Width of low-dimensional prismatoids

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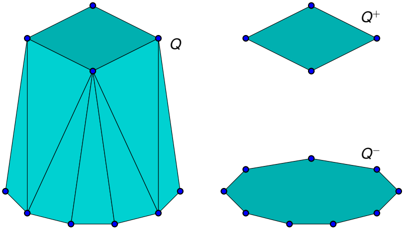
Prismatoids, the Hirsch conjecture, and pairs of maps

Prismatoids

Definition

A *prismatoid* is a polytope Q with two (parallel) facets Q^+ and Q^- containing all vertices.



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The *width* of a prismatoid is the *dual-graph* distance from Q^+ to Q^-.

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3-prismatoids have width ≤ 3.

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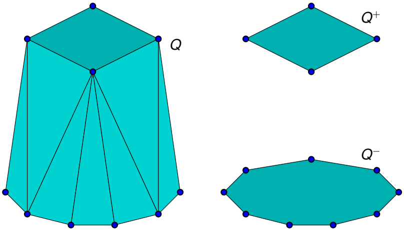

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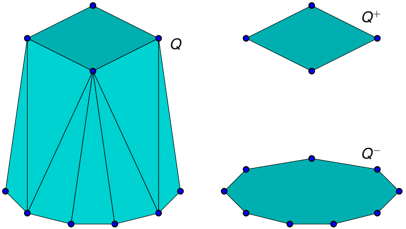

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Theorem (Strong d-step theorem, prismatoid version)

Let Q be a prismatoid of dimension d, with $n > 2d$ vertices and width δ. Then there is another prismatoid Q' of dimension $d+1$, with $n+1$ vertices and width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$.

Corollary

In particular, if a prismatoid Q has width $> d$ then there is another prismatoid Q' (of dimension $n-d$, with $2n-2d$ facets, and width $\geq \delta + n - 2d > n - d$) that violates (the dual of) the Hirsch conjecture.

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Width of prismatoids

So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant!!!*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S., July 2010].
- 5-prismatoids of width 6 exist [S., May 2010] with 28 vertices [S., Sept. 2010].

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Via gluing and products:

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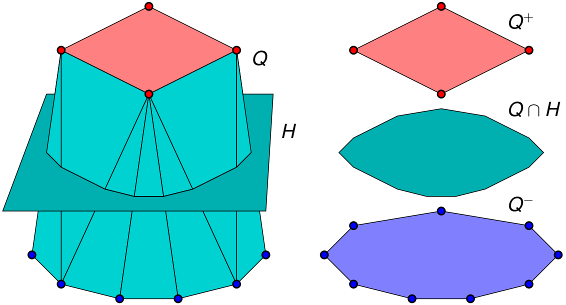
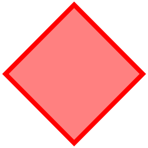
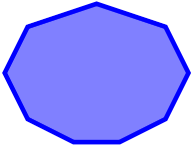
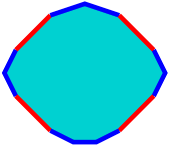
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice . . .



Combinatorics of prismatoids

...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.

$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

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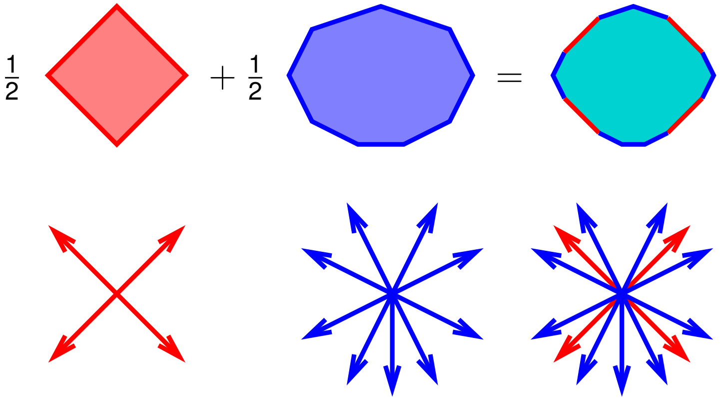

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So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Theorem

Let Q be a d-prismatoid with bases Q^+ and Q^- and let G^+ and G^- be the corresponding maps in the $(d-2)$-sphere (central projection of the normal fans of Q^+ and Q^-). Then, the width of Q equals 2 plus the minimum number of steps needed to go from a vertex of G^+ to a vertex of G^- in the (graph of) the superposition of the two maps.

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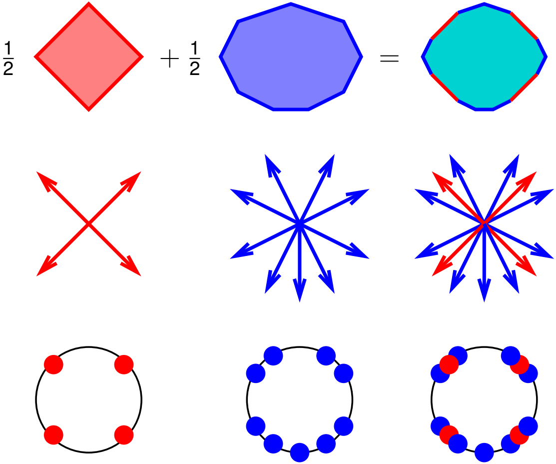

4-dimensional prismatoids

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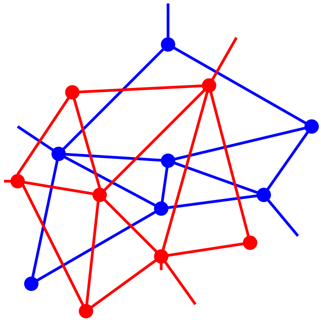

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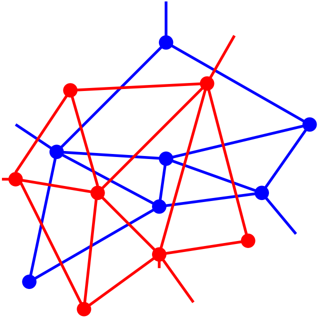

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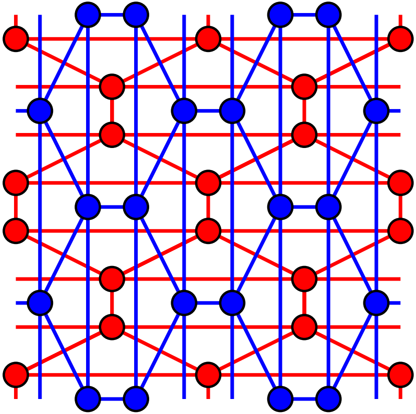
Replicating the following basic structure we can get a “non-Hirsch” periodic pair of maps in the plane:



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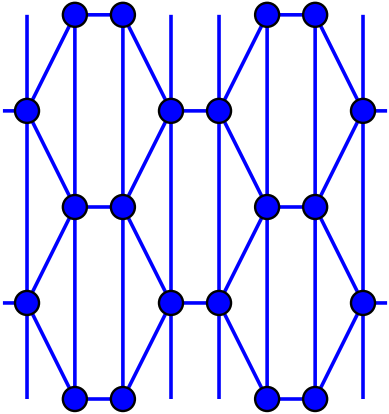

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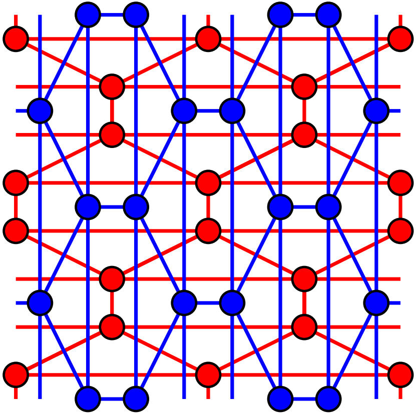

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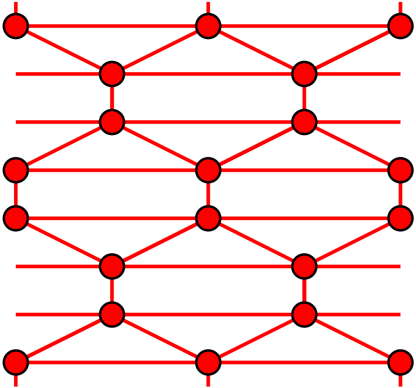

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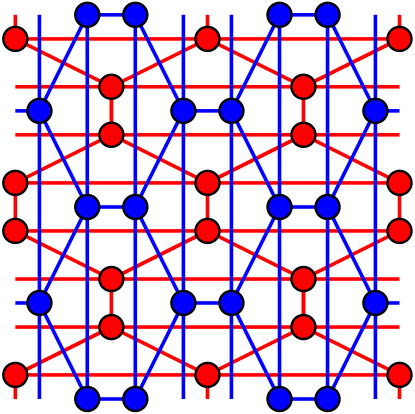

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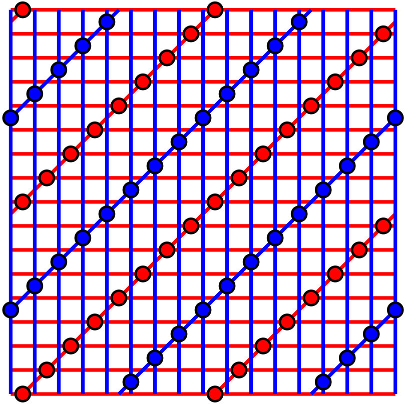

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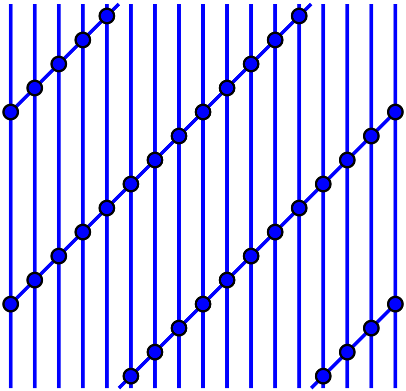

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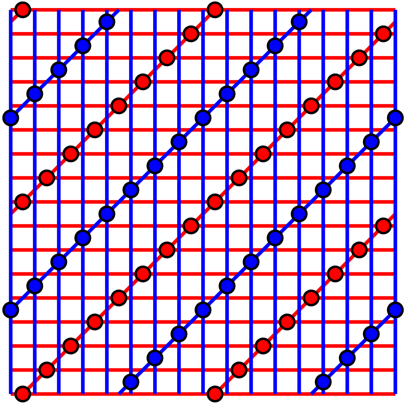

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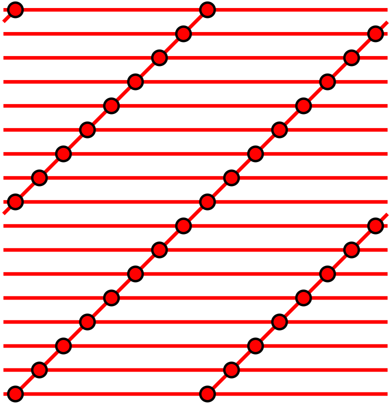

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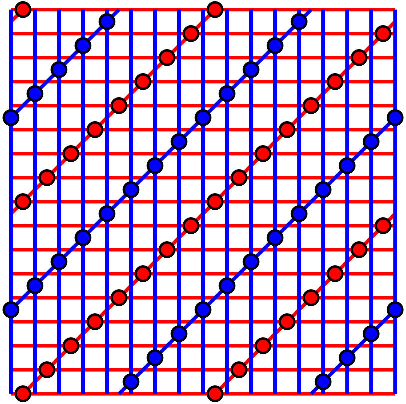

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Surprisingly enough:

Theorem (S., July 2010)

There is no "non-Hirsch" pair of maps in the 2-sphere.

To prove the theorem, we work in the general framework of pairs of maps in arbitrary surfaces. Let G^+ and G^- be two maps (a "red" and a "blue" one) in a surface S. Assume the following property:

Transversal pair of maps:

If a red edge and a blue edge intersect then they do it transversally, only once, and in their (relative) interiors.

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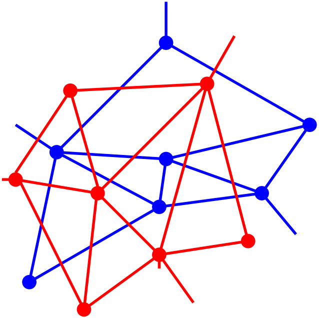

A pair of maps in a surface

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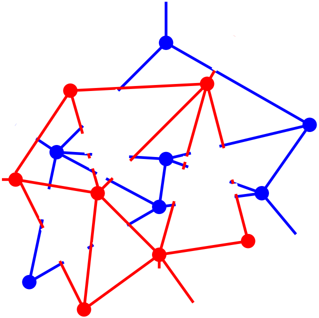

The terminal part of the common refinement graph

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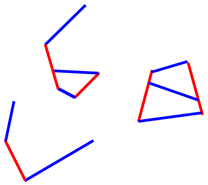

The non-terminal part of the common refinement graph

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Remark

A prismatoid Q has width 4

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In the common refinement of the corresponding pair of maps in S^2 some red terminal edge intersects a blue terminal edge.

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In the common refinement of the corresponding pair of maps in S^2 some red terminal edge intersects a blue terminal edge.

Assume, to seek a contradiction, that a certain transversal pair of maps (G^+, G^-) in the sphere S^2 does not have a terminal red edge intersect a terminal blue edge.

4-prismatoids have width < 4

Definition

A *zig-zag, color-alternating path* is a path of non-terminal edges such that whenever two consecutive edges have different colors, the path turns right from red to blue and it turns left from blue to red. A *zig-zag, color-alternating loop* is a cycle in which that happens except perhaps at the base point.

Lemma 1

Every non-terminal segment can be continued to a zig-zag, color-alternating path until the path crosses itself. At that point it produces a zig-zag, color-alternating loop.

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Choose an arbitrary point of S^2 to be “infinity”.

Lemma 2

If a zig-zag, color-alternating loop is minimal (i. e., the region it bounds contains no other such loop) then there is no other edge in its interior.

This gives a contradiction, because it implies that the boundary of some face of our pairs of maps is a zig-zag alternating loop, and a zig-zag alternating loop must contain “reflex vertices”.

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5-dimensional prismatoids

A 5-prismatoid of width > 5

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Theorem

The prismatoid Q of the next two slides, of dimension 5 and with 48 vertices, has width six.

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

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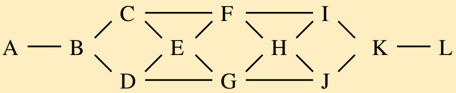
Proof 1.

It has been verified computationally that the dual graph of Q (modulo symmetry) has the following structure:



□

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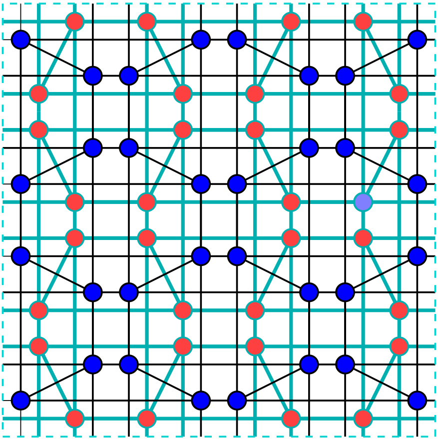
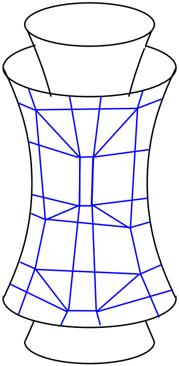
$$Q := \text{conv}\left\{ \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} \pm18 & 0 & 0 & 0 & 1 \\ 0 & \pm18 & 0 & 0 & 1 \\ 0 & 0 & \pm45 & 0 & 1 \\ 0 & 0 & 0 & \pm45 & 1 \\ \pm15 & \pm15 & 0 & 0 & 1 \\ 0 & 0 & \pm30 & \pm30 & 1 \\ 0 & \pm10 & \pm40 & 0 & 1 \\ \pm10 & 0 & 0 & \pm40 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \pm18 & -1 \\ 0 & 0 & \pm18 & 0 & -1 \\ \pm45 & 0 & 0 & 0 & -1 \\ 0 & \pm45 & 0 & 0 & -1 \\ 0 & 0 & \pm15 & \pm15 & -1 \\ \pm30 & \pm30 & 0 & 0 & -1 \\ \pm40 & 0 & \pm10 & 0 & -1 \\ 0 & \pm40 & 0 & \pm10 & -1 \end{pmatrix} \end{array} \right\}$$

A 5-prismatoid of width > 5





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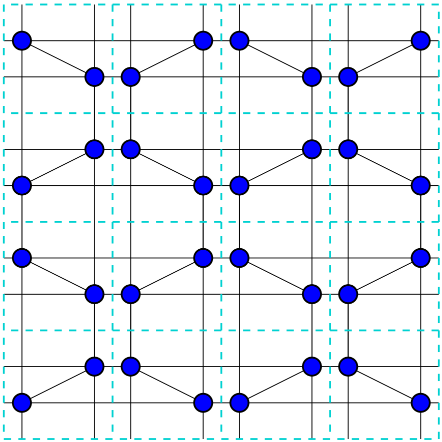

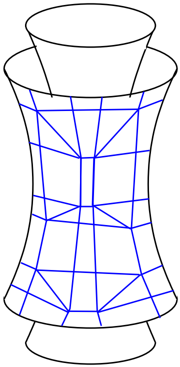

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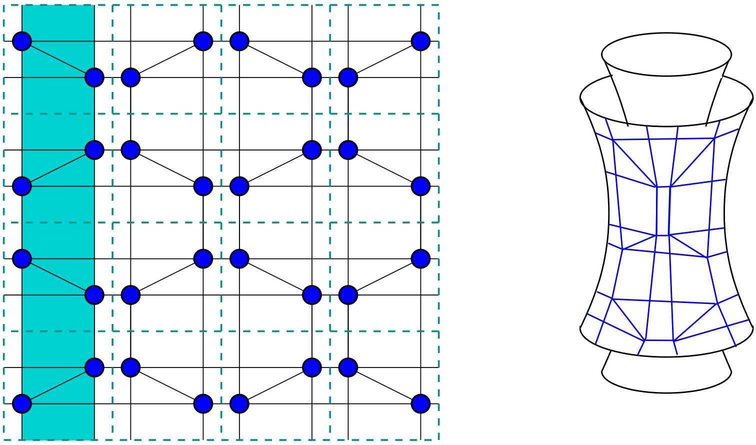

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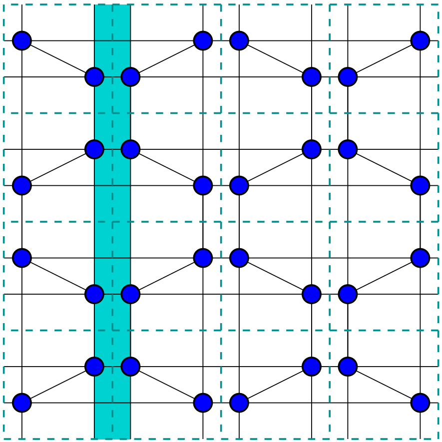

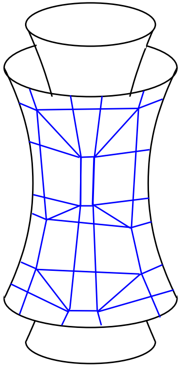

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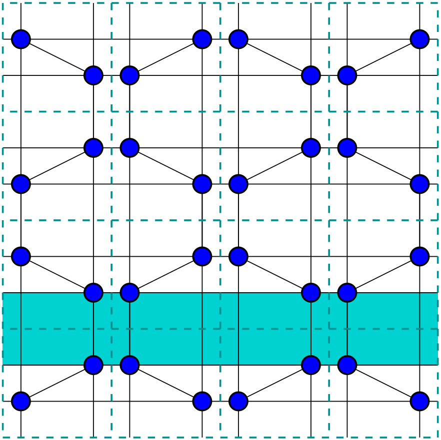

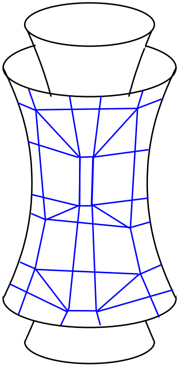

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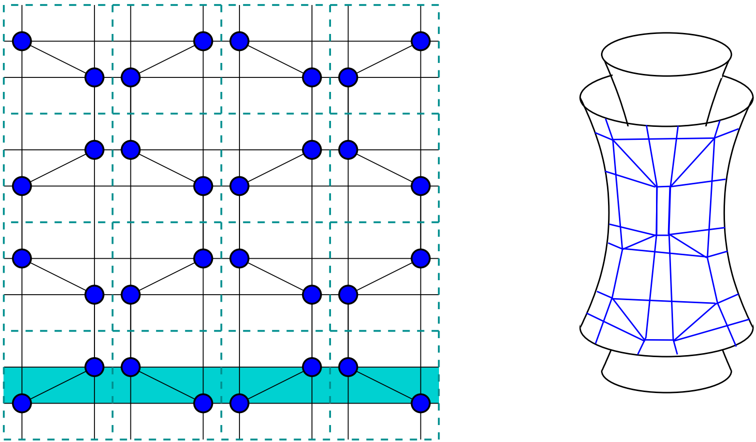

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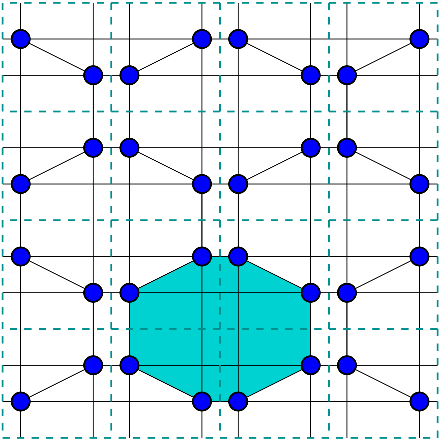

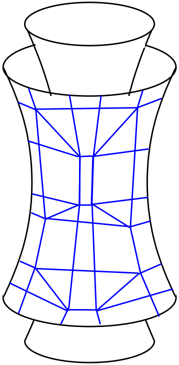

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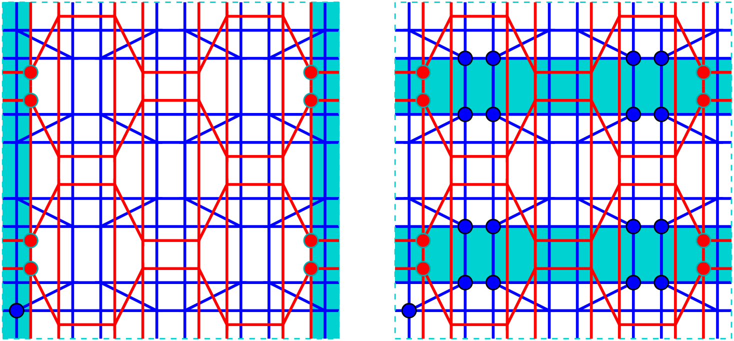
Proof 2.

Show that there are no blue vertex *a* and red vertex *b* such that *a* is a vertex of the blue cell containing *b* and *b* is a vertex of the red cell containing *a*. □



A smaller 5-prismatoid of width > 5

With the same ideas

Theorem

The following 5-prismatoid with 28 vertices (and 274 facets) has width 6.

$$Q := \operatorname{conv}\left\{ \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} \pm 18 & 0 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & 0 & 1 \\ 0 & 0 & 0 & \pm 30 & 1 \\ 0 & \pm 5 & 0 & \pm 25 & 1 \\ 0 & 0 & \pm 18 & \pm 18 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 0 & 0 & \pm 18 & 0 & -1 \\ 0 & \pm 30 & 0 & 0 & -1 \\ \pm 30 & 0 & 0 & 0 & -1 \\ \pm 25 & 0 & 0 & \pm 5 & -1 \\ \pm 18 & \pm 18 & 0 & 0 & -1 \end{pmatrix} \end{array} \right\}$$

Corollary

There is a non-Hirsch polytope of dimension 23 with 46 facets.

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If we fix the dimension d, the width of prismatoids is linear:

Theorem

The width of a d-dimensional prismatoid with n facets cannot exceed $2^{d-3}n$.

Proof.

This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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Let e be an edge obtained as the intersection of a blue 2-face and a red 2-face and such that the end points lie in a red edge and a blue edge respectively (it exists by connectivity). Along the red edge we get to a red vertex traversing at most half of the blue 3-faces. Along the blue edge we get to a blue vertex traversing at most half of the red 3-faces. □

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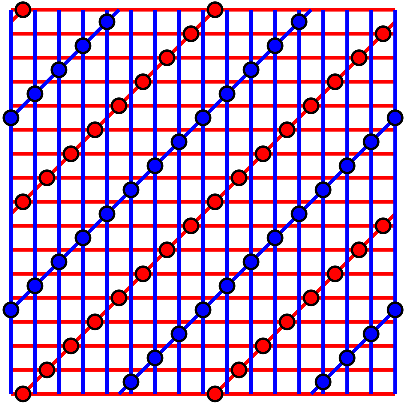

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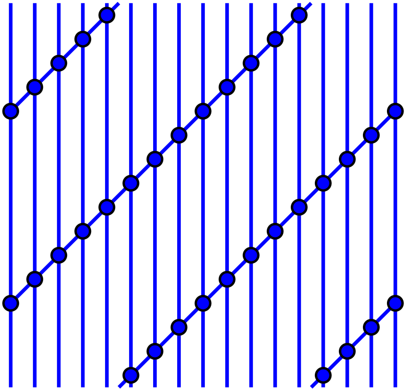

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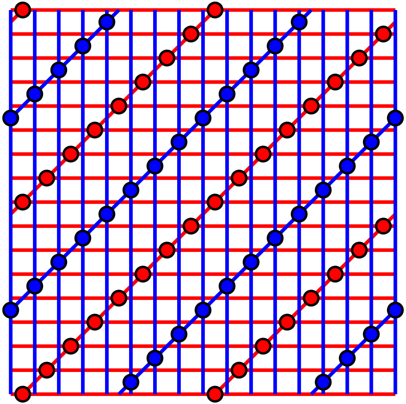

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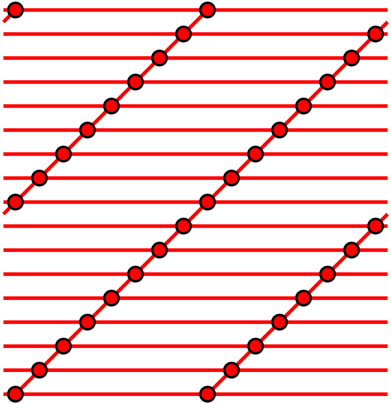

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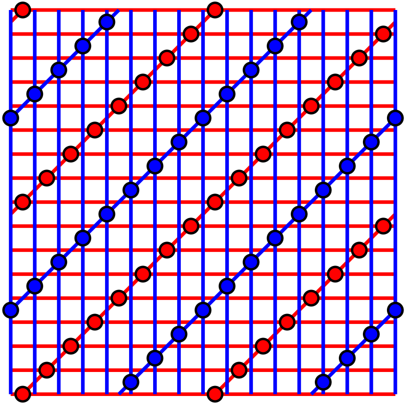
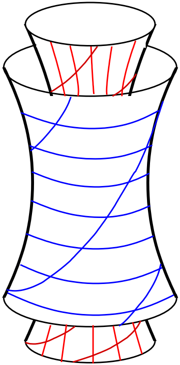


Asymptotic width in dimension five

Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps.

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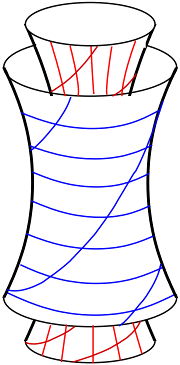

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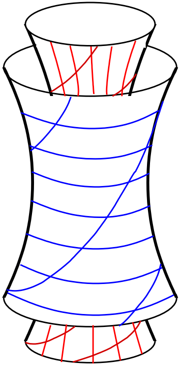

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Conclusion

- The counter-examples to the Hirsch conjecture break a "psychological barrier", but for applications they are so far irrelevant. They violate Hirsch by about 4%.
- The main open question(s) remains open: Is there a family of polytopes with superlinear diameter? Is the diameter of every polytope polynomially bounded?
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The end

THANK YOU!