Hirsch Wars Episode I
The Phantom Conjecture

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Hirsch Wars Trilogy

Slides (October 2011 version):

`http://personales.unican.es/santosf/Hirsch/`

1. Episode I: The Phantom Conjecture. (Today)
2. Episode II: Attack of the Prismatoids. (Tomorrow)
3. Episodes III and IV: Revenge of the Linear Bound, and A New Hope. (Wednesday)

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Polyhedra and polytopes

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

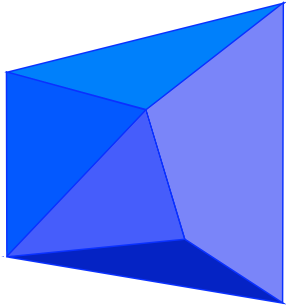
A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.

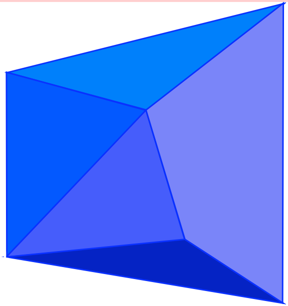


The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, every bounded polyhedron is a polytope.



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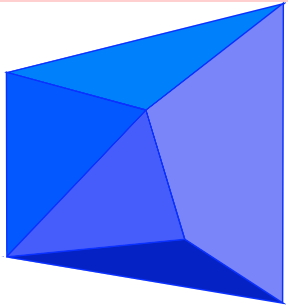

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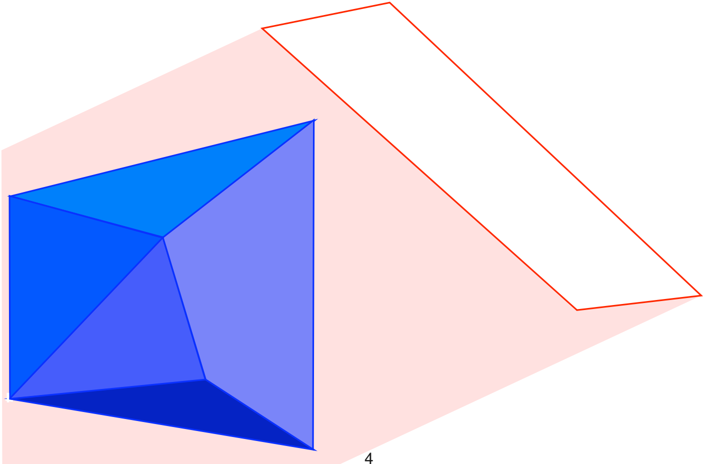
Faces of *P*

Let *P* be a polytope (or polyhedron) and let *H* be a hyperplane not cutting, but touching *P*.



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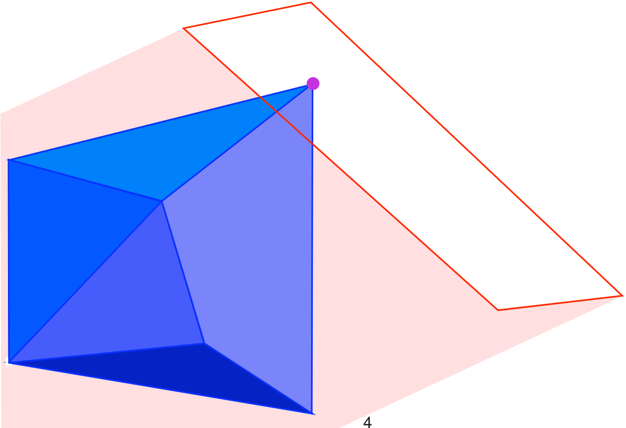
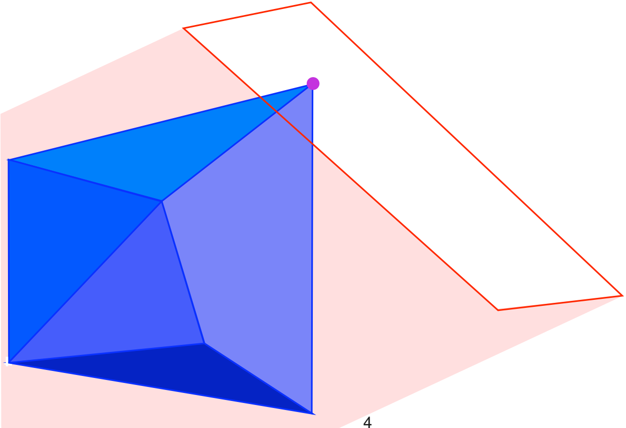


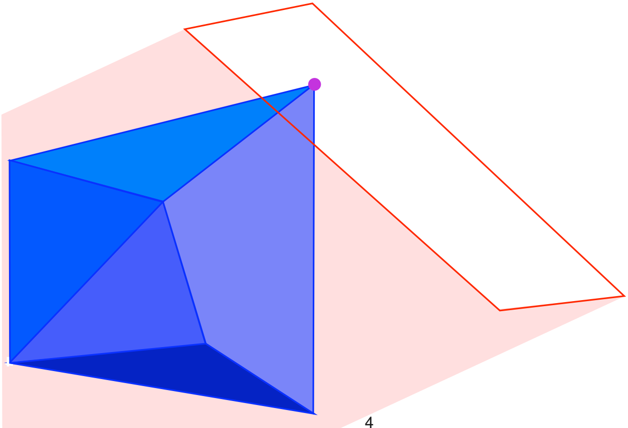
Faces of P

We say that $H \cap P$ is a face of P.



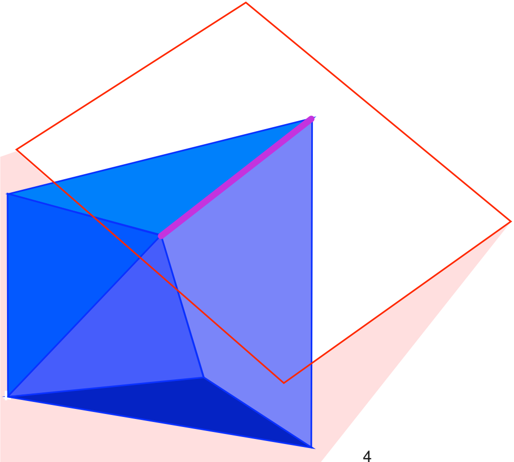
Faces of P

Faces of dimension 0 are called vertices.



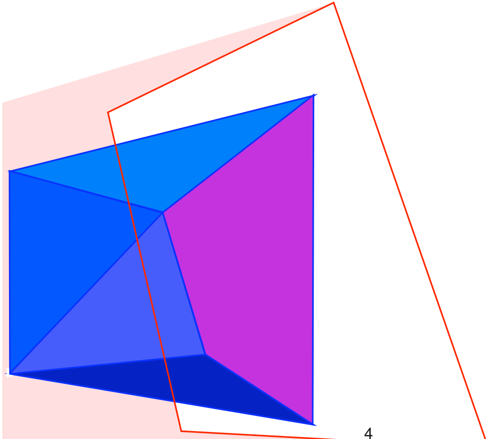
Faces of P

Faces of dimension 1 are called edges.



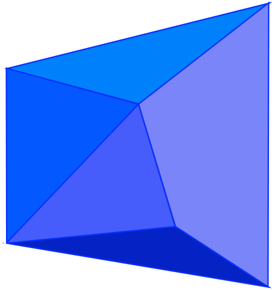
Faces of P

Faces of dimension $d-1$ are called facets.



The graph of a polytope

Vertices and edges of a polytope *P* form a graph (finite, undirected)



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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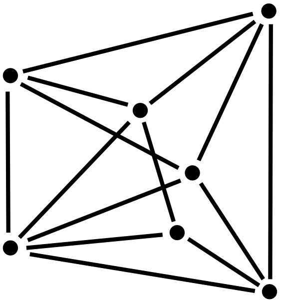

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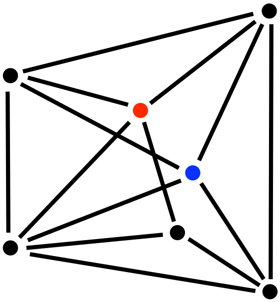

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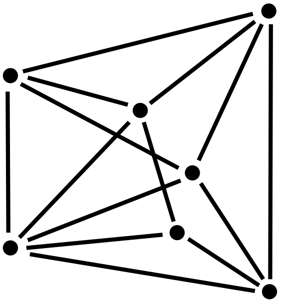

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) = \max\{d(u, v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

polytope	faces	dimension	$n-d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k+2$	3	$k-1$	$\lfloor k/2 \rfloor + 1$
n-cube	$2n$	n	n	n

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Brief history of the conjecture

1. It was communicated by W. M. Hirsch to G. Dantzig in 1957 (Dantzig had recently created the simplex method for linear programming).
2. Several special cases have been proved: $d \leq 3$, $n - d \leq 6$, $0/1$-polytopes, ...
3. But in the general case **we do not even know of a polynomial bound** for $\delta(P)$ in terms of n and d.
4. In 1967, Klee and Walkup disproved the unbounded case.
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Linear programming

A linear program is the problem of maximization (or minimization) of a linear functional subject to linear inequality constraints. That is:

Given

- a system $Mx \leq b$ of linear inequalities ($b \in \mathbb{R}^n$, $M \in \mathbb{R}^{d \times n}$), and
- an objective function $c^t \in \mathbb{R}^{d*}$

Find

- $\max\{c^t \cdot x : x \in \mathbb{R}^d, Mx \leq b\}$ (and a point x where the maximum is attained).

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Motivation: linear programming

> *Linear programming is used to allocate resources, plan production, schedule workers, plan investment portfolios and formulate marketing (and military) strategies. The versatility and economic impact of linear programming in today's industrial world is truly awesome.”*
>
> (Eugene Lawler, 1979)

Motivation: linear programming

> *If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not including database handling problems like sorting and searching) the answer would probably be linear programming.*
>
> (László Lovász, 1980)

Motivation: linear programming

> *One of these methods is called linear programming. I learned about it in 1958. I had just come to Caltech as a junior faculty member associated with the computing center. The director and I made a cross-country trip to survey the most important industrial uses of computers. In New York, we visited the Mobil Oil Company, which had just put in a multi-million-dollar computer system. We found out that Mobil had paid off this huge investment in two weeks by doing linear programming.*
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Connection to the Hirsch conjecture

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- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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Complexity of the simplex method

The simplex method is not (known to be) polynomial. More precisely, it is known to be not polynomial with the pivot rules that have been proposed so far.

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It is a cube with slanted faces in which the “biggest slope” rule is led to take an exponentially long path.

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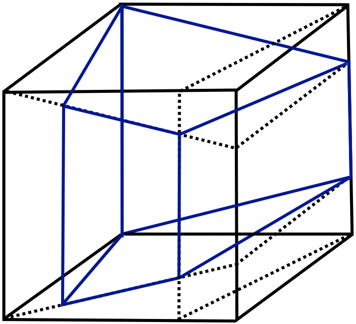

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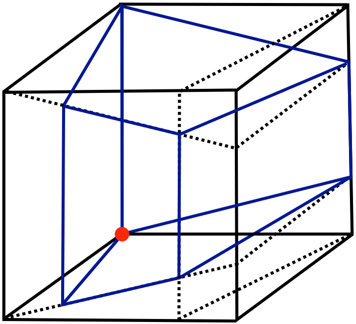

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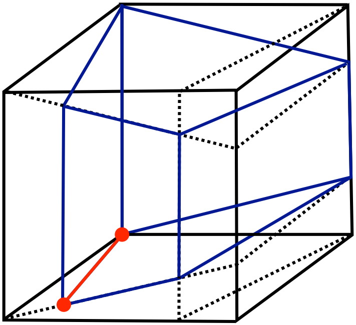

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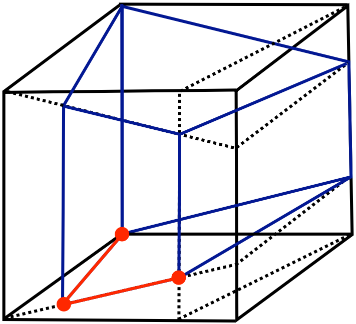

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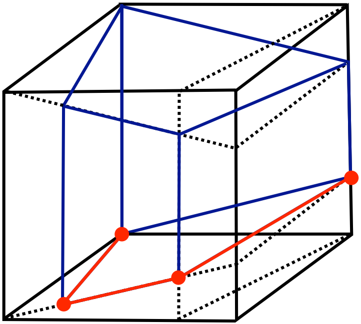

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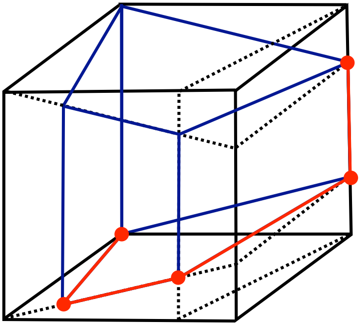

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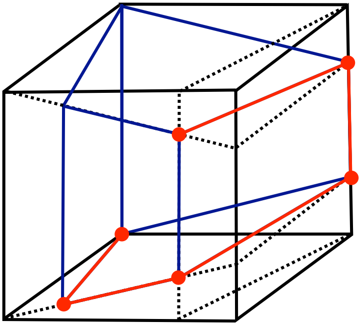

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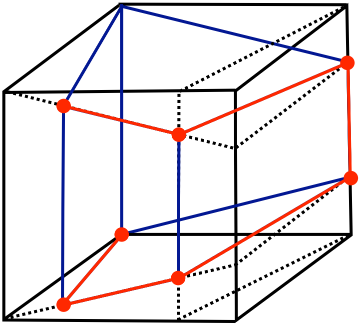

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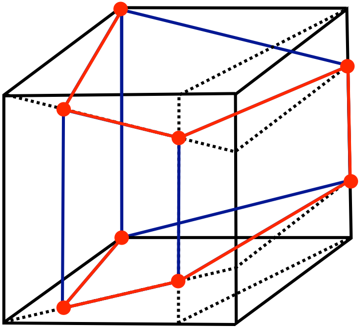

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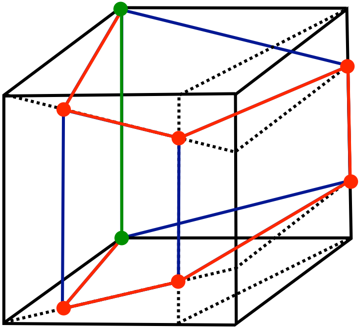

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Complexity of linear programming

Besides, the known polynomial algorithms for linear programming known are not *strongly polynomial*: They are polynomial in the bit complexity model but not in the real machine model [Blum et al. 1989]).

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Theorem [Kalai-Kleitman 1992]

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Why is $n - d$ a "reasonable" bound?

- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

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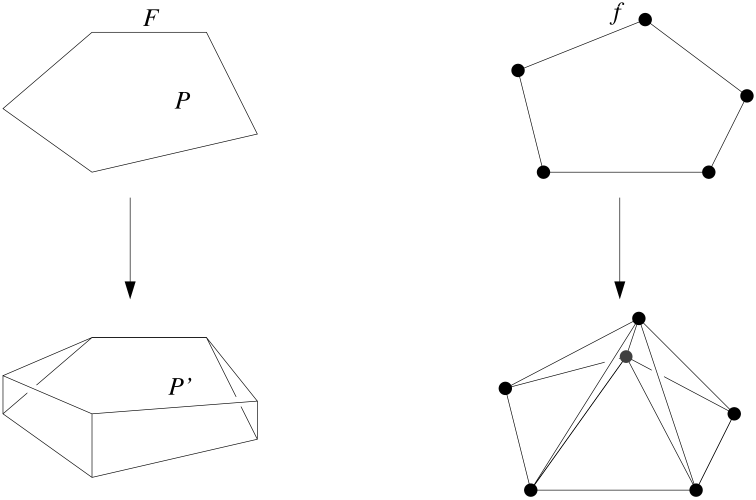

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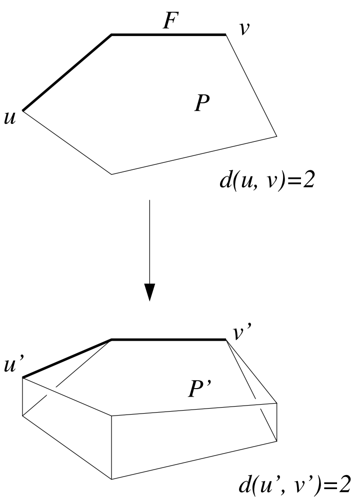

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Hirsch $\Leftrightarrow$ d-step $\Leftrightarrow$ non-revisiting path.

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Proof: Let $H(n, d) = \max\{\delta(P) : P$ is a d-polytope with n facets$\}$. The basic idea is:

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- If $n < 2d$, then $H(n, d) \leq H(n-1, d-1)$ because every pair of vertices u and v lie in a common facet F, which is a polytope with one less dimension and (at least) one less facet (induction on n and $n - d$).

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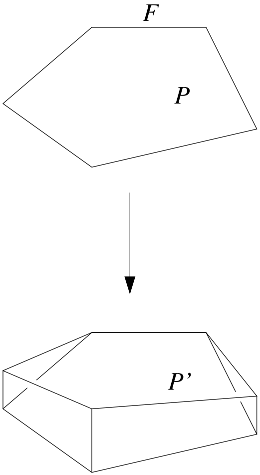
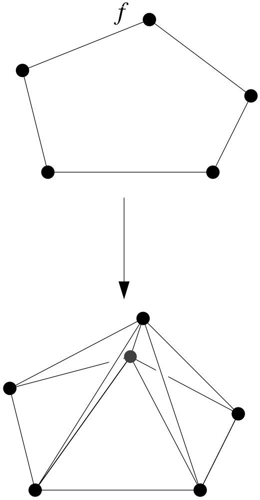
Wedging, a.k.a. one-point-suspension





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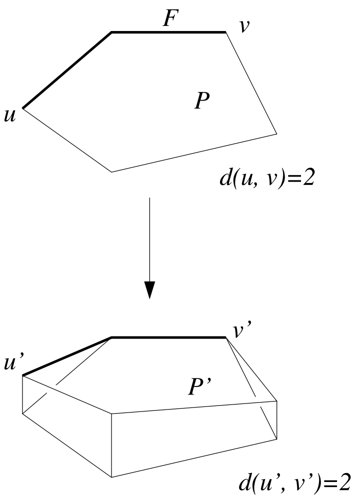

Two important remarks

The d-step Theorem follows from and implies (respectively) the following:

Lemma

For every d-polytope P with n facets and diameter δ there is a $d+1$-polytope with one more facet and the same diameter δ.

Corollary

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The feasible region of a linear program can be an unbounded polyhedron, instead of a polytope.

Unbounded version of the Hirsch conjecture:

The diameter of any polyhedron P with dimension d and n facets is at most $n - d$.

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Any of these three versions (combinatorial, monotone, unbounded) would imply the Hirsch conjecture...

... but the three were known to be false (although all known counter-examples are only by a linear factor):

- There are unbounded polyhedra of dimension 4 with 8 facets and diameter 5 [Klee-Walkup, 1967].
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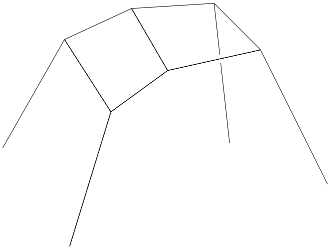

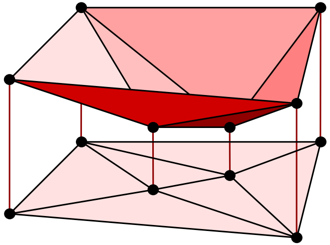

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Remember that

“The polar of an unbounded 4-polyhedron with eight facets is a regular triangulation of eight points in $\mathbb{R}^3$".

So, it suffices to show that:

Theorem

There is a regular triangulation of a 4-polytope with 8 vertices that has two tetrahedra at distance five.

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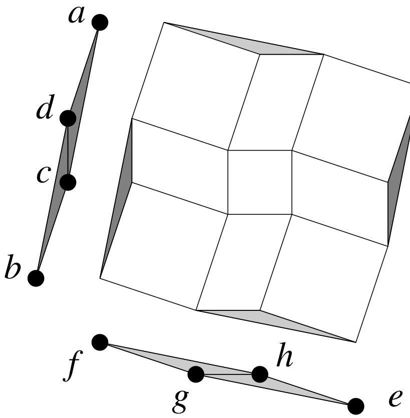
This is a (Cayley Trick view of a) 3D triangulation with 8 vertices and diameter 5:



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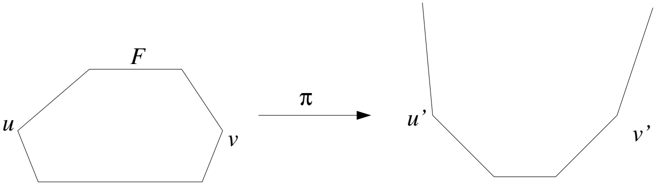

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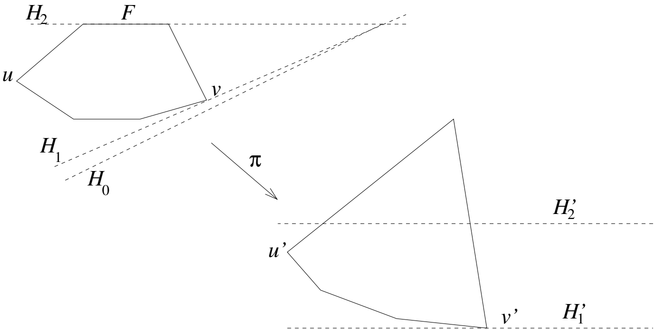

The Mani-Walkup “always revisiting” simplicial 3-sphere

Mani and Walkup constructed a simplicial 3-ball with 16 vertices and with two tetrahedra *abcd* and *mnop* with the property that any path from *abcd* to *mnop* must revisit a vertex previously abandonded.

By the (combinatorial) d-step theorem, that implies the existence of a “non-Hirsch” 11-sphere with 24 vertices ($n - d = 12$)

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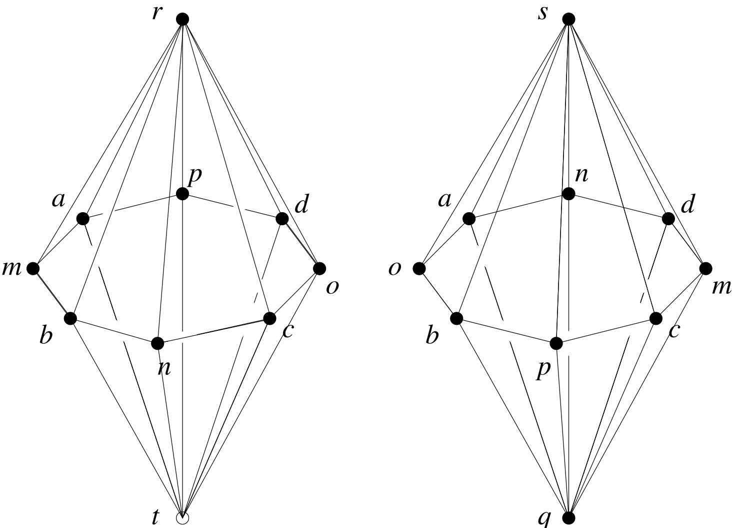

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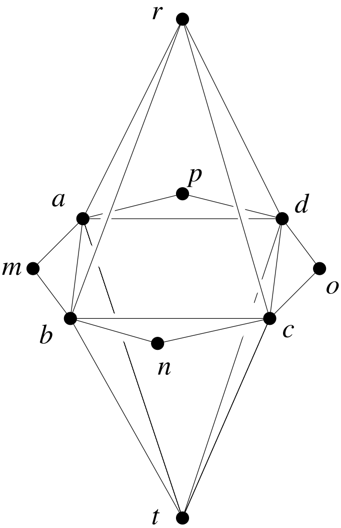

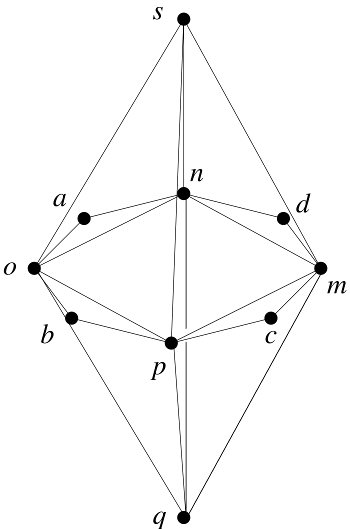

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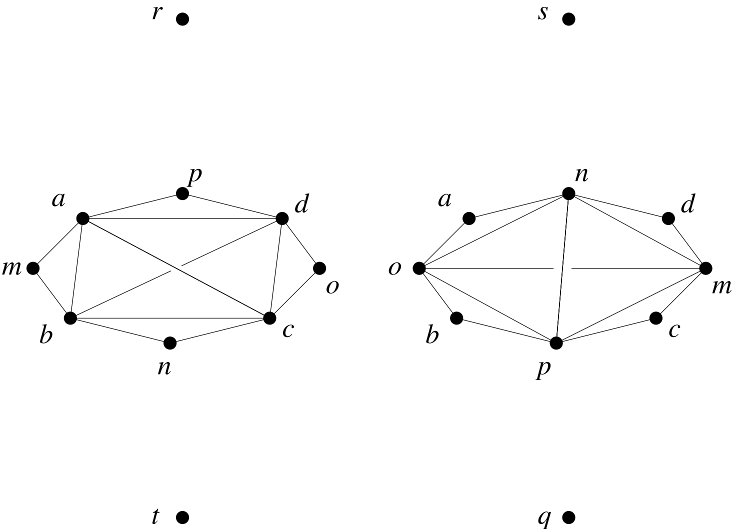

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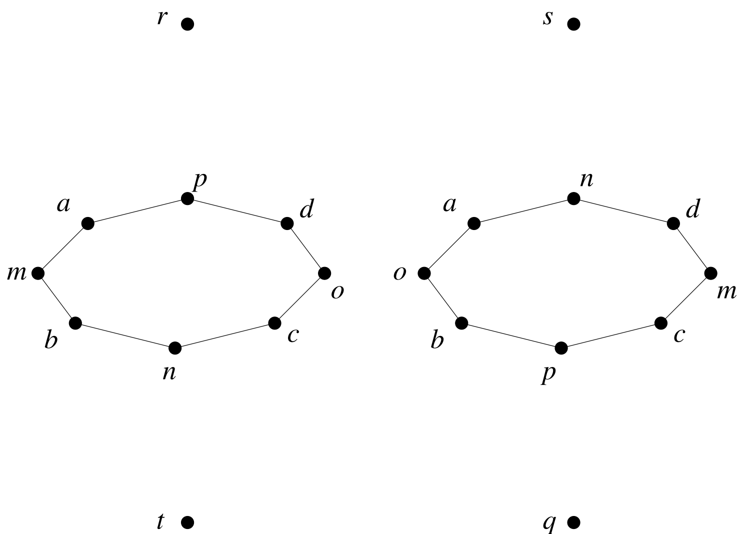

Thank you

TO BE CONTINUED