(a, b) = 2 \]

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A *spindle* is a polytope $P$ with two distinguished vertices $u$ and $v$ such that every facet contains either $u$ or $v$ (but not both).

**Definition**

The *length* of a spindle is the graph distance from $u$ to $v$. 
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Theorem (Generalized $d$-step, spindle version)

Let $P$ be a spindle of dimension $d$, with $n > 2d$ facets and length $\delta$.

Then there is another spindle $P'$ of dimension $d + 1$, with $n + 1$ facets and length $\delta + 1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle $P$ has length $> d$ then there is another spindle $P'$ (of dimension $n - d$, with $2n - 2d$ facets, and length $\geq \delta + n - 2d > n - d$) that violates the Hirsch conjecture.
Spindles

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**Prismatoids**

**Definition**

A *prismatoid* is a polytope $Q$ with two (parallel) facets $Q^+$ and $Q^-$ containing all vertices.

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Prismatoids

Theorem (Generalized \(d\)-step, prismatoid version)

Let \(Q\) be a prismatoid of dimension \(d\), with \(n > 2d\) vertices and width \(\delta\). Then there is another prismatoid \(Q'\) of dimension \(d + 1\), with \(n + 1\) vertices and width \(\delta + 1\).

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That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$. 
The generalized $d$-step Theroem

Proof.

$$Q \subset \mathbb{R}^2$$

$$Q^+$$

$$Q^-$$

$$v$$

$$w$$

$$\tilde{Q} \subset \mathbb{R}^3$$

$$\tilde{Q}^-$$

$$\tilde{Q}^+$$

$$Q^- := \text{o.p.s.}_{v}(Q^-)$$

$$\text{o.p.s.}_{v}(Q) \subset \mathbb{R}^3$$

$$w$$

$$u$$
So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension $d$ and width larger than $d$. *Its number of vertices and facets is irrelevant!!!*

**Question**

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S., July 2010].
- 5-prismatoids of width 6 exist [S., May 2010].
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Theorem 2: A non-Hirsch 4-polyhedron
Klee and Walkup, 1967
Combinatorics of prismatoids

Analyzing the combinatorics of a $d$-prismatoid $Q$ can be done via an intermediate slice . . .
... which equals the Minkowski sum $Q^+ + Q^-$ of the two bases $Q^+$ and $Q^-$. 

\[
\frac{1}{2} \quad \frac{1}{2} = \quad \frac{1}{2}
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Combinatorics of prismatoids

...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases $Q^+$ and $Q^-$. The normal fan of $Q^+ + Q^-$ equals the “superposition” of those of $Q^+$ and $Q^-$. 

\[
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\[ \quad \]

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So: the combinatorics of $Q$ follows from the superposition of the normal fans of $Q^+$ and $Q^-$. 

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Conclusion

4-prismatoids $\iff$ pairs of maps in the 2-sphere.
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Example: (part of) a 4-prismatoid

4-prismatoid of width $> 4$

pair of (geodesic, polytopal) maps in $S^2$ so that two steps do not let you go from a blue vertex to a red vertex.
Example: (part of) a 4-prismatoid

4-prismatoid of width $\geq 4$

$\iff$

pair of (geodesic, polytopal) maps in $S^2$ so that two steps do not let you go from a blue vertex to a red vertex.
Klee and Walkup, in 1967, disproved the Hirsch conjecture:

**Theorem 2 (Klee-Walkup 1967)**
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The Klee-Walkup polytope is an “unbounded 4-spindle”. What is the corresponding “superposition of two (geodesic, polytopal) maps” in a surface?
The Klee-Walkup (unbounded) 4-spindle
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A 4-dimensional prismatoid of width $> 4$?

Replicating the basic structure of the Klee-Walkup polytope we can get a “non-Hirsch” pair of maps in the plane:
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Surprisingly enough:

**Theorem (S., July 2010)**

There is no “non-Hirsch” pair of maps in the 2-sphere.

**Proof (rough idea of).**

Every pair of non-Hirsch maps on a surface necessarily contains certain “zig-zag alternating cycles”, and no such cycle can bound a 2-ball.
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A 5-prismatoid of width $> 5$

But, in dimension 5 (that is, with maps in the 3-sphere) we have room enough to construct “non-Hirsch pairs of maps”:

**Theorem**

The prismatoid $Q$ of the next two slides, of dimension 5 and with 48 vertices, has width six.
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**Theorem**

*The prismatoid Q of the next two slides, of dimension 5 and with 48 vertices, has width six.*

**Corollary**

*There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.*
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*The prismatoid $Q$ of the next two slides, of dimension 5 and with 48 vertices, has width six.*

**Proof 1.**

It has been verified with *polymake* that the dual graph of $Q$ (modulo symmetry) has the following structure:

```
A --- B
  |   |
  D   G
  |   |
  E   H
  |   |
  C   F
```

$\square$
A 5-prismatoid of width $> 5$

$$Q := \text{conv} \begin{pmatrix} 1^+ & x_1 & x_2 & x_3 & x_4 & x_5 \\ 2^+ & -18 & 0 & 0 & 0 & 1 \\ 3^+ & 0 & 18 & 0 & 0 & 1 \\ 4^+ & 0 & -18 & 0 & 0 & 1 \\ 5^+ & 0 & 0 & 45 & 0 & 1 \\ 6^+ & 0 & 0 & -45 & 0 & 1 \\ 7^+ & 0 & 0 & 0 & 45 & 1 \\ 8^+ & 0 & 0 & 0 & -45 & 1 \\ 9^+ & 15 & 15 & 0 & 0 & 1 \\ 10^+ & -15 & 15 & 0 & 0 & 1 \\ 11^+ & 15 & -15 & 0 & 0 & 1 \\ 12^+ & -15 & -15 & 0 & 0 & 1 \\ 13^+ & 0 & 0 & 30 & 30 & 1 \\ 14^+ & 0 & 0 & -30 & 30 & 1 \\ 15^+ & 0 & 0 & 30 & -30 & 1 \\ 16^+ & 0 & 0 & -30 & -30 & 1 \\ 17^+ & 0 & 10 & 40 & 0 & 1 \\ 18^+ & 0 & -10 & 40 & 0 & 1 \\ 19^+ & 0 & 10 & -40 & 0 & 1 \\ 20^+ & 0 & -10 & -40 & 0 & 1 \\ 21^+ & 10 & 0 & 0 & 40 & 1 \\ 22^+ & -10 & 0 & 0 & 40 & 1 \\ 23^+ & 10 & 0 & 0 & -40 & 1 \\ 24^+ & -10 & 0 & 0 & -40 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1^- & 0 & 0 & 0 & 18 & -1 \\ 2^- & 0 & 0 & 0 & -18 & -1 \\ 3^- & 0 & 0 & 18 & 0 & -1 \\ 4^- & 0 & 0 & -18 & 0 & -1 \\ 5^- & 45 & 0 & 0 & 0 & -1 \\ 6^- & -45 & 0 & 0 & 0 & -1 \\ 7^- & 0 & 45 & 0 & 0 & -1 \\ 8^- & 0 & -45 & 0 & 0 & -1 \\ 9^- & 0 & 0 & 15 & 15 & -1 \\ 10^- & 0 & 0 & -15 & -15 & -1 \\ 11^- & 0 & 0 & -15 & 15 & -1 \\ 12^- & 0 & 0 & -15 & -15 & -1 \\ 13^- & 30 & 30 & 0 & 0 & -1 \\ 14^- & -30 & 30 & 0 & 0 & -1 \\ 15^- & 30 & -30 & 0 & 0 & -1 \\ 16^- & -30 & -30 & 0 & 0 & -1 \\ 17^- & 40 & 0 & 10 & 0 & -1 \\ 18^- & 40 & 0 & -10 & 0 & -1 \\ 19^- & -40 & 0 & 10 & 0 & -1 \\ 20^- & -40 & 0 & -10 & 0 & -1 \\ 21^- & 0 & 40 & 0 & 10 & -1 \\ 22^- & 0 & 40 & 0 & -10 & -1 \\ 23^- & 0 & -40 & 0 & 10 & -1 \\ 24^- & 0 & -40 & 0 & -10 & -1 \end{pmatrix}$$
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Proof 2.
Show that there are no blue vertex $a$ and red vertex $b$ such that $a$ is a vertex of the blue cell containing $b$ and $b$ is a vertex of the red cell containing $a$. 
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Via glueing and products, the counterexample can be converted into an infinite family that violates the Hirsch conjecture by about 2%.

This breaks a “psychological barrier”, but for applications it is absolutely irrelevant.

Finding a counterexample will be merely a small first step in the line of investigation related to the conjecture.

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The end

THANK YOU!