The number of triangulations of the cyclic polytope $C(n, n-4)$ *

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Abstract

We show that the exact number of triangulations of the standard cyclic polytope $C(n, n-4)$ is $(n+4)2^{\frac{n-4}{2}} - n$ if n is even and $\left(\frac{3n+11}{2}\right) 2^{\frac{n-5}{2}} - n$ if n is odd. These formulas were previously conjectured by the second author.

Our techniques are based on Gale duality and the concept of *virtual chamber*. They further provide formulas for the number of triangulations which use a specific simplex. We also compute the maximum number of regular triangulations among all the realizations of the oriented matroid of $C(n, n-4)$.

Introduction

By a triangulation of a finite point set $\mathcal{A} \subset \mathbb{R}^d$ we mean a simplicial complex geometrically realized in $\mathbb{R}^d$ with vertex set contained in $\mathcal{A}$ and which covers the convex hull of $\mathcal{A}$. If $\mathcal{A}$ is the vertex set of a polytope P this definition agrees with the standard definition of triangulation of P. The collection of all triangulations of a fixed point set has attracted attention in recent years for its connections to algebraic geometry [18], combinatorial topology [5, 16] and optimization [8].

The *cyclic polytope* $C(n, d)$ of dimension d and with n vertices ($n > d$) is the convex hull of any n distinct points on the affine *moment curve* of degree d, defined as $\Gamma_d(t) := (t, t^2, \dots, t^d) \in \mathbb{R}^d$, $t \in \mathbb{R}$. Cyclic polytopes play a central role in geometric combinatorics for several reasons: They are

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neighborly, which implies that they have the maximum possible number of faces of each dimension among all polytopes of dimension d with n vertices [19, Theorem 8.23]. They are universal in the sense that for each fixed n and d there is a number $N(n,d)$ such that any $N(n,d)$ points in general position in $\mathbb{R}^d$ will contain n points which are the vertices of a cyclic polytope $C(n,d)$ (and no other polytope has this property) [7, Proposition 9.4.7]. In a context closer to this paper, the set of triangulations of a cyclic polytope has a bit more structure than the set of triangulations of an arbitrary polytope. Edelman and Reiner defined two (conjectured to be isomorphic) poset structures on this set [10], Rambau proved that all triangulations of $C(n,d)$ are connected under bistellar operations [13] (while point configurations and polytopes without this property exist [17]), and the so-called *generalized Baues problem* in the case of cyclic polytopes is essentially solved [1, 2, 15].

Our main result in this paper is a proof of the following closed formula for the number of triangulations of the cyclic polytope $C(n,n-4)$. This formula had previously been conjectured by the second author. As pointed out by Reiner [16, page 325], this and the well-known Catalan number for the number of triangulations of a convex polygon are the only known nontrivial closed formulas counting triangulations of a polytope.

Theorem 1 *The number of triangulations of the cyclic polytope $C(n,n-4)$ is:*

- $(n+4)2^{\frac{n-4}{2}} - n$, *if n is even (Theorem 2.6) and*
- $\left(\frac{3n+11}{2}\right) 2^{\frac{n-5}{2}} - n$, *if n is odd (Theorem 2.9).*

The odd case formula can be written in a way formally closer to the even case as $(\alpha n + \beta)\, 2^{\frac{n-4}{2}} - n$, with $\alpha = \frac{3}{2\sqrt{2}} = 1.061$ and $\beta = \frac{11}{2\sqrt{2}} = 3.889$. It is interesting to relate our result to the following numbers of triangulations for other parameters of the cyclic polytope:

- $C(n,0)$ (n copies of the same point) has n triangulations, if multiple points are dealt with in the natural way.
- $C(n,1)$ (n different points along a line) has 2^{n-2} triangulations.
- $C(n,2)$ (a convex n-gon) has $\frac{1}{n-1}\binom{2n-4}{n-2} = \Theta\left(\frac{2^{2n}}{n^{3/2}}\right)$ triangulations.
- $C(n,n-4)$ (this paper) has $\Theta\left(n2^{n/2}\right)$ triangulations.
- $C(n,n-3)$, $C(n,n-2)$ and $C(n,n-1)$ have, respectively, n, 2 and 1 triangulations. The case of $C(n,n-3)$ follows from [11]. The others are trivial.

Concerning $C(n, n-5)$, its number of triangulations for n up to 12 appears in [1, 15]. The numbers for $C(13, 8)$, $C(14, 9)$, $C(15, 10)$ and $C(16, 11)$ are, respectively, 35 789, 159 613, 499 900 and 2 677 865. We thank Jörg Rambau for these numbers, computed by him with his public software `TOPCOM` [14]. Dividing the number of triangulations of $C(n, n-5)$ by the number of triangulations of $C(n, 2)$ gives, for n from 5 to 16, the following intriguing sequence:

1, .875, 1, .96, 1.20, 1.16, 1.48, 1.33, 1.64, 1.30, 1.49, 0.9987.

The sequence stays surprisingly close to 1, and is neither decreasing nor increasing, even if we separate odd and even values of n.

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Our methods are based on Gale duality and oriented matroid theory. More precisely, on the concept of *virtual chamber* introduced in [8, Section 5]. This same tool was used in [4] to prove that the flip-graph of triangulations of $d+4$ points in dimension d is always connected.

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a finite point configuration in $\mathbb{R}^d$. We homogenise $\mathcal{A}$, which means that we embed $\mathbb{R}^d$ as an affine hyperplane not passing through the origin in $\mathbb{R}^{d+1}$ and consider $\mathcal{A}$ as a vector configuration in $\mathbb{R}^{d+1}$. We say that $d+1$ is the *rank* of $\mathcal{A}$ and $n-d-1$ its *corank*.

Definition 2 A *triangulation* of a vector configuration $\mathcal{A}$ is any collection of linear bases contained in $\mathcal{A}$ whose positive spans are (the maximal elements of) a simplicial fan covering the positive span of $\mathcal{A}$.

For a vector configuration obtained by homogenisation of a point configuration, the definitions of triangulation in the vector and point contexts agree.

Let $\mathcal{B}$ be a *Gale transform* of $\mathcal{A}$. This is any vector configuration $\mathcal{B} = \{b_1, \dots, b_n\} \subset \mathbb{R}^{n-d-1}$ such that the kernels of the linear maps $e_i \mapsto a_i$ and $e_i \mapsto b_i$ are orthogonal complements of one another ($\{e_1, \dots, e_n\}$ denotes the standard basis in $\mathbb{R}^n$). Equivalently, such that $\sum_{i=1}^n a_i \otimes b_i = 0$ in $\mathbb{R}^{d+1} \otimes \mathbb{R}^{n-d-1}$. Observe that there is an implicit bijection between $\mathcal{A}$ and $\mathcal{B}$ in this definition, given by the labels.

If $\mathcal{A}$ and $\mathcal{B}$ are Gale transforms of each other, then a vector $(\lambda_1, \dots, \lambda_n)$ is the sequence of values of a linear functional on one of them if and only if it is the sequence of coefficients of a linear dependence in the other. In particular, the oriented matroids of $\mathcal{A}$ and $\mathcal{B}$ are dual to each other and, hence, any information of $\mathcal{A}$ which depends only on its underlying oriented matroid such as the set of triangulations of $\mathcal{A}$ (see [8]) can be retrieved from the oriented matroid of $\mathcal{B}$. In the case of interest to us, this translates our corank 3 problem into a rank 3 one, but we have to characterize triangulations of $\mathcal{A}$ in terms of $\mathcal{B}$. This is done with the concept of *virtual chamber*, defined below. Observe that, by Gale duality, each linear basis of $\mathcal{A}$ is the complement of a linear basis of $\mathcal{B}$ and vice-versa, under the identification between the elements of $\mathcal{A}$ and $\mathcal{B}$ by their labels.

Definition 3 ([8, Section 5]) Let $\mathcal{A}$ and $\mathcal{B}$ be vector configurations which are dual to each other. A *virtual chamber* of $\mathcal{B}$ is a collection of linear bases of $\mathcal{B}$ whose complements form a triangulation of $\mathcal{A}$.

In the following result, being in general position for a vector configuration of rank k means that every k elements form a basis. Our configuration $\mathcal{B}$ will have this property since the property holds for the vertex set of a cyclic polytope and is preserved by Gale duality.

Lemma 4 ([8]) *Let $\mathcal{C}$ be a collection of linear bases of a vector configuration $\mathcal{B}$ in general position. Then, the following two conditions are equivalent:*

(i) $\mathcal{C}$ is a virtual chamber of $\mathcal{B}$.

(ii) $\mathcal{C}$ shares exactly one basis with every triangulation of $\mathcal{B}$.

The implication from (i) to (ii) holds without the general position assumption. This implication, in a sense, explains the name "virtual chamber". The *chamber fan* of the vector configuration $\mathcal{B}$ is the common refinement of all its triangulations. The same definition for a point configuration gives what is known as the *chamber complex* of the configuration. The *chambers* are the maximal cells in either case. For any given chamber, the collection of linear bases of $\mathcal{B}$ whose positive spans contain that chamber are a particular example of virtual chamber and two different chambers produce different virtual chambers. Hence, a chamber can be considered as a special case of virtual chamber. In fact, chambers of $\mathcal{B}$ are the virtual chambers corresponding to the so-called *regular* triangulations of $\mathcal{A}$ [6, 11], of interest to us in Section 4.

Following the analogy with (geometric) chambers, when a basis τ is an element of a virtual chamber $\mathcal{C}$ we will say that $\mathcal{C}$ lies on τ. With this, Lemma 4 can be rephrased as "$\mathcal{C}$ is a virtual chamber of $\mathcal{B}$ if and only if it lies on exactly one maximal simplex of every triangulation of $\mathcal{B}$". As a consequence, for every triangulation T of $\mathcal{B}$, one has that the number of virtual chambers of $\mathcal{B}$ equals $\sum_{\tau\in T}\sharp\{\text{virtual chambers lying on } \tau\}$.

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The structure of this paper is as follows. Section 1 outlines our method and translates the problem of counting triangulations of a corank 3 point configuration $\mathcal{A}$ to that of counting ideals in certain posets arising from the Gale transform of $\mathcal{A}$. Section 2 makes use of the specially nice structure of the Gale transform of $C(n, n-4)$ to prove Theorem 1. In Section 3 we compute, for even n, the number of triangulations of $C(n, n-4)$ which use any specific full-dimensional simplex (Theorem 3.3). Finally, in Section 4, we compute the exact maximum number of regular triangulations of point configurations having the same oriented matroid as $C(n, n-4)$ (Theorem 4.3). This number is a polynomial of degree four (as follows from [6, Theorem 5.7]) and contrasts with the exponential number of all triangulations. The polynomial versus exponential behaviour of regular versus non-regular triangulations of $C(n, n-4)$ had already been pointed out in [8, Section 5], although the exponential lower bound for the number of triangulations stated there is $\Omega(2^{n/4})$, instead of the $\Theta\left(n2^{n/2}\right)$ proved here.

1 Counting virtual chambers in rank 3

Virtual chambers in rank 3

Let $\mathcal{B}$ be a rank 3 vector configuration in general position, such as the Gale transform of the vertex set of $C(n, n-4)$. The oriented matroid of a vector configuration does not change under a positive scaling of an element. Hence, without loss of generality we may assume that all the elements of $\mathcal{B}$ lie on the unit 2-sphere S^2. The positive span of a subset $S \subset \mathcal{B}$ is then substituted by its spherical convex hull (i.e. the intersection of the positive span with the unit sphere) and we denote it $\operatorname{conv}(S)$. This motivates that we call *triangles* the bases of $\mathcal{B}$. We will often refer to independent sets (i.e. subsets of bases) as *simplices*. We call the simplices with one and two elements, *points* (or *vertices*) and *edges*, respectively. In this setting, a triangulation of $\mathcal{B}$ is a geometric triangulation of $\operatorname{conv}(\mathcal{B}) \subset S^2$ by (spherical) triangles of the configuration.

The *relative interior* of a subset S of $\mathcal{B}$ is the sphere S^2 intersected with the relatively open cone of strictly positive linear combinations of elements of S. We say that two simplices of a configuration *overlap* if their relative interiors have nonempty intersection. If two edges l_1 and l_2 of $\mathcal{B}$ overlap, then their relative interiors meet in a single point. We say that l_1 and l_2 *cross* each other.

A simplex σ of $\mathcal{B}$ is said to be *empty* if $\operatorname{conv}(\sigma) \cap \mathcal{B} = \sigma$. It is clear that if an empty edge l overlaps an empty triangle τ, then either l crosses two edges of τ or $l \cap \tau$ is a vertex of both l and τ and l crosses the opposite edge of τ. Note also that since $\mathcal{B}$ is in general position, all its edges are empty.

Definition 1.1 Let $\mathcal{B}$ be a rank 3 configuration. An empty triangle τ of $\mathcal{B}$ is said to be *admissible* with respect to one of its vertices p if no edge of $\mathcal{B}$ which overlaps τ has p as a vertex.

Remark 1.2 If τ has an edge l which is not crossed by any other edge of the configuration, then τ is admissible with respect to the vertex opposite to l. The converse is not true.

An edge l of $\mathcal{B}$ defines two *sides*, which are the two hemispheres in which S^2 is divided by the unique great circle which contains l. A virtual chamber $\mathcal{C}$ will be said to lie on a certain side of an edge l if there is a triangle of $\mathcal{C}$ contained in that (closed) side of l. By part (i) of the following Lemma, a virtual chamber cannot lie on both sides of an edge. But it can lie on neither of the two sides of it.

Lemma 1.3 *Let $\mathcal{C}$ be a virtual chamber of a rank 3 configuration $\mathcal{B}$ in general position. Then:*

(i) Any two triangles of $\mathcal{C}$ overlap.

(ii) If $S \subset \mathcal{B}$ contains a triangle of $\mathcal{C}$, then any triangulation of S contains an element of $\mathcal{C}$.

(iii) If an edge l overlaps an empty triangle of $\mathcal{C}$, then $\mathcal{C}$ lies on some side of l.

Proof: Parts (i) and (iii) are Lemma 1.4 and Proposition 3.2 in [4], respectively. Part (ii) is the specialization of Lemma 2.7 in [3] to triangulations. $\square$

Virtual chambers in rank 3 as poset ideals

Given a poset (partially ordered set) $(\mathcal{P}, <)$ and two subsets $I, F \subset \mathcal{P}$, we say that I is an *ideal* if for every pair of elements $x, y \in \mathcal{P}$ with $x < y$ and $y \in I$, it holds that $x \in I$.

Let $\tau := \{p, q, r\}$ be a triangle of a rank 3 configuration $\mathcal{B}$, admissible with respect to p. We denote by $\Omega(\tau)$ the set of edges which overlap τ and we define the following binary relation in $\Omega(\tau)$: $l_1 \prec_p l_2$ if $l_1 \neq l_2$ and l_2 does not overlap the triangle $l_1 \cup \{p\}$. Observe that if $l_1 \prec_p l_2$, then the rays from p through q and from p through r meet $\mathrm{conv}(l_1)$ before $\mathrm{conv}(l_2)$. The converse is not true, but this property implies that the relation $\prec_p$ does not have cycles. Hence, its transitive closure is a partial order, that we denote $<_p$. If $l_1 <_p l_2$ we say that l_1 is *closer to p* than l_2. A chain $l_1 <_p \cdots <_p l_k$ of edges in $(\Omega(\tau), <_p)$ will be called a *strong chain* if every consecutive pair of edges share a vertex.

Let $\mathcal{C}$ be a virtual chamber lying on τ. By part (iii) of Lemma 1.3, $\mathcal{C}$ lies on a side of every edge of $\Omega(\tau)$. Let $I(\mathcal{C})$ be the set of edges in $\Omega(\tau)$ which have p and $\mathcal{C}$ on opposite sides. Equivalently, $I(\mathcal{C}) := \{l \in \Omega(\tau) : l \cup \{p\} \notin \mathcal{C}\}$.

Proposition 1.4 *For every virtual chamber $\mathcal{C}$ lying on τ, $I(\mathcal{C})$ is an ideal of $(\Omega(\tau), <_p)$. Moreover, the correspondence $\mathcal{C} \mapsto I(\mathcal{C})$ is a bijection between virtual chambers which lie on τ and ideals of $(\Omega(\tau), <_p)$.*

Proof: Let us see that $I(\mathcal{C})$ is an ideal. Let $l_1, l_2 \in \Omega(\tau)$ with $l_1 <_p l_2$ and let $l_2 \in I(\mathcal{C})$. First suppose that $l_1 \prec_p l_2$. We consider a triangulation T of $\tau \cup l_2$ which uses the triangle $l_2 \cup \{p\}$. The remaining triangles of T are contained in the side of l_2 opposite to p. Since $\mathcal{C}$ lies on the side of l_2 opposite to p, $\mathcal{C}$ lies on some of the remaining triangles. The condition $l_1 \prec_p l_2$ implies that none of the remaining triangles overlaps $l_1 \cup \{p\}$, which implies $l_1 \in I(\mathcal{C})$. Hence, $\mathcal{C} \mapsto I(\mathcal{C})$ is a well-defined map from virtual chambers lying on τ to ideals in $(\Omega(\tau), <_p)$.

Conversely, for each ideal I of $\Omega(\tau)$, let $\mathcal{C}(I)$ be the following collection of triangles of $\mathcal{B}$: (a) Triangles whose convex hull contains τ. (b) Triangles containing p in their convex hulls and with exactly one edge in $\Omega(\tau) \setminus I$. (c) Triangles not containing p in their convex hulls and with exactly one edge in $\Omega(\tau) \cap I$. We claim that $\mathcal{C}(I)$ is a virtual chamber, i.e. that it contains one triangle from each triangulation T of $\mathcal{B}$.

If no edge of a triangulation T overlaps τ, then the triangle of T containing τ in its convex hull is the only triangle of T in $\mathcal{C}(I)$. Otherwise, since τ is empty and admissible with respect to p, the collection of edges of T which overlap τ forms a a strong chain $l_1 <_p \cdots <_p l_k$. Observe that there is a unique triangle σ of T containing p in its convex hull and overlapping τ, that σ must have a unique edge in $\Omega(\tau)$ and that such an edge must be l_1. If $l_1 \notin I$ then σ is the only triangle of T in $\mathcal{C}(I)$. Finally, if $l_1 \in I$ then let i be the biggest index such that $l_i \in I$. Clearly, the triangle of T incident to l_i on the side opposite to p is the only triangle of T in $\mathcal{C}(I)$.

The fact that the correspondences $\mathcal{C} \mapsto I(\mathcal{C})$ and $I \mapsto \mathcal{C}(I)$ are inverse of each other is straightforward. $\square$

Remark 1.5 As a conclusion, there is a bijection between ideals in the poset $(\Omega(\tau), <_p)$ and triangulations of $\mathcal{A}$ using the simplex $\mathcal{A} \setminus \tau$. We will not work out the details, but this bijection extends to a bijection between *polyhedral subdivisions* of $\mathcal{A}$ that use $\mathcal{A} \setminus \tau$ and *pairs of ideals* $I_1 \subset I_2$ such that $I_2 \setminus I_1$ is an antichain.

Remark 1.6 Proposition 1.4 translates the problem of counting virtual chambers in an admissible triangle to counting ideals in a certain poset. This will suffice for counting triangulations of corank 3 cyclic polytopes since, as we will see, their Gale transforms can be triangulated with admissible triangles.

But, in fact, the technique can be modified to deal with non-admissible triangles as well. Let τ be a non-admissible triangle in a rank 3 configuration $\mathcal{B}$. We can assume that τ is empty, otherwise we triangulate its convex hull by empty triangles. Let p be a vertex of τ and let $l = \{q, r\}$ be the opposite edge. We consider the ordered sequence (from q to r) of edges $\{p, p_i\}$, $i = 2, \ldots, k$ which overlap τ. Set $p_1 := q$ and $p_{k+1} := r$. The triangles

$\tau_i := \{p, p_i, p_{i+1}\}$, $i = 1, \dots, k-1$ are admissible with respect to p, they intersect properly and, since τ is empty, they cover τ. In rank 3, any set of triangles which intersect properly can be completed to a triangulation, which implies that each virtual chamber of $\mathcal{B}$ lying on τ lies on exactly one of the triangles τ_i. Moreover, the bijection between virtual chambers of $\mathcal{B}$ lying on τ_i and ideals of $(\Omega(\tau_i), <_p)$ restricts to a bijection between virtual chambers lying both on τ_i and τ and ideals of $(\Omega(\tau_i), <_p)$ not containing l. Summing up, we can count virtual chambers in τ by adding the numbers of ideals of $(\Omega(\tau_i), <_p)$ which do not contain the edge l. An analogous generalization can be done for the case of subdivisions and pairs of ideals mentioned in the previous remark.

2 Counting virtual chambers of $C(n, n-4)^*$

Combinatorial structure of $C(n, n-4)^*$

A sign sequence of length n is any element of $\{-1, 0, +1\}^n$. The support of a sign sequence is its set of non-zero coordinates. Recall that in oriented matroid theory the *circuits* of a vector configuration $\mathcal{A}$ are the sign sequences with minimal support produced by the coefficients of linear dependences in $\mathcal{A}$, and the cocircuits of $\mathcal{A}$ are the sign sequences with minimal support produced by the values of non-zero linear functionals on $\mathcal{A}$. Either circuits or cocircuits suffice to characterize the oriented matroid of $\mathcal{A}$ and two oriented matroids are dual to each other if and only if the circuits of one are the cocircuits of the other.

Let $C(n, n-4) = \{a_1, \dots, a_n\}$ denote the vertex set of a cyclic polytope of dimension $n-4$ with n vertices. Let $p_1, \dots, p_n$ be points in a non-great circle γ in S^2, taken in order along the circle. Let $C(n, n-4)^* := \{b_1, \dots, b_n\}$, where $b_i := (-1)^i p_i$, for $i = 1, \dots, n$. Part (ii) of the following statement appears in [19, Exercise 6.13], and part (i) is essentially Gale evenness criterion for cyclic polytopes.

Lemma 2.1 *(i) For each quadruple $\{b_{i_1}, b_{i_2}, b_{i_3}, b_{i_4}\}$ with $i_1 < i_2 < i_3 < i_4$ the signs sign$(b_{i_j}) = (-1)^{i_j + j}$, $j \in \{1, 2, 3, 4\}$ give one of the two (opposite) circuits with support in that quadruple.*

(ii) The oriented matroids of $C(n, n-4)$ and $C(n, n-4)^*$ are dual to each other. □

Observe that, in general, $C(n, n-4)^*$ is not going to be a Gale transform of $C(n, n-4)$ according to our definition, in which the points of $C(n, n-4)$ are taken along the moment curve. But it has the same oriented matroid as a Gale transform, and hence the same collection of triangulations and virtual chambers.

To simplify the notation, from now on we will refer to each point $b_i \in C(n, n-4)^*$ by its label $i \in \{1, \ldots, n\}$. $C(n, n-4)^*$ is contained in two opposite circles in S^2 one with the set of "odd points" $\{1, 3, 5, \ldots\}$ and the other with the "even points" $\{2, 4, 6, \ldots\}$. The points in each circle define a spherical polygon, so that the sphere is divided into two polygons (*odd* and *even*) plus a topological band between them.

A triangle τ of $C(n, n-4)^*$ will be said to lie on one of the polygons if its convex hull is contained in that polygon and will be said to lie on the band if its convex hull is contained in the band.

Lemma 2.2 *(i) An edge in the boundary of one of the polygons is crossed by no other edge of $C(n, n-4)^*$. In particular, it overlaps no empty triangle.*

(ii) Every triangulation of $C(n, n-4)^*$ by empty triangles uses all the boundary edges of the two polygons. Hence, all its triangles lie either on one of the polygons or on the band.

(iii) A triangle lies on one of the polygons if and only if its three vertices are in that polygon.

(iv) For a triangle τ which has two vertices in one polygon and the third in the other polygon the following conditions are equivalent:

(a) τ is admissible with respect to this third vertex.

(b) τ is empty.

(c) τ lies on the band.

(d) τ has one edge in the boundary of one of the polygons and τ is not one of the following triangles: $\{i, i+1, i+2\}$ with $i \in \{1, \ldots, n-2\}$ or, if n is even, $\{n, 1, 2\}$ or $\{n-1, n, 1\}$.

Proof: That an edge l in the boundary of the polygons does not cross any other edge can be proved geometrically. Or it can be derived from Lemma 2.1 that l cannot be the positive part of a circuit of $C(n, n-4)^*$. The second part of part (i) holds because any edge overlapping an empty triangle must cross at least one of the edges of the triangle.

Part (ii) is a consequence of part (i): let l be a boundary edge of one of the polygons and let x be a point in the relative interior of l. If T is a triangulation by empty simplices, x must lie on the relative interior of a unique simplex of T. This simplex cannot be a triangle or an edge other than l itself, by part (i).

Part (iii) is trivial. In part (iv), the equivalence of (d) with any of (a), (b) or (c) is easy to establish: let l be the edge of τ having its two vertices in the same polygon. For τ to be either empty, admissible or to lie on the band it is clearly necessary that l be an edge of that polygon. The converse is true unless τ is one of the triangles excluded in part (d), whose convex hull contains the whole polygon. $\square$

Let $\mathcal{C}$ be a virtual chamber and let T be a triangulation of $C(n, n-4)^*$ all of whose triangles lie either on one of the polygons or on the band (we will see later that such triangulations exist). We say that $\mathcal{C}$ lies on the odd polygon, on the even polygon or on the band if it lies on a triangle of T on the odd polygon, the even polygon, or the band, respectively. According to part (ii) of Lemma 1.3 applied to the two polygons this definition is independent of T. In what follows we will count the number of virtual chambers on each polygon and on the band.

Virtual chambers of a polygon

Let M be the configuration consisting of the vertices of one of the two polygons of $C(n, n-4)^*$.

Lemma 2.3 *There is a natural bijection between virtual chambers of $C(n, n-4)^*$ lying on M and virtual chambers of M (as a configuration by itself).*

Proof: By part (i) of Lemma 2.2, every triangle of M containing an edge in the boundary of M is admissible with respect to the opposite vertex, both in M and in $C(n, n-4)^*$. Clearly, M can be triangulated with triangles of that type: take any point p in M and triangulate M by coning p to every boundary edge of M not containing p.

Moreover, only edges in M overlap $\operatorname{conv}(M)$ and, hence, for those triangles, the poset $\Omega(\tau)$ is the same in M and in $C(n, n-4)^*$. By Proposition 1.4 the number of virtual chambers of M and of $C(n, n-4)^*$ lying on each of those triangles is the same. $\square$

Lemma 2.3 allows us to forget $C(n, n-4)^*$ for a while and speak rather of a polygon P whose vertices are labelled from 1 to m. We will compute the number of virtual chambers in P by adding the virtual chambers which lie on the triangles $\tau_i := \{1, i-1, i\}$ for $i = 3, \dots, m$ since these triangles are admissible with respect to 1 and define a triangulation of P. In $\Omega(\tau_i)$ we consider the ordering $<_1$ of "being closer to 1". Our task is to count ideals of $(\Omega(\tau_i), <_1)$.

Note that

$$\Omega(\tau_i) = \{\{j, k\} \ : \ 2 \leq j \leq i-1, i \leq k \leq m\} \setminus \{\{i-1, i\}\},$$

with the partial order $\{j, k\} \leq_1 \{j', k'\}$ if and only if $j \leq j'$ and $k \geq k'$. We extend $\Omega(\tau_i)$ to a larger poset $\widehat{\Omega(\tau_i)} := \Omega(\tau_i) \cup \{\{i-1, i\}\}$ by setting $\{i-1, i\}$ as the maximum of $\widehat{\Omega(\tau_i)}$. The ideals of $\widehat{\Omega(\tau_i)}$ are those of $\Omega(\tau_i)$ plus $\widehat{\Omega(\tau_i)}$ itself. The Hasse diagram of $\widehat{\Omega(\tau_i)}$ is shown in part (a) of Figure 1.

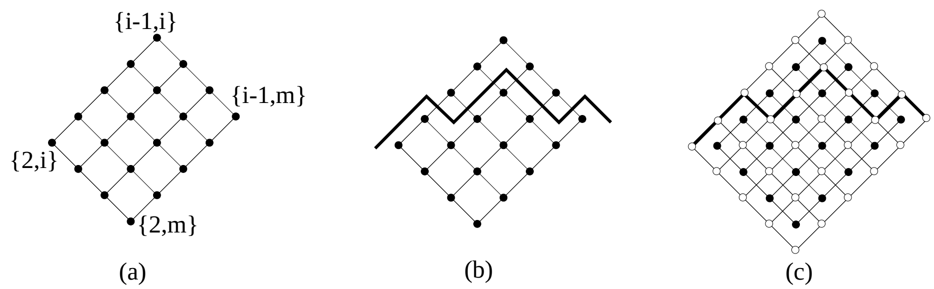

Figure 1: Part (a) is the Hasse diagram of the poset $\widehat{\Omega(\tau_i)}$. Part (b) represents an ideal of $\widehat{\Omega(\tau_i)}$ as a path through the edges of its Hasse diagram (black dots). The ideal is the set of elements below the path. It corresponds to a path through the vertices of the "dual diagram" (white dots) represented in (c).

Proposition 2.4 *(i) The poset $\Omega(\tau_i)$ has $\binom{m-1}{i-2} - 1$ ideals.*

(ii) A polygon with m vertices has $2^{m-1} - m$ virtual chambers.

Proof: Ideals in $\widehat{\Omega(\tau_i)}$ are in bijection with maximal left-to-right monotone paths through the lattice points in a $(i-2) \times (m-i+1)$ rectangle as shown in Figure 1. The number of such paths is $\binom{(m-i+1)+(i-2)}{i-2} = \binom{m-1}{i-2}$. This proves part (i). For part (ii), add the virtual chambers in each triangle τ_i, i.e., the number of ideals in each poset $\Omega(\tau_i)$. □

The number $2^{m-1} - m$ equals the number of *maximal straightline thrackles* with vertices in a convex m-gon, computed in [9]. There is actually a bijection between these thrackles and virtual chambers in the polygon, in which edges of the thrackle correspond to "flippable edges" of the virtual chamber.

Virtual chambers on the band: the even case

We now assume n to be even and let $m = n/2$. All indices will be regarded modulo n. Let T_{even} be the following set of triangles, all lying on the band according to part (iv) of Lemma 2.2.

$$T_{even} = \{\{2i+1, 2i+2, 2i+4\}, \{2i+1, 2i+3, 2i+4\} : 0 \le i \le m-1\}.$$

These triangles form a triangulation of the band, by which we mean that adding to them triangulations of the odd and even polygons we get a triangulation of $C(n, n-4)^*$.

Figure 2 displays T_{even}. In this figure and the subsequent ones, the following flattened, twisted, planar representation of the band is used: the odd and even points are placed on two parallel lines in the plane, with $2i-1$

and $2i$ facing each other. The sequence of points is meant to be repeated infinitely, or the left and right ends of the figure be identified. For a given odd and a given even vertex there are different ways to draw a straight line segment joining them. We choose to take the one of greatest positive slope, considering a vertical line as having infinite positive slope. With this choice, two edges cross in the representation if and only if they cross in $C(n, n-4)^*$.

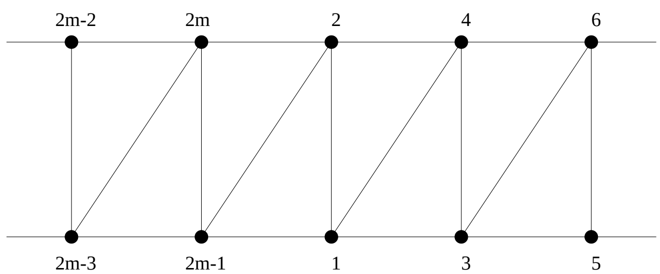

Figure 2: The set of triangles T_{even} in our twisted representation of the band.

Since T_{even} is a "triangulation of the band", in order to count the virtual chambers in the band we just add the virtual chambers in the triangles of T_{even}. Due to the simmetry of T_{even}, this gives the same result as multiplying by $2m$ the number of virtual chambers in $\{1, 2, 4\}$.

As stated in Lemma 2.2, the triangle $\tau = \{1, 2, 4\}$ is admissible with respect to the vertex 1. Therefore, our task is to compute the number of ideals of $(\Omega(\tau), <_1)$. Figure 3 represents the triangle τ (with $m = 5$) and the edges in $\Omega(\tau)$. These edges are those crossing either $\{1, 2\}$ or $\{1, 4\}$ which, by Lemma 2.1, are respectively $\{\{2i, 2j+1\} : 2 \leq i \leq j \leq m-1\}$ and $\{\{2i, 2j+1\} : 3 \leq i \leq j \leq m-1\}$. Since the second set is contained in the first, we have

$$\Omega(\tau) = \{\{2i, 2j+1\} \ : \ 2 \leq i \leq j \leq m-1\}$$

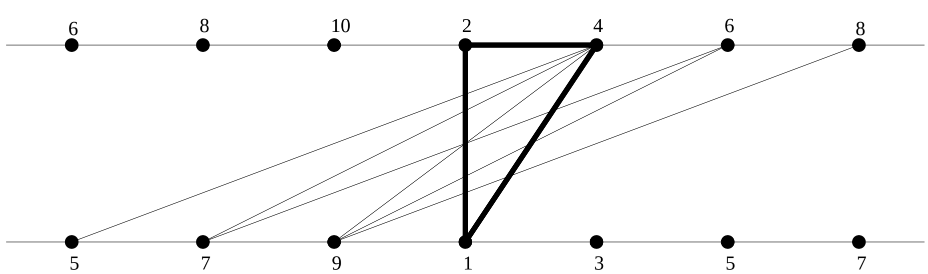

Figure 3: All the edges which cross $\{1, 2, 4\}$ for $m = 5$.

The poset structure in $\Omega(\tau)$ is that $(2i, 2j+1) \leq_1 (2i', 2j'+1)$ if and only if $i \geq i'$ and $j \geq j'$. Hence, the Hasse diagram of $(\Omega(\tau), <_1)$ is as depicted in part (a) of Figure 4.

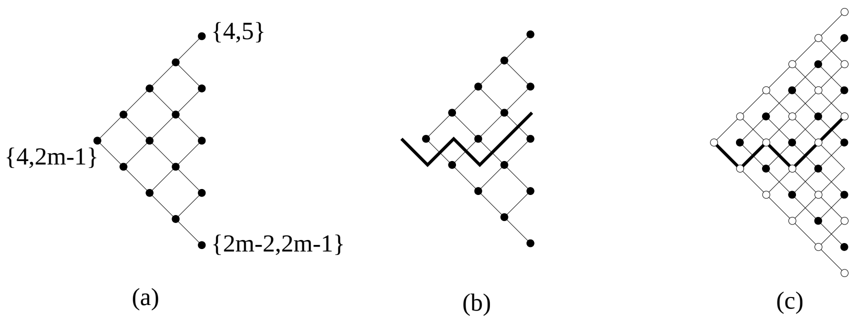

Figure 4: Part (a) is the Hasse diagram of $\Omega(\tau)$. Parts (b) and (c) represent the same as in figure 1 but for this new poset.

Proposition 2.5 *(i) The number of virtual chambers lying on the triangle $\{1,2,4\}$ of $C(2m, 2m-4)^*$ is 2^{m-2}.*

(ii) The number of virtual chambers of $C(2m, 2m-4)^*$ lying on the band is $m2^{m-1}$.

(iii) The total number of virtual chambers of $C(2m, 2m-4)^*$ is $(m+2)2^{m-1} - 2m$.

Proof: Let $\tau = \{1,2,4\}$. Ideals of $(\Omega(\tau), <_1)$ are in bijection with maximal left-to-right monotone paths in the "dual diagram" shown in part (c) of Figure 4. These, in turn, are in bijection with maximal monotone paths in the complete binary tree of depth $m-2$. This proves part (i) and the symmetry remarks stated above prove part (ii).

For part (iii) we have to add the $m2^{m-1}$ virtual chambers on the band to the $2^{m-1}-m$ on each of the two polygons, which gives the stated number. □

Theorem 2.6 *The cyclic polytope $C(2m, 2m-4)$ has $(m+2)2^{m-1} - 2m$ triangulations.* □

Virtual chambers on the band: the odd case

The configuration $C(2m-1, 2m-5)^*$ (up to oriented matroid equivalence) can be obtained from $C(2m, 2m-4)^*$ by deleting the element $2m$. We intend to apply the same technique to $C(2m-1, 2m-5)^*$ as in the even case. We start by choosing a triangulation of the band (see Figure 5):

$$T_{odd} = \{\{2i+1, 2i+2, 2i+4\}, \{2i+1, 2i+3, 2i+4\} : 0 \le i \le m-3\} \cup$$

$$\cup \{\{2m-3, 2m-2, 2\}, \{2m-3, 2m-1, 2\}, \{2m-1, 1, 2\}\}$$

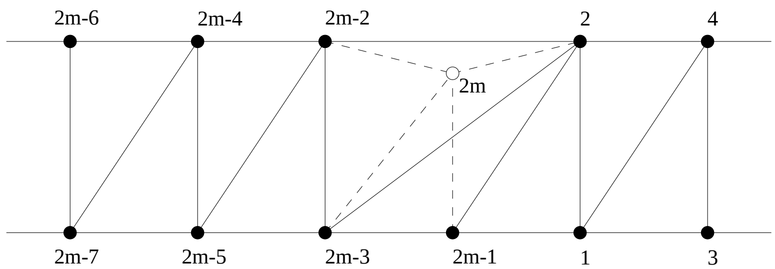

Figure 5: The set of triangles T_{odd}. The deleted vertex $2m$ has been drawn in white. Dashed lines show the edges of T_{even} which were incident to $2m$.

The triangles of T_{odd} are all admissible (this is automatic, by Lemma 2.2(iv)). Hence, we can compute the number of virtual chambers in each triangle of T_{odd} using Proposition 1.4. The only new difficulty with respect to the even case is that T_{odd} is not preserved under any non-trivial oriented matroid symmetry of $C(2m-1, 2m-5)^*$. In fact, $C(2m-1, 2m-5)^*$ has only one non-trivial symmetry: the reversal of indices.

Lemma 2.7 *The number of virtual chambers of $C(2m-1, 2m-5)^*$ in each triangle of T_{odd} is:*

(i) $2^{m-2} - \sum_{k=0}^{i-1} \binom{m-3}{k}$ *for* $\{2i+1, 2i+2, 2i+4\}$*, with* $i \in \{0, \dots, m-3\}$.

(ii) $2^{m-2} - \sum_{k=0}^{i} \binom{m-3}{k}$ *for* $\{2i+1, 2i+3, 2i+4\}$*, with* $i \in \{0, \dots, m-3\}$.

(iii) 2^{m-2} *for* $\{2m-1, 1, 2\}$.

(iv) 2^{m-3} *for* $\{2m-3, 2m-2, 2\}$ *and for* $\{2m-3, 2m-1, 2\}$.

Proof: $C(2m-1, 2m-5)^*$ can be obtained from $C(2m, 2m-4)^*$ by deleting any element. Triangles of parts (i), (ii) and (iii) belong to T_{even}, so we can assume them to be $\{1, 2, 4\}$ as long as we choose the appropriate vertex to be removed (instead of $2m$). That is, we remove the element $2m-2i$ for (i), $2i+5$ for (ii) and 3 for (iii). In each case, the poset of edges crossing $\{1, 2, 4\}$ is the same as in the even case, except for the edges using the removed element, which are missing.

For part (i), the Hasse diagram is in fact the same one obtained by removing the edges $\{2j, 2j+1\}$ for $m-i \le j \le m-1$, as shown in Figure 6. Consider the vertices of its dual diagram. Some of the ones which would have two rightwards neighbours in the even case, now have one or none. We label them from 0 to $i-1$, as shown in Figure 6(c). For each path ending at one of the labelled vertices there would be two ways of extending it rightwards in the even case. Thus now the number of paths is obtained by subtracting to 2^{m-2} the number of paths which end at each labelled vertex.

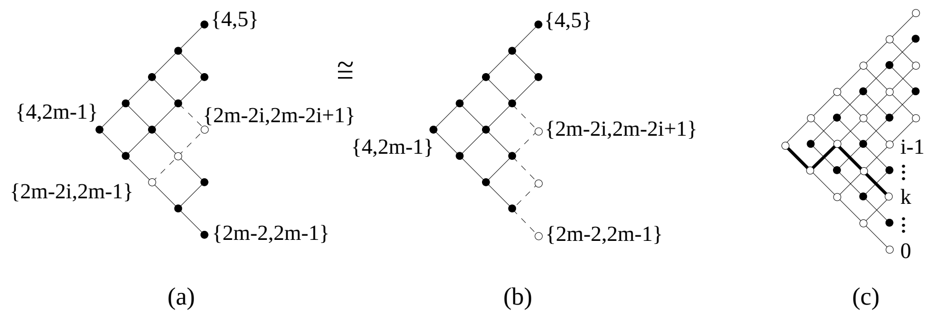

Figure 6: Part (a) shows the Hasse diagram of $\Omega(\tau)$ for the triangles of case (i) (in black) together with the missing elements with respect to the even case (in white). This poset is isomorphic to the one obtained by removing the elements $\{2j, 2j+1\}$, for $m-i \leq j \leq m-1$, represented in part (b). Ideals of the poset are in bijection with monotone left-to-right paths through the white dots in (c). Some of these dots, which we label from 0 to $i-1$, had two rightwards neighbours in the even case, while now they have one or none.

For the k-th vertex this number is $\binom{m-3}{k}$. This finishes part (i). Parts (ii) and (iii) are analogous.

For part (iv), let $\tau_1 = \{2m-3, 2m-2, 2\}$. The edges overlapping τ_1 are exactly those crossing the edge $\{2m-3, 2m-2\}$. By Lemma 2.1, the edges of $C(2m-1, 2m-5)^*$ crossing $\{2m-3, 2m-2\}$ are those of the form $\{2i, 2j+1\}$ with $1 \leq i \leq j \leq m-3$. The order is $\{2i, 2j+1\} \leq_{2m-3} \{2i', 2j'+1\}$ if and only if $i \geq i'$ and $j \geq j'$. This is the same poset we would have for the triangle $\{1,2,4\}$ in $C(2m-2, 2m-6)^*$, except for a shift of two units in all the indices. Hence, the number of ideals is 2^{m-3}. $\square$

Proposition 2.8 *The number of virtual chambers of $C(2m-1, 2m-5)^*$ lying on triangles of T_{odd} is $(3m-2)2^{m-3}$.*

Proof: By Lemma 2.7 the number N of virtual chambers in T_{odd} is

$$\sum_{i=0}^{m-3}\left(2^{m-2} + 2^{m-2} - \sum_{k=0}^{i}\left(\binom{m-3}{k} + \binom{m-3}{k-1}\right)\right) + 2^{m-2} + 2\cdot 2^{m-3} =$$

$$= (4m-4)2^{m-3} - \sum_{i=0}^{m-3}\sum_{k=0}^{i}\binom{m-2}{k}$$

If we call

$$A_i = \sum_{k=0}^{i}\binom{m-2}{k}$$

we have $2^{m-2} = A_i + A_{m-i-3}$ which implies $(m-2)2^{m-2} = 2(\sum_{i=0}^{m-3} A_i)$. Hence

$$N = (4m-4)2^{m-3} - (m-2)2^{m-3} = (3m-2)2^{m-3}.$$

$\square$

Theorem 2.9 *The number of triangulations of the cyclic polytope $C(2m-1, 2m-5)$ is $(3m+4)2^{m-3} - (2m-1)$.*

Proof: To the number obtained in Proposition 2.8 we have to add the numbers of virtual chambers of the two polygons defined by $C(2m-1, 2m-5)^*$, which have m and $m-1$ vertices, respectively. By Proposition 2.4, these numbers are $2^{m-1} - m$ and $2^{m-2} - (m-1)$, respectively. $\square$

3 Triangulations of $C(2m, 2m-4)$ which use a fixed simplex

In this section we will count the number of virtual chambers in any particular triangle of $C(2m, 2m-4)^*$, although the technique will still use some particular properties of this configuration rather than the general method outlined in Remark 1.6. One can do analogous calculations in the case of $C(2m-1, 2m-5)^*$ but there are many more cases to be studied due to much less symmetry, so we prefer to restrict our study to the even case.

We first prove two additional results:

Lemma 3.1 *Let M be a convex m-gon. Let S be a subconfiguration consisting of k consecutive vertices of M. Then, the number of virtual chambers of M lying on S (i.e. lying on triangles of any triangulation of S) equals*

$$\sum_{l=0}^{k-2} \binom{m-1}{l} - (k-1).$$

Proof: Consider the vertices of M labelled from 1 to m, and without loss of generality suppose that $S = \{1, \dots k\}$. Then, the following is a triangulation of S:

$$\{\{1, l-1, l\} : l \in \{3, \dots, k\}\}.$$

By Proposition 2.4, the triangle $\{1, l-1, l\}$ contains $\binom{m-1}{l-2} - 1$ virtual chambers of M. Adding this number for l from 3 to k gives

$$\sum_{l=3}^{k} \binom{m-1}{l-2} - (k-2) = \sum_{l=1}^{k-2} \binom{m-1}{l} - (k-2)$$

which coincides with the number stated. $\square$

Lemma 3.2 *Let τ be any empty triangle in $C(2m, 2m-4)^*$ not contained in one of the polygons. Then, there are exactly 2^{m-2} virtual chambers of $C(2m, 2m-4)^*$ in τ.*

Proof: The subgroup of combinatorial symmetries of $C(2m, 2m-4)^*$ generated by $i \mapsto i+2$ and $i \mapsto 2m-i+1$ (which has $2m$ elements), applied to τ, produces $2m$ empty triangles not contained in either of the polygons, i.e. lying on the band by Lemma 2.2. It is easy to check that these $2m$ triangles form a triangulation of the band, very similar to the triangulation T_{even} depicted in Figure 2. By symmetry, all the $2m$ triangles contain the same number of virtual chambers, i.e. the $m2^{m-1}$ virtual chambers in the band divided by $2m$. □

Theorem 3.3 *Let τ be a triangle in $C(2m, 2m-4)^*$.*

(i) If τ is contained in one of the two polygons, let $\tau = \{i, j, k\}$ with $i < j < k$. The number of virtual chambers of $C(2m, 2m-4)^*$ lying on τ equals

$$2^{m-1} - \sum_{l=0}^{a} \binom{m-1}{l} - \sum_{l=0}^{b} \binom{m-1}{l} - \sum_{l=0}^{c} \binom{m-1}{l},$$

where a, b and c are the numbers of points in the polygon and between each two vertices of τ. I.e., $a = \frac{j-i}{2} - 1$, $b = \frac{k-j}{2} - 1$ and $c = \frac{2m+i-k}{2} - 1$.

(ii) If τ does not lie on either of the polygons, then there is a combinatorial symmetry of $C(2m, 2m-4)^*$ which sends τ to a triangle $\tau' = \{i, j, k\}$ with $i < j < k$ and with i odd and j and k even. Then the number of virtual chambers of $C(2m, 2m-4)^*$ lying on τ equals

$$\binom{\frac{k-j}{2}}{2} (2^{m-2} - 1) + \sum_{l=0}^{\frac{k-j}{2}-1} \binom{m-1}{l}$$

Proof: The case of τ lying on a polygon is easy in the light of Lemma 3.1: Joining to τ triangulations of three configurations as the one in Lemma 3.1 with the parameter k taking the values $a+2$, $b+2$ and $c+2$ produces a triangulation of the whole polygon, which has $2^{m-1} - m$ virtual chambers. Hence, the number of virtual chambers in τ equals

$$2^{m-1} - m - \sum_{l=0}^{a} \binom{m-1}{l} + (a+1) -$$

$$- \sum_{l=0}^{b} \binom{m-1}{l} + (b+1) - \sum_{l=0}^{c} \binom{m-1}{l} + (c+1),$$

as desired since $a+b+c=m-3$.

In part (ii), τ has two vertices in one polygon and the third vertex in the other polygon. The combinatorial symmetry of the statement can be taken as one of the two which send this third vertex to the vertex 1. Without loss of generality we assume in the rest of the proof that $\tau=\{i,j,k\}$ with i odd, j and k even and $i<j<k$.

A point l is in the interior of the triangle τ exactly when the circuit with support in $\{i,j,k,l\}$ has the same sign in i, j and k and the opposite sign in l. By Lemma 2.1 this happens if and only if l is even and between j and k. We consider the subconfiguration $S=\{i,j,j+2,j+4,\ldots,k\}$ consisting of the vertices of τ and its interior points. We can triangulate the part of $\operatorname{conv}(S)$ in the even polygon as in Lemma 3.1, which gives

$$\sum_{l=0}^{\frac{k-j}{2}-1}\binom{m-1}{l}-\frac{k-j}{2}$$

virtual chambers, and the part in the band with the $\frac{k-j}{2}$ empty triangles $\{\{i,l,l+2\}:l\in\{j,j+2,j+4,\ldots,k-2\}\}$, each containing 2^{m-2} virtual chambers by Lemma 3.2. □

Hence, the following table gives the number of triangulations of the cyclic polytope $C(2m,2m-4)$ which use the simplex $C(2m,2m-4)\setminus\{i,j,k\}$ under the assumption that $i<j<k$ and depending on the parities of $k-j$ and $j-i$. The first two rows are just the formulas in Theorem 3.3, where the value of a, b and c can be found. The last two rows are the translation of the second row to the case in which $k-j$ is odd.

$j-i$ and $k-j$ even	$2^{m-1}-\sum_{l=0}^{a}\binom{m-1}{l}-\sum_{l=0}^{b}\binom{m-1}{l}-\sum_{l=0}^{c}\binom{m-1}{l}$
$j-i$ odd, $k-j$ even	$\frac{k-j}{2}(2^{m-2}-1)+\sum_{l=0}^{\frac{k-j}{2}-1}\binom{m-1}{l}$
$j-i$ even, $k-j$ odd	$\frac{j-i}{2}(2^{m-2}-1)+\sum_{l=0}^{\frac{j-i}{2}-1}\binom{m-1}{l}$
$j-i$ odd, $k-j$ odd	$\frac{2m+i-k}{2}(2^{m-2}-1)+\sum_{l=0}^{\frac{2m+i-k}{2}-1}\binom{m-1}{l}$

4 Regular triangulations of $C(n,n-4)$

A *regular triangulation* of a d-dimensional point configuration $\mathcal{A}$ is a triangulation of $\mathcal{A}$ which can be obtained as the orthogonal projection of the lower envelope of a $(d+1)$-dimensional polytope (see [6] or [19] for details). The bijection between triangulations of $\mathcal{A}$ and virtual chambers of its Gale transform $\mathcal{B}$ sends the regular triangulations to the geometric chambers of $\mathcal{B}$, i.e. the full-dimensional cones in the chamber fan that we defined in the introduction.

The chamber fan of a configuration does not depend only on the oriented matroid. We intend here to compute the maximum possible number of chambers among all the coordinatizations of the oriented matroid of $C(n, n-4)^*$. That is to say, the maximum possible number of regular triangulations of a polytope realizing the oriented matroid of $C(n, n-4)$.

For the next lemma, we recall that in oriented matroid theory a vector configuration is called totally cyclic if its positive span is the whole space. The statement holds equally if $\mathcal{B}$ is not totally cyclic, except that the formula changes by 1 because of the use of Euler's formula for a ball instead of a sphere.

Lemma 4.1 *Let $\mathcal{B}$ be a rank 3 totally cyclic vector configuration in general position. Let N be the number of (opposite pairs of) circuits of $\mathcal{B}$ having two positive and two negative elements. Then, the maximum number of chambers produced by realizations of the oriented matroid of $\mathcal{B}$ is*

$$N + \binom{n}{2} - n + 2.$$

Moreover, the maximum is achieved in any realization in which no three edges cross in a common point. This happens if $\mathcal{B}$ is sufficiently generic among the realizations of its oriented matroid.

Proof: We first prove that there is a realization of the oriented matroid of $\mathcal{B}$ in which no three edges have a common crossing. Indeed, let k be the number of triplets of edges crossing in a point. If $k \leq 1$ then a slight perturbation of the coordinates of one of the six vertices involved in a triple crossing decreases the number of such crossings, and does not change the oriented matroid because of our general position assumption. On the other hand, if $k = 0$ then sufficiently small perturbations cannot create triple crossings. This proves the assertion.

Next we will prove that if $\mathcal{B}$ has no triple crossings then it has exactly the stated number of chambers. This, together with the fact that small perturbations cannot decrease the number of chambers, implies that the stated number is indeed the maximum.

Embedding $\mathcal{B}$ in the 2-sphere as we have done in this paper, the chamber complex of $\mathcal{B}$ (i.e. the intersection of the chamber fan with the unit sphere) is a polyhedral subdivision of the sphere whose numbers of cells of dimensions 2, 1 and 0 we denote f_2, f_1 and f_0. The number of chambers equals f_2. The number f_0 equals n plus the number of crossing points between edges of $\mathcal{B}$, which under the assumption of no triple crossings equals N.

$$f_0 = N + n.$$

On the other hand, $2f_1$ equals the number of incidences between 0-cells and 1-cells in the cell decomposition. That is to say,

$$2f_1 = 4N + n(n-1)$$

where the term $n(n-1)$ comes from the fact that $n-1$ edges are incident at each point of $\mathcal{B}$. By Euler's formula for the 2-sphere,

$$f_2 = f_1 - f_0 + 2 = N - n + \binom{n}{2} + 2. \qquad \square$$

Proposition 4.2 *The number of (pairs of) circuits with two elements of each sign in $C(n, n-4)^*$ equals*

- $6\binom{m}{4} + 3\binom{m}{3}$ *if $n = 2m$ is even, and*
- $6\binom{m}{4} + \binom{m}{2} - m + 1$ *if $n = 2m - 1$ is odd.*

Proof: Suppose first that $n = 2m$ is even. We know that no edge of $C(2m, 2m-4)^*$ crosses the boundary of any of the two polygons defined by $C(2m, 2m-4)^*$, so if two edges cross, then either both are edges of the band or both are edges of one of the polygons. Let B and P be the numbers of crossings of edges of the band and of the even polygon, respectively. Then

$$N = B + 2P.$$

The number P is the number of crossings between edges of an m-gon, but any four vertices of an m-gon define a unique crossing, so

$$P = \binom{m}{4}.$$

For computing the number B, we first compute the number of edges of $C(2m, 2m-4)^*$ crossing a certain edge $\{a, b\}$ of the band. Let us assume $a = 1$ and, hence, $b = 2j$ is even, in order that $\{a, b\}$ be in the band. Let $\{a', b'\}$ be another edge in the band, and assume that a' is odd and b' is even. According to Lemma 2.1, under these assumptions $\{1, b\}$ and $\{a', b'\}$ cross each other if and only if $1 < a' < b' < b$ or $1 < b < b' < a'$. Taking into account the parities of a', b' and b, the first case gives $\binom{j-1}{2}$ possibilities and the second gives $\binom{m-j}{2}$ (for the first number, observe for example that each pair of indices $1 \leq i' < j' \leq j-1$ gives the edge having $a' = 2i' + 1$ and $b' = 2j'$).

Adding up these numbers for $j \in \{1, \ldots, m\}$ we conclude that the total number of crossings between edges of the band one of which contains the point 1 equals

$$\sum_{j=1}^{m} \binom{j-1}{2} + \sum_{j=1}^{m} \binom{m-j}{2} = \binom{m}{3} + \binom{m}{3} = 2\binom{m}{3}$$

Now, by the symmetries of $C(2m, 2m-4)^*$ the same is valid for any other vertex: the number of crossings in the band between edges one of which contains any specific vertex is $2\binom{m}{3}$. Since each crossing uses 4 vertices, the number of crossings in the band equals

$$B = \frac{2m}{4} 2\binom{m}{3} = m\binom{m}{3} = m\binom{m-1}{3} + m\binom{m-1}{2} = 4\binom{m}{4} + 3\binom{m}{3}.$$

Hence,

$$N = B + 2P = 6\binom{m}{4} + 3\binom{m}{3}.$$

For the odd case, remember that $C(2m-1, 2m-5)^*$ can be obtained from $C(2m, 2m-4)^*$ by deleting the point $2m$. Then, the number of crossings in $C(2m-1, 2m-5)^*$ equals the total number of crossings in $C(2m, 2m-4)^*$ minus the crossings involving the vertex $2m$. This number is $2\binom{m}{3}$ crossings in the band plus $\binom{m-1}{3}$ crossings in the even polygon. Hence, in the odd case we have

$$N = 6\binom{m}{4} + 3\binom{m}{3} - 2\binom{m}{3} - \binom{m-1}{3} = 6\binom{m}{4} + \binom{m}{2} - m + 1.$$

$\square$

Theorem 4.3 *The number of regular triangulations of $C(n, n-4)$ is at most:*

(i) $6\binom{m}{4} + 3\binom{m}{3} + 4\binom{m}{2} - m + 2$ *if $n = 2m$ for some positive integer m.*

(ii) $6\binom{m}{4} + 5\binom{m}{2} - 4m + 5$ *if $n = 2m-1$ for some positive integer m.*

Moreover, these formulas give the exact number of regular triangulations in any sufficiently generic coordinatization of the oriented matroid of $C(n, n-4)$.

Proof: This is straightforward from Lemma 4.1 and Proposition 4.2, taking into account that $\binom{n}{2} - n + 2 = \frac{n^2-3n+4}{2}$ which if $n = 2m$ gives

$$2m^2 - 3m + 2 = 4\binom{m}{2} - m + 2$$

and if $n = 2m-1$ gives

$$\frac{4m^2 - 4m + 1 - 6m + 3 + 4}{2} = 2m^2 - 5m + 4 = 4\binom{m}{2} - 3m + 4.$$

$\square$

Remark 4.4 One may ask about the minimal, instead of maximal, number of regular triangulations in realizations of the oriented matroid of $C(n, n-4)$. This would correspond to computing the number of chambers in the "least generic" realization of the dual oriented matroid. It is reasonable to think that, if $n = 2m$ is even, the realization in which each half of the points form a regular m-gon is a good approximation of this "least-generic" case. The number of chambers in a regular m-gon has been computed in [12]. The result is $\frac{m^4}{24} \pm \Theta(m^3)$, exactly as in the most-generic case. This leads to the conjecture that the number of regular triangulations in every realization of $C(n, n-4)$ is $\frac{n^4}{64} \pm \Theta(n^3)$, as in the generic case.

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