ON THE TOPOLOGY OF THE BAUES POSET OF POLYHEDRAL SUBDIVISIONS

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Abstract. Given an affine projection $\pi : P \to Q$ of convex polytopes, let $\omega(P, \pi)$ be the refinement poset of proper polyhedral subdivisions of Q which are induced by π, in the sense of Billera and Sturmfels. Let $\omega_{\mathrm{coh}}(P, \pi)$ be the spherical subposet of π-coherent subdivisions. It is proved here that the inclusion of the latter poset into the former induces injections in homology and homotopy. In particular, the poset $\omega(P, \pi)$ is homotopically nontrivial. As a corollary, the equivalence of the weak and strong forms of the generalized Baues problem of Billera, Kapranov and Sturmfels is established. As special cases, these results apply to the refinement poset of proper polyhedral subdivisions of a point configuration and to the extension poset of a realizable oriented matroid.

1. Introduction

Let $\pi : P \to Q$ be an affine projection of convex polytopes. The Baues poset $\omega(P, \pi)$ is a combinatorial model for the space of all continuous sections of π which lie in the boundary of P. It arose in the theory of fiber polytopes of Billera and Sturmfels [7]. Roughly speaking, $\omega(P, \pi)$ consists of the non-trivial polyhedral subdivisions of Q whose cells are projections of faces of P and is partially ordered by refinement. It can also be described as the set of face bundles of π which are "locally π-coherent". For a precise definition see Section 2.2.

Baues posets provide a unified approach to the study of various objects of interest in polyhedral combinatorics, the most important arising in the following three special cases; see [19] and [24, Lecture 9]. When P is a simplex, $\omega(P, \pi)$ is the poset of all (non-trivial) polyhedral subdivisions of the point configuration $\pi(\mathrm{vertices}(P))$. When P is a cube, it is the poset of all zonotopal tilings of the zonotope Q. When $\dim(Q) = 1$, it is the poset of all cellular strings on the polytope P in the direction orthogonal to the fibers of the projection. In particular, the connectivity of $\omega(P, \pi)$ when P is a simplex or a cube is closely related to the connectivity of the set of triangulations of a point configuration by geometric bistellar operations [14] or that of cubical tilings of a zonotope by mutations [12].

The topology of the poset $\omega(P, \pi)$ (see Section 2.1) will be the object of study in this paper. The special case in which P is a zonotope and $\dim(Q) = 1$ appeared in the work of Baues in the theory of loop spaces [6], where the poset was conjectured to have the homotopy type of a sphere of dimension $\dim(P) - 2$. The proof of this conjecture, as well as the fact that $\omega(P, \pi)$ always contains a geometrically defined subposet $\omega_{\mathrm{coh}}(P, \pi)$ which is homeomorphic to a sphere of dimension

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$\dim(P) - \dim(Q) - 1$ (see Section 2.3), led Billera et al. [10] to formulate the *generalized Baues problem.* This problem asks to determine whether the subposet $\omega_{\text{coh}}(P,\pi)$ is a strong deformation retract of $\omega(P,\pi)$ (strong GBP) or whether, at least, $\omega(P,\pi)$ has the homotopy type of a sphere of dimension $\dim(P)-\dim(Q)-1$ (weak GBP). Examples in which $\omega(P,\pi)$ is disconnected have been constructed explicitly with $\dim(P) - \dim(Q) = 3$ [18] and probably exist if P is a simplex of sufficiently large dimension [21]. On the other hand, the (strong) GBP was resolved affirmatively when $\dim(Q) = 1$ in [10] and when $\dim(P) - \dim(Q) \leq 2$ in [18]. In many other positive results the proofs available establish, in principle, only the weak version. For instance, this has been the case when P is either a cube or a simplex and $\dim(Q) \leq 2$ or $\dim(P) - \dim(Q) \leq 3$ [5, 13, 23], when $\dim(Q) = 2$ and π(vertices(P)) is contained in the boundary of Q [1] and when π is the natural projection between two cyclic polytopes [4, 17] or cyclic zonotopes [2, 23]. See [19, Section 4] for an overview of recent results.

Our main result clarifies the topological relation between $\omega(P,\pi)$ and its spherical subposet $\omega_{\text{coh}}(P,\pi)$. It implies that the inclusion of $\omega_{\text{coh}}(P,\pi)$ into $\omega(P,\pi)$ induces injections in homology and homotopy (in particular that $\omega(P,\pi)$ is always homotopically nontrivial) and that the weak and strong forms of the generalized Baues problem are equivalent. The former statement gives the first result on the topology of the Baues poset since [7, 10] valid for all projections of polytopes.

Our starting point is the fact that the fiber $\pi^{-1}(x)$ of π over a point x in the relative interior of Q is a polytope of dimension $\dim(P) - \dim(Q)$.

Theorem 1.1. *Let $\pi : P \to Q$ be an affine projection of polytopes, x be a generic point in the relative interior of Q and let ω_x denote the poset of proper faces of the fiber $\pi^{-1}(x)$.*

The map which sends a subdivision $S \in \omega(P,\pi)$ to its unique cell containing x induces an order preserving map $f : \omega(P,\pi) \to \omega_x$ such that the composition

$$\omega_{\text{coh}}(P,\pi) \xrightarrow{\iota} \omega(P,\pi) \xrightarrow{f} \omega_x$$

of f with the inclusion map $\iota : \omega_{\text{coh}}(P,\pi) \to \omega(P,\pi)$ is a homotopy equivalence between $\omega_{\text{coh}}(P,\pi)$ and ω_x.

We will define the map f more carefully in Section 3 for any point x in the relative interior of Q and prove the statement of the previous theorem under this assumption. The map f (described differently) has been used several times in the literature to prove the weak GBP in special instances; see Remark 3.3.

We now list the main consequences of Theorem 1.1.

Corollary 1.2. *Let $\pi : P \to Q$ be an affine projection of polytopes of dimensions d' and d, respectively, and let $\iota : \omega_{\text{coh}}(P,\pi) \to \omega(P,\pi)$ and $f : \omega(P,\pi) \to \omega_x$ be as in Theorem 1.1. We have:*

(i) *The map ι induces injections of the homology and homotopy groups (with arbitrary coefficients) of the $(d'-d-1)$-sphere $\omega_{\text{coh}}(P,\pi)$ into the corresponding groups of $\omega(P,\pi)$.*
(ii) *The map f induces surjections of the homology and homotopy groups (with arbitrary coefficients) of $\omega(P,\pi)$ onto the corresponding groups of the $(d'-d-1)$-sphere ω_x.*
(iii) *The Baues poset $\omega(P,\pi)$ is not contractible.*

(iv) *The following statements are equivalent: (a) $\omega(P,\pi)$ is homotopy equivalent to the sphere $S^{d'-d-1}$; (b) ι is a homotopy equivalence; (c) f is a homotopy equivalence and (d) $\omega_{\mathrm{coh}}(P,\pi)$ is a strong deformation retract of $\omega(P,\pi)$. In particular, the weak and strong GBP are equivalent.*

Theorem 1.1 and Corollary 1.2 are proved in Section 3 after the necessary definitions are provided in Section 2. In Section 4 we study some natural maps between different Baues posets and prove that they are homotopy equivalences if the Baues posets are spherical. Section 5 deals with extension spaces of oriented matroids, which appear in one of the main open cases of the GBP. Although Theorem 1.1 applies implicitly, we show a more direct and explicit way to construct the map f in this particular case.

2. Preliminaries

2.1. **Poset topology.** The *order complex* $\Delta\omega$ of a finite *poset* (short for partially ordered set) ω is the simplicial complex of *chains* (totally ordered subsets) of ω. It is customary to refer to the topology of the underlying space $|\Delta\omega|$ as the topology of ω [11]. The motivation for this is that if ω is (the face poset of) a regular cell complex, then $\Delta\omega$ is the first barycentric subdivision of ω and, in particular, the underlying spaces $|\omega|$ and $|\Delta\omega|$ are homeomorphic. Also, order preserving maps of posets induce simplicial maps between the associated order complexes.

For $y \in \omega$ we write $\omega_{\leq y} = \{x \in \omega : x \leq y\}$. The following lemma, known as the "Quillen Fiber Lemma" [11, Theorem 10.5 (i)], is a standard tool in topological combinatorics.

Lemma 2.1. (Quillen [16]) *Let $g : \omega' \to \omega$ be an order preserving map of posets. If $g^{-1}(\omega_{\leq y})$ is contractible for every $y \in \omega$ then g induces a homotopy equivalence between ω' and ω.*

2.2. **Polyhedral subdivisions.** Let $\pi : P \to Q$ be an affine surjection of full dimensional polytopes $P \subseteq \mathbb{R}^{d'}$ and $Q \subseteq \mathbb{R}^d$ and let $\mathcal{A} := \pi(\text{vertices}(P))$ be the point configuration of projections of the vertices of P in $\mathbb{R}^d$, under the map π. Thus $\mathcal{A}$ is a multiset of points, each labeled with the corresponding vertex of P, and $\mathrm{conv}(\mathcal{A}) = Q$.

A *face* of a subconfiguration $\sigma \subseteq \mathcal{A}$ is a subconfiguration of σ consisting of all points on which some linear functional in $\mathbb{R}^d$ takes its minimum over σ. By the relative interior of σ we mean the relative interior of its convex hull. We call σ *spanning* if $\mathrm{conv}(\sigma)$ has dimension d. Two subconfigurations σ_1, σ_2 of $\mathcal{A}$ *intersect properly* if

(i) $\mathrm{conv}(\sigma_1) \cap \mathrm{conv}(\sigma_2) = \mathrm{conv}(\sigma_1 \cap \sigma_2)$ and
(ii) $\sigma_1 \cap \sigma_2$ is either empty or a face of both σ_1 and σ_2.

Following [8], we define a *(polyhedral) subdivision* of $\mathcal{A}$ as a finite collection S of spanning subconfigurations of $\mathcal{A}$, called the *cells* of S, such that any two elements of S intersect properly and

(iii) $\bigcup_{\sigma \in x} \mathrm{conv}(\sigma) = Q$.

A subdivision S of $\mathcal{A}$ is called *π-induced* [7] if each cell of S is the projection under π of the vertex set of a face of P. The set of all π-induced subdivisions of $\mathcal{A}$ can be ordered by refinement, i.e. by letting $S \leq T$ if for every $\sigma \in S$ there exists a $\tau \in T$ with $\sigma \subseteq \tau$. The unique maximal element of this poset is the *trivial* or *improper* subdivision, consisting of the single cell $\mathcal{A}$. The subposet of proper π-induced subdivisions of $\mathcal{A}$ is the *Baues poset* $\omega(P,\pi)$ of the projection π, introduced implicitly in [7] and explicitly in [10]. If P is a simplex then any subdivision of $\mathcal{A}$ is π-induced and $\omega(P,\pi)$ is the refinement poset of all proper polyhedral subdivisions of $\mathcal{A}$, denoted $\omega(\mathcal{A})$.

2.3. **Coherent subdivisions.** Let $\pi : P \to Q$ be as in Section 2.2. A linear functional $\psi \in \ker(\pi)^*$ defines a π-induced subdivision S^ψ of $\mathcal{A}$ as follows. Given $x \in Q$, let $\pi^{-1}(x)^\psi$ be the face of the fiber $\pi^{-1}(x)$ minimized by ψ and $\mathcal{F}_x^\psi$ be the unique minimal face of P which contains $\pi^{-1}(x)^\psi$. The cells of S^ψ are the spanning subconfigurations of $\mathcal{A}$ which are projections of faces of P of the form $\mathcal{F}_x^\psi$. A π-induced subdivision S of $\mathcal{A}$ is called *π-coherent* [7, Section 2] if $S = S^\psi$ for some ψ. We denote by $\omega_{\mathrm{coh}}(P,\pi)$ the subposet of $\omega(P,\pi)$ consisting of all proper π-coherent subdivisions of $\mathcal{A}$. The following lemma is an easy consequence of the definition of coherence. For normal fans of polytopes see [24, Section 7.1].

Lemma 2.2. *Let $\mathcal{N}$ be the common refinement in* $\ker(\pi)^*$ *of the normal fans of the fibers $\pi^{-1}(x)$ for $x \in Q$. We have:*

(i) $S^{\psi_1} = S^{\psi_2}$ *if and only if ψ_1 and ψ_2 lie in the same open cone of $\mathcal{N}$.*
(ii) *The map $\psi \to S^\psi$ induces an isomorphism between the poset of proper faces of $\mathcal{N}$ and $\omega_{\mathrm{coh}}(P,\pi)$.*

The previous lemma implies that $\omega_{\mathrm{coh}}(P,\pi)$ is topologically a sphere of dimension $\dim(P) - \dim(Q) - 1$. Actually, the common refinement $\mathcal{N}$ is the normal fan of the fiber polytope $\Sigma(P,\pi)$ of the projection [7]. In particular, $\omega_{\mathrm{coh}}(P,\pi)$ is isomorphic to the poset of proper faces of $\Sigma(P,\pi)$.

3. Proof of the main results

In this section we prove Theorem 1.1 and its corollary. Let $\pi : P \to Q$ be as in Section 2.2 and let x be a point in the relative interior of Q.

Given $S \in \omega(P,\pi)$, let σ be the unique face (possibly trivial) of a cell of S which contains x in its relative interior and $\mathcal{F}_\sigma$ be the face of P which projects to σ under π. We define the map f of Theorem 1.1 to send S to the intersection $\mathcal{F}_\sigma \cap \pi^{-1}(x)$, which is a face of $\pi^{-1}(x)$. If S is a π-coherent subdivision S^ψ then ψ lies in the relative interior of a unique cone in the normal fan of $\pi^{-1}(x)$ and $f(S)$ is the face of $\pi^{-1}(x)$ which corresponds to this cone.

Proof of Theorem 1.1. Let $f : \omega(P,\pi) \to \omega_x$ be as before. To see that f is well defined note that $\pi^{-1}(x)$ cannot be contained in a proper face of P and hence if S is proper then $f(S)$ is a proper face of $\pi^{-1}(x)$. The map f is order preserving by construction.

Let ω' and ω denote the posets $\omega_{\mathrm{coh}}(P,\pi)$ and ω_x, respectively, $y \in \omega$ and $g = f \circ \iota : \omega' \to \omega$. By Lemma 2.1 it suffices to show that $g^{-1}(\omega_{\leq y})$ is contractible as a subposet of ω'. Let $\mathcal{N}$ be the common refinement in $\ker(\pi)^*$ of the normal fans $\mathcal{N}_q$ of the fibers $\pi^{-1}(q)$ for all $q \in Q$, as in Lemma 2.2. Since x is a point in the relative interior of Q, $\pi^{-1}(x)$ has dimension $d' - d$ and hence $\mathcal{N}_x$ is a pointed fan in $\ker(\pi)^*$, i.e. its faces are pointed cones. Let C_y be the closure of the open cone in $\mathcal{N}_x$ corresponding to y. A π-coherent subdivision S^ψ is in $g^{-1}(\omega_{\leq y})$ if and only if ψ is nonzero and lies in C_y. It follows from Lemma 2.2 that $g^{-1}(\omega_{\leq y})$ is isomorphic to the poset of proper faces of the restriction of $\mathcal{N}$ to C_y. Since C_y is pointed, this poset is contractible. $\square$

Proof of Corollary 1.2. Since $f \circ \iota : \omega_{\mathrm{coh}}(P,\pi) \to \omega_x$ is a homotopy equivalence by Theorem 1.1, it induces isomorphisms $(f \circ \iota)_* = f_* \circ \iota_*$ of homology and homotopy groups. This implies parts (i) and (ii). Part (iii) follows from either one of parts (i) or (ii). In (iv), the equivalence (b)⇔(c) holds because $f \circ \iota$ is a homotopy equivalence. Since (d)⇒(b)⇒(a) is trivial, we only need to prove that (a)⇒(b)⇒(d).

Suppose that (a) holds and let $h : \omega(P,\pi) \to S^{d'-d-1}$ be a homotopy equivalence. The maps $h \circ \iota$ and $f \circ h^{-1}$ from $S^{d'-d-1}$ to itself must have degrees 1 or -1, since their composition is a homotopy equivalence. By the Brower degree theorem (see Corollary 7 in [22, Section 7.5]) $h \circ \iota$ and $f \circ h^{-1}$ are homotopy equivalences, and hence ι and f are homotopy equivalences. That $\omega_{\mathrm{coh}}(P,\pi)$ is a strong deformation retract of $\omega(P,\pi)$ is a consequence of the following technical lemma, since ι is the inclusion map of a subcomplex in a simplicial complex. $\square$

Lemma 3.1. *Let L be a subcomplex of a simplicial complex K. If the inclusion map $\iota : |L| \to |K|$ is a homotopy equivalence, then $|L|$ is a strong deformation retract of $|K|$.*

Proof. Let $X = |K|$ and $A = |L|$. Both (X, A) and $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$ are relative CW-complexes [22, §7.6] and, hence, they have the homotopy extension property (HEP) with respect to any space [22, §7.6, Theorem 12]. By the HEP of (X, A), if the inclusion ι is a homotopy equivalence then A is a deformation retract of X [22, §1.4, Corollary 10]. By the HEP of $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$, this implies that ι is a strong deformation retract as well [22, §1.4, Theorem 11]. $\square$

Remark 3.2. The statements and proofs of Theorem 1.1 and Corollary 1.2 are valid word by word if the poset $\omega(P,\pi)$ is replaced with any subposet containing $\omega_{\mathrm{coh}}(P,\pi)$. An interesting example is the poset of (proper) lifting subdivisions, discussed in Section 4, of the point configuration $\mathcal{A} = \pi(\mathrm{vertices}(P))$ when P is a simplex. It is not known whether this poset is always homotopy equivalent to a sphere, or whether it is connected.

Remark 3.3. The map f which appears in our main theorem is a surjective order preserving map from $\omega(P,\pi)$ onto a spherical poset. It was shown to be a homotopy equivalence in various special instances [1, 3, 5, 10], thus resolving the GBP in the affirmative, except that the poset ω_x and the map f appeared in disguise. We briefly describe these instances next. Let $\pi : P \to Q$ be a polytope projection and $\mathcal{A} := \pi(\mathrm{vertices}(P))$. The *chamber complex* of $\mathcal{A}$ [8] is the common refinement of all π-induced subdivisions of $\mathcal{A}$. Let x be a generic point in Q, i.e. a point lying in the interior of a full-dimensional chamber C_x. Observe that the poset ω_x of proper faces of $\pi^{-1}(x)$ is isomorphic to the poset of proper faces of P whose projection contains x in the relative interior (equivalently, contains C_x). In what follows ω_x is understood in this sense.

(a) Suppose that P is a simplex, so that ω_x is the poset of subconfigurations of $\mathcal{A}$ whose convex hulls contain the chamber C_x. By the duality associated with the theory of secondary polytopes [8, 9], the complements of these subconfigurations form a regular triangulation T_x of the Gale transform $\mathcal{A}^*$ of $\mathcal{A}$. Hence, f can be interpreted as the (order reversing) map $\omega(P,\pi) \to T_x$ which sends each subdivision

S of $\mathcal{A}$ to the complement of the unique cell of S covering x. This map appears in [5] and is proved to be a homotopy equivalence if $\dim(P) - \dim(Q) = 3$.

(b) Now let P be arbitrary and let $\emptyset = Q_{-1} \subset Q_0 \subset \cdots \subset Q_d = Q$ be a complete flag of faces of Q. Let C_x be the unique chamber "incident" to this flag, meaning that (the closure of) C_x intersects $Q_i \setminus Q_{i-1}$ for every i or, equivalently, that $C_x \cap Q_i$ has dimension i for each i. Let x be a point in that chamber and $\mathcal{F}_i = \pi^{-1}(Q_i)$ be the maximal face of P projecting to Q_i. Then ω_x is the poset of proper faces of P which intersect $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ for all $0 \leq i \leq d$. The map f sends each subdivision S to the unique face in ω_x projecting to a cell of S. This map appears in [1], where it is shown to be a homotopy equivalence if $\dim(Q) = 2$ and $\mathcal{A} \subset \partial Q$. Moreover, if $\dim(Q) = 1$ and Q_0 is the π-minimum vertex of the segment Q then C_x is the π-initial chamber and f sends each cellular string in $\omega(P, \pi)$ to its π-initial cell. In this case f was shown to be a homotopy equivalence in [10] and, in the more general context of shellable regular CW-spheres, in [3, Section 5].

4. Functorial properties of Baues posets

Let $\pi : P \to Q$ be a polytope projection as in Section 2.2. Let $Q' \subseteq Q$ be a polytope contained in Q which intersects the relative interior of Q and let $P' = \pi^{-1}(Q')$. We can consider π as a projection from P' to Q' as well. Then every π-induced subdivision of Q restricts to a π-induced subdivision of Q'. This induces a natural order preserving map of posets $h : \omega(P, \pi) \to \omega(P', \pi)$ which reduces to the map f of Theorem 1.1 if Q' is a single point.

Let x be a point in the relative interior of Q', hence also in the relative interior of Q, and ω_x be the poset of proper faces of $\pi^{-1}(x)$. Consider the maps $f : \omega(P, \pi) \to \omega_x$ and $f' : \omega(P', \pi) \to \omega_x$ of Theorem 1.1. Clearly $f' \circ h = f$. Hence, whenever two of the three maps are homotopy equivalences, the third one is a homotopy equivalence too.

There are at least the following two special cases of interest.

Lifting subdivisions and zonotopal tilings. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be a point configuration of dimension d embedded in the plane $x_1 = 1$ in $\mathbb{R}^{d+1}$, let $Z_{\mathcal{A}}$ be the zonotope

$$Z_{\mathcal{A}} = \{\sum_{i=1}^{n} t_i a_i : 0 \leq t_i \leq 1\}$$

in $\mathbb{R}^{d+1}$ and $P = I^n = [0,1]^n$, $P' = \Delta^{n-1} = \mathrm{conv}(\{e_1, \ldots, e_n\})$ be the standard unit n-cube and $(n-1)$-simplex in $\mathbb{R}^n$, respectively. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{d+1}$ be the linear map which sends the ith coordinate vector e_i in $\mathbb{R}^n$ to a_i for each i. Hence $\mathcal{A}$ is the projection of the vertex set of P' and $Z_{\mathcal{A}} = \pi(P)$. The map h is the natural, order preserving map of Baues posets

$$h_{\mathcal{A}} : \omega(I^n, \pi) \ \longrightarrow \ \omega(\Delta^{n-1}, \pi)$$

sending an element of $\omega(I^n, \pi)$ (which is a zonotopal subdivision of $Z_{\mathcal{A}}$) to its link at the origin (which is a polyhedral subdivision of $\mathcal{A}$). The image of $h_{\mathcal{A}}$ is the set of (proper) *lifting subdivisions* [12, Section 9.6] [20] of $\mathcal{A}$.

Corollary 4.1. *Let $\mathcal{A}$ be a point configuration of dimension d with n elements and let $Z_{\mathcal{A}}$ be the associated zonotope. Any two of the following three statements imply the third:*

(i) *The poset $\omega(\mathcal{A})$ of all proper subdivisions of $\mathcal{A}$ has the homotopy type of a sphere of dimension $n-d-2$.*
(ii) *The poset of all proper zonotopal tilings of Z has the homotopy type of a sphere of dimension $n-d-2$.*
(iii) *The order preserving map $h_{\mathcal{A}}$ sending each proper zonotopal tiling of $Z_{\mathcal{A}}$ to the corresponding lifting subdivision of $\mathcal{A}$ is a homotopy equivalence.*

For example, if $\mathcal{A}$ is the vertex set of a cyclic polytope, the first two statements in the previous corollary are known to hold [2, 23, 17] and the third was conjectured by Reiner [19, Conjecture 6.7]. For cyclic polytopes and zonotopes and their subdivisions see [2, 4, 17].

Corollary 4.2. *If $\mathcal{A}$ is the vertex set of a cyclic d-polytope with n vertices then the map $h_{\mathcal{A}}$ is a homotopy equivalence.*

Contraction of point configurations and zonotopes. Let $\mathcal{A}$ be a point configuration with n points and dimension d, let P be an $(n-1)$-simplex and let π be the projection sending the vertices of P to the points in $\mathcal{A}$. Hence, $\omega(P,\pi)$ is the poset $\omega(\mathcal{A})$ of all proper subdivisions of $\mathcal{A}$.

Suppose that $\mathcal{A}$ has at least one extremal point a_0, i.e. a point which can be separated from the rest of $\mathcal{A}$ by an affine hyperplane H. In particular, a_0 is a vertex of $\operatorname{conv}(\mathcal{A})$. Let $P' = \pi^{-1}(H)$ and let $\mathcal{A}'$ be the image under π of the vertex set of P'. Our hypotheses on H imply that P' is the vertex figure of P at the vertex $\pi^{-1}(a_0)$, i.e. an $(n-2)$-simplex. In oriented matroid terms, $\mathcal{A}'$ is the contraction of $\mathcal{A}$ at a_0, whose oriented matroid (and hence whose poset of subdivisions) is independent of the choice of H. The map $h : \omega(P,\pi) \to \omega(P',\pi)$ sends each polyhedral subdivision of $\mathcal{A}$ to its link at a_0, which is (combinatorially) a subdivision of the contraction $\mathcal{A}'$. The map h need not be surjective.

Corollary 4.3. *Let $\mathcal{A}$ be a point configuration of dimension d with n elements and $\mathcal{A}'$ its contraction at an extremal point a_0. Any two of the following three statements imply the third:*
(i) *The poset $\omega(\mathcal{A})$ of all proper subdivisions of $\mathcal{A}$ has the homotopy type of a sphere of dimension $n-d-2$.*
(ii) *The poset $\omega(\mathcal{A}')$ of all proper subdivisions of $\mathcal{A}'$ has the homotopy type of a sphere of dimension $n-d-2$.*
(iii) *The order preserving map sending each proper subdivision of $\mathcal{A}$ to its link at a_0 is a homotopy equivalence between $\omega(\mathcal{A})$ and $\omega(\mathcal{A}')$.*

Similar reasoning applies to the posets of lifting subdivisions of $\mathcal{A}$ and $\mathcal{A}'$. Stated differently, let Z be the d-zonotope $Z_{\mathcal{V}}$ associated to the configuration $\mathcal{V}$ of n vectors in $\mathbb{R}^d$, let $P = I^n$ be the standard n-cube and let π be the projection sending the coordinate vectors of $\mathbb{R}^n$ to the vectors in $\mathcal{V}$. Hence, $\omega(P,\pi)$ is the poset of all proper zonotopal tilings of Z.

Let $v \in \mathcal{V}$ be a nonzero generator of Z, which we can rescale without affecting the relevant Baues posets. If the length of v is sufficiently large then there is an affine hyperplane H such that the zonotope $Z' = Z \cap H$ has the oriented matroid of the contraction of $\mathcal{V}$ at v and $P' = \pi^{-1}(Z')$ is an $(n-1)$-cube. The map $h : \omega(P,\pi) \to \omega(P',\pi)$ is the order preserving map sending a proper zonotopal tiling of Z to its link (contraction) at v, which is (combinatorially) a zonotopal tiling of Z'.

Corollary 4.4. *Let Z be a zonotope of dimension d with n generators and let Z' be its contraction at a nonzero generator v. Any two of the following three statements imply the third:*

(i) *The poset of all proper zonotopal tilings of Z has the homotopy type of a sphere of dimension $n-d-1$.*
(ii) *The poset of all proper zonotopal tilings of Z' has the homotopy type of a sphere of dimension $n-d-1$.*
(iii) *The order preserving map sending each proper zonotopal tiling of Z to its contraction at v is a homotopy equivalence.*

Once again, if $\mathcal{A}$ is the vertex set of a cyclic polytope (and a_0 is the last vertex of the linear order of the cyclic polytope) or Z is a cyclic zonotope (and v is any generator) then the first two assertions in Corollaries 4.3 and 4.4, respectively, are known to hold but the third one was not known before, in both cases.

5. Extension spaces of oriented matroids

Let $\mathcal{M}$ be an oriented matroid on a ground set E. A single element extension of $\mathcal{M}$ is any oriented matroid $\mathcal{M}'$ on a ground set $E \cup \{p\}$ whose restriction to E coincides with $\mathcal{M}$. We say that $\mathcal{M}'$ is a non-trivial extension of $\mathcal{M}$ if p is neither a loop nor a coloop. The *extension poset* $\mathcal{E}(\mathcal{M})$ of $\mathcal{M}$ is the set of its (non-trivial, single element) extensions, ordered by weak maps. See [12, Chapter 7] for details on extensions and extension posets in oriented matroid theory.

The *Bohne-Dress Theorem* [12, 24] asserts that if $\mathcal{M}$ is realizable of rank d with n elements then $\mathcal{E}(\mathcal{M})$ is the Baues poset of a projection $\pi : P \to Q$, where P is the standard cube of dimension n and Q is a zonotope of dimension $n-d$. Conversely, every such projection produces as Baues poset the extension poset of a realizable oriented matroid. Hence, the *extension space conjecture* of oriented matroid theory (asserting that $\mathcal{E}(\mathcal{M})$ is homotopically a $(d-1)$-sphere for any realizable $\mathcal{M}$ of rank d [17], [12, Section 7.2]) is equivalent to the (weak) GBP restricted to the case of P being a cube. Disconnected extension posets of non-realizable oriented matroids of rank 4 are known to exist [15].

It is not clear at all how to describe the map f of Theorem 1.1 directly from the oriented matroid. Our goal here is to give a more transparent map that can be used instead. Let $\Lambda_d := \{-,0,+\}^d \setminus (0,\dots,0)$ be the set of nonzero sign vectors of length n, partially ordered by letting $(a_1,a_2,\dots,a_d) \leq (b_1,b_2,\dots,b_d)$ if and only if, for all i, $a_i \neq 0$ implies $a_i = b_i$. Thus Λ_d is the poset of proper faces of a d-dimensional cube. Our starting point is that if $\mathcal{M}$ is the coordinate oriented matroid of rank d (realized by any d independent vectors) then the poset $\mathcal{E}(\mathcal{M})$ is isomorphic to Λ_d. More precisely, there is a natural poset isomorphism $f_0 : \mathcal{E}(\mathcal{M}) \to \Lambda_d$ which takes each extension $\mathcal{M} \cup \{p\}$ to the sequence of signs in the unique circuit of $\mathcal{M} \cup \{p\}$, oriented so that p is positive.

Let now $\mathcal{M}$ be an arbitrary realizable oriented matroid of rank d on n elements, realized by a vector configuration $\mathcal{V}$ in $\mathbb{R}^d$. Let $\mathcal{E}(\mathcal{V})$ denote the subposet of $\mathcal{E}(\mathcal{M})$ consisting of the oriented matroids obtained by adding a new vector to $\mathcal{V}$. The poset $\mathcal{E}(\mathcal{V})$ is isomorphic to the chamber complex of the vector configuration $\mathcal{V} \cup (-\mathcal{V})$ and, hence, it is a sphere of dimension $d-1$.

Theorem 5.1. *Let $\mathcal{M}$ be an oriented matroid of rank d realized by a vector configuration $\mathcal{V}$ and $B \subseteq \mathcal{V}$ be one of its bases. Let $\mathcal{M}(B)$ denote the oriented matroid $\mathcal{M}$ restricted to B, which is the coordinate oriented matroid of rank d.*

The composition $f_B : \mathcal{E}(\mathcal{M}) \to \Lambda_d$ of the forgetful map $\mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{M}(B))$ with the previously described map $f_0 : \mathcal{E}(\mathcal{M}(B)) \to \Lambda_d$ is order preserving and the map

$$\mathcal{E}(\mathcal{V}) \xrightarrow{\iota} \mathcal{E}(\mathcal{M}) \xrightarrow{f_B} \Lambda_d$$

obtained by composing f_B with the inclusion map $\iota : \mathcal{E}(\mathcal{V}) \to \mathcal{E}(\mathcal{M})$ is a homotopy equivalence between $\mathcal{E}(\mathcal{V})$ and Λ_d.

Proof. Suppose without loss of generality that B is the standard basis in $\mathbb{R}^d$. We identify the elements of Λ_d with the different relatively open, non-zero orthants of $\mathbb{R}^d$ in the natural way. Then $f_B \circ \iota$ can be described directly as the map which takes each realized extension $\mathcal{V} \cup \{v\}$ to the open orthant containing v. This map is well defined (independent of the choice of v for a given extension), order preserving and surjective. That it is a homotopy equivalence can be proved following the ideas of the proof of Theorem 1.1. $\square$

From Theorem 5.1 we can derive a corollary analogous to Corollary 1.2, which we will not state.

Remark 5.2. The map f_B of Theorem 5.1 is again a disguised version of the map f of Theorem 1.1. Observe that $f_B \circ \iota$ is the natural map from the chamber complex of $\mathcal{V} \cup (-\mathcal{V})$ to that of $B \cup (-B)$. The faces of the chamber complex of $\mathcal{V} \cup (-\mathcal{V})$ form the poset of all proper coherent subdivisions of the Gale transform $(\mathcal{V} \cup (-\mathcal{V}))^*$ (see [9]) and hence the situation reminds the one in part (a) of Remark 3.3. In fact, $(\mathcal{V} \cup (-\mathcal{V}))^*$ is the vertex set of a Lawrence polytope Λ and the results in Section 4.3 of [20] imply that its poset of subdivisions equals $\mathcal{E}(\mathcal{M})$. It remains to decide which point x in the relative interior of Λ turns f_B into the map f of Theorem 1.1. If $\mathcal{F}$ is the face of Λ with vertex set the complement of $B \cup (-B)$, we claim that x can be chosen as any point sufficiently close to a generic point in the relative interior of $\mathcal{F}$. The details are left to the interested reader.

Remark 5.3. The procedure of Theorem 5.1 can be applied to any oriented matroid which has an "adjoint" [12, Section 5.3], a case more general than that of realizable oriented matroids. If $\mathcal{M}$ has an adjoint $\mathcal{M}^{ad}$, then the poset of covectors of $\mathcal{M}^{ad}$ is a spherical subposet of $\mathcal{E}(\mathcal{M})$ and its inclusion into $\mathcal{E}(\mathcal{M})$ composed with the map f_B can be easily proved to be a homotopy equivalence.

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