THE GENERALIZED BAUES PROBLEM FOR CYCLIC POLYTOPES I

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ABSTRACT. An important special case of the Generalized Baues Problem asks whether the order complex of all proper polyhedral subdivisions of a given point configuration, partially ordered by refinement, is homotopy equivalent to a sphere. In this paper, an affirmative answer is given for the vertex sets of cyclic polytopes in all dimensions. This yields the first non-trivial class of point configurations with neither a bound on the dimension, the codimension, nor the number of vertices for which this is known to be true. Moreover, it is shown that all triangulations of cyclic polytopes are lifting triangulations. This contrasts the fact that in general there are many non-regular triangulations of cyclic polytopes. Beyond this, we find triangulations of $C(11, 5)$ with flip deficiency. This proves—among other things—that there are triangulations of cyclic polytopes that are non-regular for every choice of points on the moment curve.

1. INTRODUCTION

Polyhedral subdivisions of point configurations and their combinatorial properties have attracted considerable attention during the past decade. One direction of research is the so-called Generalized Baues Problem posed by Billera, Kapranov, and Sturmfels [5]. This is a question arising in the theory of fiber polytopes (see [6], [24, Lecture 9]) and connected with several classical objects of study in polytope theory such as monotone paths, zonotopal tilings, and triangulations. See [17] and the recent survey [20] for an overview.

The special case of the Generalized Baues Problem investigated in this paper asks whether the order complex of all proper polyhedral subdivisions of a given point configuration, partially ordered by refinement, is homotopy equivalent to a sphere. In [9] it is shown that the Generalized Baues Problem has an affirmative answer for cyclic polytopes in dimensions not exceeding three. We show that this is actually true in all dimensions.

Theorem 1.1. *For all $d > 0$ and $n > d$ the Baues poset $\omega(C(n, d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n, d)$ is homotopy equivalent to an $(n - d - 2)$-sphere.*

The proof is done in Section 4 by generalizing the deletion construction for triangulations in [18] to arbitrary subdivisions of cyclic polytopes. Our results in this section and, in particular, Theorem 1.1 have been extended in [1] to prove sphericity of the Baues poset $\omega(C(n, d') \to C(n, d))$ of subdivisions of $C(n, d)$ which are induced by the natural projection from $C(n, d')$ to $C(n, d)$ which forgets the last $d' - d$ coordinates.

The cyclic polytope $C(n, d)$ is the convex hull of any n pairwise distinct points on the moment curve $\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\}$ in $\mathbb{R}^d$. Its combinatorial type does not depend on the choice of the points along the moment curve, since its face lattice is combinatorially determined by Gale's evenness condition (see [24, p. 14]). In fact, not only the face lattice of $C(n, d)$ is independent of the choice of points along

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the moment curve but also the oriented matroid of affine dependences between its vertices. It is the so-called alternating uniform oriented matroid of rank $d+1$ on n elements (cf. [8, Section 9.4]). This has some importance for us since the concepts appearing in this paper depend only on the oriented matroid. Thus, our results hold for any polytope whose vertices have the alternating oriented matroid, although we will assume our cyclic polytopes to be realized with vertices along the moment curve in some of the proofs. Note that not every polytope combinatorially equivalent to a cyclic polytope has the oriented matroid of a cyclic polytope. Our proofs would not be valid for those polytopes.

Cyclic polytopes are important in polytope theory because they are neighborly and because they have the largest number of faces of every dimension among all polytopes of a fixed dimension and number of vertices. In the context of triangulations and the Baues problem the vertex sets of cyclic polytopes are the best understood non-trivial point configurations so far. Edelman and Reiner [10] introduced a natural poset structure (actually *two* natural poset structures, which are conjectured to coincide: the two Stasheff-Tamari posets) on the collection of triangulations of $C(n,d)$. Using this structure Rambau [18] has proved that the set of triangulations of a cyclic polytope is connected under bistellar flips and that every triangulation of $C(n,d)$ is shellable. More recently Edelman et al. [9] have used these ideas to prove our Theorem 1.1 for the case $d \leq 3$ (and a similar statement on the Stasheff-Tamari posets valid in every dimension and codimension). Finally, Athanasiadis et al. [2] have studied the fiber polytopes produced by projections between cyclic polytopes and, among other things, have determined exactly for what values of n, d and d' ($n > d' > d$) the Baues poset of the natural projection $C(n,d') \to C(n,d)$ is isomorphic to the face lattice of a polytope.

However, triangulations of cyclic polytopes also present "bad behavior" sometimes. For example, starting with $C(9,3)$, $C(9,4)$, and $C(9,5)$—as the minimal cases with respect to dimension and/or codimension—cyclic polytopes have non-regular triangulations (see [2]). Even more, the number of non-regular triangulations of the cyclic polytope $C(n,n-4)$ is known to grow exponentially with n, while the number of regular ones grows polynomially [15]. One of our results reflects this bad behavior: triangulations of cyclic polytopes may have "flip deficiency":

Theorem 1.2. *There are* 4 *(out of* $51,676$*) triangulations of* $C(11,5)$ *with only four bistellar flips, while the dimension of the secondary polytope is five.*

We provide one example in Section 5, found by a computer program. This result is important because of the following: The secondary polytope of a point configuration with n points in d-space is an $(n-d-1)$-polytope whose vertices are in one-to-one correspondence to the regular triangulations of the configuration and whose edges are in one-to-one correspondence to the bistellar operations (flips). In particular, every regular triangulation has at least $n-d-1$ bistellar neighbors. A non-regular triangulation may have fewer bistellar neighbors (see [16] and [22]); in this case, we say that it has *flip deficiency*. For us, the fact that triangulations of cyclic polytopes may have flip deficiency implies that flip deficiency has to be considered a natural phenomenon to occur in a Baues poset and not a "pathology" of some "bad polytopes."

It is interesting to observe that cyclic polytopes are "universal" subpolytopes of every point configuration: for any given integers $n > d \geq 2$ there is an integer $N = N(n,d)$ such that any generic point configuration in $\mathbb{R}^d$ with at least N points contains the vertices of a cyclic polytope $C(n,d)$ ([8, Proposition 9.4.7]). The case $d = 2$ is the classic Erdös-Szekeres theorem (1935). This has the following consequence: since $C(d+6,d)$ has non-regular triangulations for every $d \geq 3$, every generic point configuration in $\mathbb{R}^d$ with at least $N(d+6, d)$ points has non-regular triangulations. Theorem 1.2 seems to indicate that any generic point configuration in $\mathbb{R}^5$ with more than $N(11, 5)$ points has triangulations with flip-deficiency, although this is not a straightforward conclusion.

Our third result concerns the class of *lifting subdivisions*, introduced in [8, Section 9.6] and studied in [21]. This class is a combinatorial analogue—and a generalization—of regular subdivisions. It turns out that all triangulations of cyclic polytopes belong to this class:

Theorem 1.3. *Every triangulation of* $C(n, d)$ *is a lifting subdivision.*

We will prove this result in Section 3 by using a characterization Theorem from [21], which we state in Section 2 below. Although this result is probably true for arbitrary subdivisions and not only triangulations, we do not have a proof of it.

This result and Theorem 1.1 are related to the extension space of alternating oriented matroids, studied by Sturmfels and Ziegler [23]. The extension space of an oriented matroid M is the poset of all single-element extensions of M, ordered by weak-maps (see [8, Chapter 7]). It is conjectured that this poset is homotopy equivalent to a sphere of dimension one less than the rank of M for a realizable oriented matroid. For non-realizable oriented matroids this conjecture is known to be false. For the relation of this conjecture to the Generalized Baues conjecture see [21, Section 4] or [20]. Sphericity of the extension space is proved in [23] for the class of *strongly Euclidean oriented matroids*, which include the cases of rank at most 3 and also the alternating oriented matroids of arbitrary rank or number of elements—i.e. the oriented matroids of cyclic polytopes, as well as their duals.

Let P be a polytope with vertex set A and let M denote the oriented matroid of affine dependences of A. Lifting subdivisions of P are defined via the so-called *lifts* of the oriented matroid M. Since lifts and extensions are dual concepts in oriented matroid theory, there is a natural order preserving map from the extension space of the oriented matroid M^* dual to M and the Baues poset of P, whose image is precisely the subposet of lifting subdivisions of P (compare with Exercise 9.30, in [8, page 414]).

For cyclic polytopes, our results that all triangulations are lifting and that the Baues poset is spherical suggest the conjecture that all subdivisions are lifting as well and that the order-preserving map mentioned above is a homotopy equivalence (if all subdivisions are lifting then the map is automatically surjective). This would follow if we had proved what Reiner [20] calls the "strong generalized Baues conjecture" for cyclic polytopes, namely that the subposet of regular subdivisions—i.e. the face poset of the secondary polytope—is a deformation retract of the Baues poset.

In the same context, we have to mention that our proof of Theorem 1.1 reminds (and is inspired by) the proof of sphericity in [23]. In fact, analyzing our proof one finds that it is based upon the following two particular properties of cyclic polytopes, apart from induction on the number of vertices:

- The existence of inseparable pairs of vertices in the polytope, which provides two pushing subdivisions corresponding to "almost opposite" extensions of the dual oriented matroid. This is used to create a suspension of a sphere in Definition 4.1 —while [23] uses two opposite extensions for doing this same thing.
- The property of "stackability in a certain direction," proved in Corollary 2.16 and used in Theorem 4.5, which is reminiscent of strong Euclideanness.

Incidentally, for proving sphericity of a Baues poset of dimension d we use the second of the properties mentioned above (stackability) in dimension $d-1$. Since stackability is trivially true in dimension 2, the ideas in Section 4 might be useful for proving sphericity of the Baues poset in dimension 3, a case which is still open. However, inseparability is a rather restrictive property even in dimension 3, so some new ideas are still needed.

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The following are immediate consequences of our results. The first and the second item answer questions recently posed in [2].

Corollary 1.4. (i) *There are triangulations of* $C(11,5)$ *that are non-regular for every choice of points on the moment curve.*
(ii) *There are lifting triangulations with flip deficiency.*
(iii) *There are spherical Baues posets containing triangulations with flip deficiency.*

2. Preliminaries

2.1. **Subdivisions.** We consider the following combinatorial framework for subdivisions. If $A \subset \mathbb{R}^d$ is a point configuration, we will use the words *independent*, *spanning*, and *basis* applied to subsets of A meaning that the subset is affinely independent, that it affinely spans A, or both things at the same time, respectively. A subset τ of a subset $\sigma \subseteq A$ is a *face* of σ if it is the set of all points where the maximum over σ of some linear functional in $(\mathbb{R}^d)^*$ is attained. Note that it is not sufficient for τ to be contained in such a maximizing set. In other words, τ is a face of σ if it is the intersection of σ with a face of the polytope $\operatorname{conv}(\sigma)$. For convenience, the empty set is always considered a face.

Following [4] and [11], we define our main objects of study.

Definition 2.1 (Subdivision)**.** A *subdivision* of A is a collection S of spanning subsets (*cells*) of A satisfying:

- The union of all $\operatorname{conv}(\sigma)$ for $\sigma \in S$ equals $\operatorname{conv}(A)$,
- $\sigma \cap \tau$ is a face of both σ and τ for all $\sigma, \tau \in S$ and $\operatorname{conv}(\tau \cap \sigma) = \operatorname{conv}(\tau) \cap \operatorname{conv}(\sigma)$ (σ and τ *intersect properly*).

Cells sharing a common facet are *adjacent*. A *triangulation* is a subdivision all of whose cells are bases. A *subdivision of a polytope* is a subdivision of its vertex set.

We say that a subdivision S_1 *refines* a subdivision S_2 if

$$S_1 \le S_2 \qquad :\Longleftrightarrow \qquad \forall \sigma_1 \in S_1 \, \exists \sigma_2 \in S_2 : \; \sigma_1 \subset \sigma_2.$$

Refinement of subdivisions is a partial order. The poset of subdivisions of A has a unique maximal element which is the trivial subdivision $\{A\}$. The poset of all non-trivial subdivisions of A is called the Baues poset of A and denoted by $\omega(A)$. The generalized Baues conjecture posed by Billera, Kapranov, and Sturmfels had as one of its implications that the poset $\omega(A)$ is homotopically equivalent to a sphere of dimension $\#A - \mathtt{dim}(A) - 2$. The conjecture itself has been disproved by Rambau and Ziegler [19], but the special case of subdivision posets of point configurations as considered in this paper is still open.

Every subdivision can be refined to a triangulation. This is true in general, but especially obvious when A is in general position; in this case, in order to refine a subdivision we can just triangulate each of its cells independently.

The following lemma gives a combinatorial characterization of subdivisions:

Lemma 2.2. *Let* A *be a point configuration. Let* S *be a collection of full-dimensional subsets of* A *which intersect pairwise properly. Then, the following conditions are equivalent:*

(i) S *is a subdivision (i.e.,* S *covers* $\operatorname{conv}(A)$*).*
(ii) *For every* $\sigma \in S$ *and for every facet* τ *of* σ*, either* τ *lies in a facet of* A *or there is another* $\sigma' \in S$ *of which* τ *is a facet.*

Proof. Easy (see, e.g., [18, Proposition 2.2]. Observe that the cell σ' in part (ii) will automatically be unique and lie in the opposite side of τ as σ, or otherwise σ and σ' do not intersect properly. □

2.2. **Lifting Subdivisions.** A special class of subdivisions is obtained by lifts of the oriented matroid (of affine dependences) $M(A)$ of a point configuration A. For background on oriented matroids, see [8]. We recall here that a *lift* at $p \notin E$ of an oriented matroid M on a ground set E is an oriented matroid $\widehat{M}$ on the ground set $E \cup \{p\}$ such that the contraction M/p of p in $\widehat{M}$ is M. The *facets* of an oriented matroid are the zero-sets of positive cocircuits. The set of all facets of M is denoted by $F(M)$. In the case of an oriented matroid arising from some point configuration (a *realizable* oriented matroid) these facets coincide with those of the (convex hull of the) point configuration. A lift of a realizable oriented matroid, however, need not be realizable; and we need the facets of arbitrary lifts for the definition of lifting subdivisions. The following definition, together with the proof that lifting subdivisions are indeed subdivisions in the sense of Definition 2.1, appeared for the first time in [8, Section 9.6].

Definition 2.3 (Lifting Subdivision)**.** Let A be a point configuration with oriented matroid $M(A)$. Moreover, let $\widehat{M(A)}$ be a lift of $M(A)$. Then

$$S_{\widehat{M(A)}} := \left\{ B \in F(\widehat{M(A)}) \,:\, p \notin B \right\}$$

is the *lifting subdivision* of A defined by $\widehat{M(A)}$.

The definition becomes intuitively plausible if one restricts attention to the following construction: For a point configuration A, assign a height to every point $a \in A$. Add an additional point p in interior of $\operatorname{conv} A$ and assign a very large height to p. Then the vertex figure of p in $\operatorname{conv}(A \cup \{p\})$ is affinely equivalent to $\operatorname{conv} A$, i.e., we constructed an *affine lift*. All *lower* facets of $\operatorname{conv}(A \cup \{p\})$ (those that can be seen from a point with very large negative height) are exactly those facets not containing p. These form the so-called *regular* subdivision induced by the heights on A (regular subdivisions are lifting, but not conversely). The following is a special kind of regular subdivisions (see [13] for a survey of elementary constructions).

Definition 2.4 (Pushing Subdivision)**.** Let A be a point configuration, S be a regular subdivision obtained by a height function $\alpha : A \to \mathbb{R}$, and $a \in A$. Then the subdivision $S_{\text{push } a}$ obtained by *pushing* a is the regular subdivision induced by increasing ("pushing") the height $\alpha(a)$ of a by some "very small" positive ε.

In the definition, "very small" means small enough such that $S_{\text{push } a}$ is a refinement of S.

For example, if $S = \{A\}$ is the trivial subdivision of A, the subdivision $S_{\text{push } a}$ obtained by pushing an element $a \in A$ is the unique subdivision which contains $A \setminus \{a\}$ as a cell. All the other cells in $S_{\text{push } a}$ (if any) contain a and can be thought of as "cones" with apex a over the facets of $A \setminus \{a\}$ whose exterior is visible from a.

Given a subdivision S of A, it is not trivial to construct a lifting of $M(A)$ that induces S or prove that no such lifting exists. In [21], Santos gave a characterization of lifting subdivisions of oriented matroids, which is in particular valid for point configurations. This characterization concerns subdivisions of subconfigurations of A [12, 21]. The crucial definition is the following:

Definition 2.5. Let A be a point configuration. Moreover, let $\mathcal{S} = \{\, S_B \,:\, B \subset A \,\}$ be a collection of subdivisions, one for each subset $B \subset A$. $\mathcal{S}$ is *consistent* if for every subset $B \subset A$ the following properties are satisfied:

(i) For every cell $\tau \in S_B$ and for every $B' \subset B$ the set $\tau \cap B'$ is a face of a cell of $S_{B'}$.
(ii) If σ is an affine basis of $\mathbb{R}^d$ which is contained in B and contained in a cell of $S_{\sigma\cup\{b\}}$ for every $b \in B\backslash\sigma$, then σ is contained in a cell of S_B as well.

We can now state the following theorem from [21]. The form of the theorem we state below appears in [12].

Theorem 2.6. *Let S be a subdivision of a point configuration A. Then, S is a lifting subdivision if and only if there is a consistent collection of subdivisions $\{ S_B : B \subset A \}$ with $S_A = S$.*

For our purposes, it will be useful to reformulate the definition of consistency:

Lemma 2.7. *Conditions (i) and (ii) in the definition of a consistent collection of subdivisions are equivalent to:*

(i′) *For every cell $\tau \in S_B$ and for every $b \in B$ the set $\tau\backslash\{b\}$ is a face of a cell of $S_{B\backslash\{b\}}$.*
(ii′) *If σ is an affine basis of $\mathbb{R}^d$ which is contained in B and contained in cells of both $S_{B\backslash\{b\}}$ and $S_{B\backslash\{c\}}$ for some pair of elements $b, c \in B\backslash\sigma$ with $b \neq c$, then σ is contained in a cell of S_B as well.*

Proof. That (i) implies (i′) is obvious. Also, (i) easily follows from (i′) and (ii) from (ii′) by recursion. We have to proof that (i) and (ii) imply (ii′).

Let σ be an affine basis contained in B and let $b, c \in B\backslash\sigma$, with $b \neq c$. Condition (i) applied to $B\backslash\{b\}$ implies that for every $b' \in B\backslash\sigma$ other than b, σ lies in a cell of $S_{\sigma\cup\{b'\}}$. Condition (i) applied to $B\backslash\{c\}$ implies the same for $b' = b$, and then condition (ii) implies that σ is in a cell of S_B. □

We will only be interested in the case where A is generic (no $d + 1$ points lie in a hyperplane). In this case property (i′) can be simplified further:

Lemma 2.8. *If the point configuration A is generic, then condition* (i′) *of Lemma 2.7 is equivalent to the following one:*

(i″) *For every cell $\tau \in S_B$ and for every $b \in B$, if $\tau\backslash\{b\}$ is spanning then it is a cell of $S_{B\backslash\{b\}}$.*

Proof. That statement (i′) implies (i″) is trivial. For the converse, let $\tau \in S_B$ be a (spanning) cell in S_B. If $\tau\backslash\{b\}$ is spanning, then statement (i″) is equivalent to (i′).

If $\tau\backslash\{b\}$ is not spanning then it has codimension 1 and τ is a basis (a simplex in S_B). We have two possibilities: if there is a $\sigma \in S_B$ containing $\tau\backslash\{b\}$ other than τ, then σ cannot contain b (otherwise it contains τ) and thus $\tau\backslash\{b\} = \sigma \cap \tau$ is a facet of σ. Property (i″) implies that $\sigma \in S_{B\backslash\{b\}}$; thus, (i′) holds for τ.

Otherwise τ is the unique cell of S_B containing $\tau\backslash\{b\}$. By Lemma 2.2, $\tau\backslash\{b\}$ lies in a facet of B and since B is generic $\tau\backslash\{b\}$ is a facet of B. But then it is a facet of $B\backslash\{b\}$ as well, so that it is a facet of a cell of every subdivision of $B\backslash\{b\}$. □

2.3. **Cyclic Polytopes.**

We start with a definition of cyclic polytope based upon the structure of its oriented matroid (see [8]).

Definition 2.9 (Cyclic Polytope)**.** The d-dimensional *cyclic polytope* with n vertices $C(n, d)$ is a point configuration whose oriented matroid of affine dependences is the alternating oriented matroid. As the *geometric standard embedding* of $C(n, d)$ we consider the points

$$(i, i^2, \ldots, i^d) \in \mathbb{R}^d, i = 1, \ldots, n.$$

The concept of upper and lower facets of a cyclic polytope allows us to define a partial order on the cells of every subdivision of a cyclic polytope.

Upper and lower facets can be characterized combinatorially. In all what follows, the i-th vertex of $C(n, d)$ will be denoted by i. See [18] for a formally worked out setup that distinguishes carefully between embeddings of cyclic polytopes and the corresponding combinatorially invariant objects.

Definition 2.10 (Upper and Lower Facets)**.** For a subset F of the vertex set $[n] := \{1, \ldots, n\}$ of $C(n, d)$ an *odd gap* (resp. *even gap*) is a vertex $i \in [n] \setminus F$ with an odd (resp. even) number of elements in F larger than i. The *upper* (resp. *lower*) facets of $C(n, d)$ are the subsets of $[n]$ with d elements which only have odd (resp. even) gaps.

By Gale's Evenness Criterion (see, e.g., [24, Thm. 0.7]), upper and lower facets are indeed facets of $C(n, d)$ and they are all the facets.

In the standard embedding of $C(n, d)$, upper (resp. lower) facets are facets that can be seen from a point in $\mathbb{R}^d$ with "very large" positive (resp. negative) coordinate. That these are all facets of $C(n, d)$ means that there are no "vertical" facets. The natural projection $C(n, d+1) \to C(n, d)$ which forgets the last coordinate shows that the lower facets of $C(n, d+1)$ form a (regular) subdivision of $C(n, d)$ (and the same holds for the upper facets). The fact that $C(n, d+1)$ is simplicial implies that this subdivisions are in fact triangulations. Thus:

Lemma 2.11. *The set of all lower (resp. upper) facets of* $C(n, d+1)$ *is a triangulation of* $C(n, d)$*.*

Since every cell in a subdivision of $C(n, d)$ is a cyclic polytope itself, we can speak of upper and lower facets of cells in a subdivision. If two adjacent cells in $C(n, d)$ are properly intersecting then it is easy to see that the intersection of them must be a lower facet of one of them and an upper facet of the other. Hence, we get the following relation on adjacent cells of a subdivision of $C(n, d)$.

Definition 2.12. Let S be a subdivision of $C(n, d)$ and σ_1 and σ_2 be adjacent cells in S. Then σ_1 is *below* σ_2—in formula: $\sigma_1 < \sigma_2$—if $\sigma_1 \cap \sigma_2$ is an upper facet of σ_1 (and thus a lower facet of σ_2). In this case, we also say that σ_2 is *above* σ_1.

A notion of central technical importance for usis the following:

Definition 2.13. A subdivision S of $C(n, d)$ is *stackable* if the transitive closure of the relation "σ_1 is below σ_2" is a partial order.

Equivalently, a subdivision S is stackable if one can number its cells so that whenever two cells σ_1 and σ_2 are adjacent the one above has the higher label.

Rambau [18] has proved that all triangulations of $C(n, d)$ are stackable. For this he defines the following total order on subsets of $[n]$. Then he shows that the relation "being above" defined in the collection of simplices of a triangulation of $C(n, d)$ is compatible with the total order, which implies that triangulations are stackable. We will use this same total order to extend the result to all subdivisions of $C(n, d)$.

Definition 2.14. For a subset F of $[n]$ let $\gamma_{[n]}(F)$ be the following string on n characters: the i-th character $(\gamma_{[n]}(F))_i$ of $\gamma_{[n]}(F)$ is 'o' if i is an odd gap, 'e' if i is an even gap, and '$*$' if i is an element of F. The order "$<_{(o*e)}$" on the subsets F of $[n]$ is defined to be the lexicographic order on the strings $\gamma_{[n]}(F)$ according to the chain $o < * < e$.

Lemma 2.15. F_1 *and* F_2 *be a lower and upper facet of* $C(n, d)$*, respectively. Then there is a triangulation of* $C(n, d)$ *in which the simplices* σ_1 *and* σ_2 *incident to* F_1 *and* F_2 *satisfy* $\sigma_1 \leq_{(o*e)} \sigma_2$*.*

Proof. We will add one element to F_1 and one to F_2 to obtain d-simplices σ_1 and σ_2 such that the relation $\sigma_1 \leq_{(o*e)} \sigma_2$ holds and that σ_1 and σ_2 are either both upper or both lower facets of $C(n, d+1)$. Then σ_1 and σ_2 are contained in the triangulation of $C(n, d)$ consisting of all upper resp. lower facets of $C(n, d+1)$ (see Lemma 2.11), and we are done.

The lower facet F_1 has only even gaps (see Definition 2.10); the upper facet F_2 has only odd gaps. Let i be the smallest index which is not in both of F_1 and F_2.

We first show that i cannot be missing in both F_1 and F_2. Assume, without loss of generality, that i is an even gap in F_1. We show that in this case i is in F_2. Indeed: if i were not in F_2 then it would be an odd gap in F_2. Hence, there would be an even number of elements larger than i in F_1 and an odd number of elements larger than i in F_2. All elements smaller than i are contained in both F_1 and F_2, thus, this would imply that F_1 and F_2 have different cardinalities. But since $C(n, d)$ is simplicial, all facets have cardinality d: contradiction.

This shows that there are only the following two cases: Either i is in F_1 or in F_2.

CASE 1: $i \in F_1$. Then i is not in F_2. Let i_1 be the smallest index not in F_1 and let i_2 be the largest index not in F_2. By construction of i we have $i \leq i_1$ and $i \leq i_2$. Define

$$\sigma_1 := F_1 \cup \{i_1\} \quad \text{and} \quad \sigma_2 := F_2 \cup \{i_2\}.$$

Filling the smallest gap in F_1 implies that the "parity" of all remaining gaps stays the same. In other words: σ_1 has only even gaps. Filling the largest gap in F_2 changes the "parity" of all remaining gaps. That means that σ_2 has only even gaps as well. Thus, both σ_1 and σ_2 are lower facets of $C(n, d+1)$, as desired.

It remains to show that $\sigma_1 \leq_{(o*e)} \sigma_2$. If $i_2 = i$ then $\sigma_2 = C(n, d) = \sigma_1$, and we are done. If $i_2 \neq i$ then, by construction, $i_2 > i$, and the fact that i is an even gap in σ_2 while i is contained in σ_1 proves that $\sigma_1 \leq_{(o*e)} \sigma_2$.

CASE 2: $i \in F_2$. This case is completely analogous except that now we define σ_1 by adding to F_1 the *largest* index not in F_1 and σ_2 by adding to F_2 the *smallest* index not in F_2. Both σ_1 and σ_2 are then upper facets of $C(n, d+1)$ and $\sigma_1 \leq_{(o*e)} \sigma_2$ follows by arguments similar to case 1. □

Corollary 2.16. *Any subdivision of a cyclic polytope is stackable.*

Proof. Let S be a subdivision of $C(n, d)$. We want to prove that the relation "being above" defined on pairs of adjacent cells of S has no cycles. Suppose by way of contradiction that $\sigma_0, \sigma_1, \dots, \sigma_k = \sigma_0$ is a sequence of cells (with no repetitions) with σ_i adjacent and above σ_{i-1} for each $i = 1, \dots, k$. Let τ_i be the common facet of σ_i and σ_{i-1}. The cell σ_i $(i = 1, \dots, k)$ is itself a cyclic polytope and τ_i and τ_{i+1} are a lower and an upper facet of it respectively. By Lemma 2.15 we can refine the subdivision S to a triangulation T with the following property: if σ_{i-1}^+ and σ_i^- denote the simplices incident to τ_i below and above respectively, then $\sigma_i^- \leq_{(o*e)} \sigma_i^+$, for each $i = 0, \dots, k$. Then, by Rambau's result in [18], $\sigma_{i-1}^+ <_{(o*e)} \sigma_i^-$, for each i, so we get a directed cycle in the total order "$<_{(o*e)}$", which is impossible. □

2.4. **Miscellaneous.** We will need the following standard constructions on polyhedral complexes. For sets S, T of subsets $\sigma \subseteq A$ we define

$$\begin{aligned}
\mathrm{spanning}(S) &:= \{\, \sigma \in S \,:\, \sigma \text{ is spanning} \,\}, && \text{(spanning subsets)}\\
\mathrm{ast}_S(i) &:= \{\, \sigma \in S \,:\, i \notin \sigma \,\}, && \text{(antistar)}\\
\mathrm{lk}_S(i) &:= \{\, \sigma \setminus i \,:\, \sigma \in S, i \in \sigma \,\}, && \text{(link)}\\
S * T &:= \{\, \sigma \cup \tau \,:\, \sigma \in S, \tau \in T \,\}. && \text{(join)}
\end{aligned}$$

3. All Triangulations of $C(n, d)$ are Lifting Triangulations

In this section we present a commutative family of deletion constructions for subdivisions of cyclic polytopes, based on the deletion construction for triangulations which appears in [18]. As a consequence we get a canonical collection of subdivisions of the cyclic polytope $C(n, d)$ from any subdivision S of $C(n, d)$, and we will prove this collection to be consistent if S is a triangulation. This implies that all triangulations of cyclic polytopes are lifting triangulations. Although the construction of the family is valid for non-simplicial subdivisions as well, its consistency is not. Thus, we do not have a proof of liftingness for non-simplicial subdivisions. However, some of the constructions in this section will be used in Section 4 in the non-simplicial case.

First we recall the deletion of n in a triangulation of $C(n, d)$. We will use that the vertex figure of a cyclic polytope $C(n, d)$ on the last vertex n is a cyclic polytope $C(n-1, d-1)$. This is not always geometrically true if we require our cyclic polytopes to be realized with vertices along the moment curve: there is a choice of points on the moment curve such that the vertex figure has not all of its vertices on the moment curve. For a particular choice of points on the curve, however, this cannot happen (see [2, Lemma 4.8]). This implies—since the vertex figure construction translates to a well-defined operation on the oriented matroid: the *contraction*—that the oriented matroid of the vertex figure is always the alternating oriented matroid, which is good enough for our purposes. In particular, we have the following.

Lemma 3.1. *The link* $\mathrm{lk}_S(n)$ *at the vertex* n *of a subdivision* S *of* $C(n, d)$ *is a subdivision of* $C(n-1, d-1)$.

Theorem 3.2 (Deletion of n [18])**.** *Let* T *be a triangulation of* $C(n, d)$. *Then*

$$T \backslash n := \mathrm{ast}_T(n) \cup \left(\mathrm{ast}_{\mathrm{lk}_T(n)}(n-1) * \{n-1\}\right)$$

is a triangulation of $C(n-1, d)$ *that coincides with* T *on the antistar of* n *in* T.

Moreover, $T \backslash n$ *may be obtained by sliding vertex* n *to vertex* $n-1$ *in* T.

This result motivates the following generalization to subdivisions and to arbitrary vertices.

The fact that any subset of the vertices of a cyclic polytope $C(n, d)$ is the set of vertices of a cyclic polytope as well will be crucial for the rest of this section. For any subset $A \subset [n]$ we denote by $C(A, d)$ the cyclic polytope having as vertices those vertices of $C(n, d)$ with labels in A (here we are assuming a particular embedding of $C(n, d)$, although what the embedding is will not really be important).

Theorem 3.3 (Deletion in Subdivisions)**.** *Let* S *be a subdivision of the cyclic polytope* $C(n, d)$. *Then*

$$S^{i \to i-1} := \mathrm{ast}_S(i) \cup \mathrm{spanning}\big(\mathrm{lk}_S(i) * \{i-1\}\big)$$

is a subdivision of $C([n] \backslash i, d)$, $\forall 1 < i \leq n$,

$$S^{i \to i+1} := \mathrm{ast}_S(i) \cup \mathrm{spanning}\big(\mathrm{lk}_S(i) * \{i+1\}\big)$$

is a subdivision of $C([n] \backslash i, d)$, $\forall 1 \leq i < n$.

Observe that if T is a triangulation and $i = n$, then the definition of $T^{n \to n-1}$ coincides with that of $T \backslash n$ in Theorem 3.2. Even if S is not a triangulation we will denote $S \backslash n := S^{n \to n-1}$ in Section 4.

Proof. We will only prove the case of $S^{i\to i-1}$. The other one is analogous. The set of cells $S^{i\to i-1}$ may be constructed from S in the following geometric way: slide vertex i continuously to vertex $i-1$ in $C(n,d)$ along the moment curve to obtain $C([n]\backslash i,d)$. Let the time interval in which this happens be $[0,1]$. At any time $0\le t<1$, the point configuration is still a cyclic polytope $C(n,d)$ and the subdivision S combinatorially stays the same. At time $t=1$

- all cells not containing i are still the same;
- all cells containing i and $i-1$ have collapsed to cells with one vertex less, where d-simplices of this type have collapsed to $(d-1)$-simplices;
- in cells containing i and not $i-1$, i is replaced by $i-1$.

To see that the final stage of the slide yields a subdivision consider the following two d-volumes:

- the d-volume of the part of $C(n,d)$ resp. $C([n]\backslash i,d)$ that is covered by the interior of more than one d-cell;
- the d-volume of the part of $C(n,d)$ resp. $C([n]\backslash i,d)$ that is covered by the interior of less than one d-cell.

Both volumes are continuous functions of the vertex coordinates, thus of the slide time t. Since both volumes are zero for S, both volumes are zero for all $0\le t<1$. By continuity, both volumes are zero for $t=1$ as well. But this, together with genericity of the set of vertices of a cyclic polytope, means that $S^{i\to i-1}$ is a subdivision. □

Let S be a subdivision of the cyclic polytope $C(n,d)$. We can now define a collection of subdivisions of the subsets of vertices of $C(n,d)$ recursively: we define $S_{[n]}=S$ and for each subset $A=\{a_1,\dots,a_{\#A}\}\subset[n]$ and $a_i\in A$ we define $S_{A\backslash a_i}=S_A^{a_i\to a_{i-1}}$ if $i\ne 1$ and $S_{A\backslash a_1}=S^{a_1\to a_2}$. We will call *DEL*$(S)$ the collection of subdivisions so obtained. The following commutativity relations imply that S_A is well-defined in the sense that it is independent of the order in which we eliminate the elements of $[n]$ in order to arrive at A.

Theorem 3.4. *Let* $A=\{a_1,\dots,a_{\#A}\}\subseteq[n]$ *and let* S *be a subdivision of* $C(A,d)$*. Then*

$$\begin{aligned}
(S^{a_i\to a_{i-1}})^{a_j\to a_{j-1}} &= (S^{a_j\to a_{j-1}})^{a_i\to a_{i-1}} && \forall 2\le i<j-1\le \#A-1;\\
(S^{a_i\to a_{i-1}})^{a_{i-1}\to a_{i-2}} &= (S^{a_{i-1}\to a_{i-2}})^{a_i\to a_{i-2}} && \forall 3\le i\le \#A;\\
(S^{a_i\to a_{i-1}})^{a_1\to a_2} &= (S^{a_1\to a_2})^{a_i\to a_{i-1}} && \forall 3\le i\le \#A;\\
(S^{a_2\to a_1})^{a_1\to a_3} &= (S^{a_1\to a_2})^{a_2\to a_3}.
\end{aligned}$$

Proof. The assertions are easily observed by considering the corresponding slides. □

Theorem 3.5. *If* S *is a triangulation of a cyclic polytope* $C(n,d)$*, then the collection of subdivisions DEL*(S) *obtained in this way from* S *is consistent. Thus, any triangulation of a cyclic polytope is a lifting subdivision.*

Proof. We first observe that if S is a triangulation then the construction $S^{i\to i-1}$ (same for $S^{i\to i+1}$) produces a triangulation as well. This is true because the cells in $S^{i\to i-1}$ are either cells of S or spanning sets of the form $\tau\cup\{i-1\}$ where $\tau\cup\{i\}$ is a (simplicial) cell in S.

Thus, the family *DEL*(S) is, in fact, a family of triangulations. We will prove that it satisfies property (i") of Lemma 2.8 and property (ii') of Lemma 2.7.

For a triangulation S_B and a cell $\tau\in S_B$ the only way in which a $\tau\backslash\{b\}$ can be spanning is that $b\notin\tau$. In this case, $\tau\in \mathrm{ast}_{S_B}(i)\subset S_{B\backslash b}$. Thus, property (i") holds.

For proving property (ii') of Lemma 2.7, let $B\subset[n]$ and let $b,c\in B$. Moreover, let $\sigma\subseteq B$ be a basis contained in cells of both $S_{B\backslash\{b\}}$ and $S_{B\backslash\{c\}}$. Because all subdivisions are triangulations we have

$$\sigma \in S_{B\setminus\{b\}} \cap S_{B\setminus\{c\}}.$$

We claim that σ is in S_B. Assume—for the sake of contradiction—that σ is not in S_B. Then there is a cell $\sigma_b \in S_B$ with $b \in \sigma_b$ such that b slides to some $b' \in \sigma$ during the construction of $S_{B\setminus\{b\}}$. Similarly, there is a cell $\sigma_c \in S_B$ with $c \in \sigma_c$ slides to some $c' \in \sigma$ during the construction of $S_{B\setminus\{c\}}$.

Because σ_b and σ_c are both contained in S_B they must be equal. Indeed: since σ_b and σ_c intersect in σ after the slides, the continuity of the sliding process implies that the volume of their intersection during the whole slide must have been non-zero. This together with proper intersection in S_B is only possible if $\sigma_b = \sigma_c$. Because c is in $\sigma_c = \sigma_b$ and σ is in $S_{B\setminus\{b\}}$ we know that c is also in σ; on the other hand, σ in $S_{B\setminus\{c\}}$ implies that c is not in σ: contradiction. Therefore, property (ii) holds. ☐

This proves Theorem 1.3.

Remark 3.6. We present a simple example showing that if the original subdivision S is not a triangulation then the construction of the family of subdivisions need not be consistent. Let $n = 5$, $d = 2$, and let $S = \{1235, 345\}$ be the original subdivision of $C(5, 2)$. Then, the slide of the vertex 5 to 4 produces the trivial subdivision $S^{5\to4} = \{1234\}$ of $C(4, 2)$. If we now take $B = \{12345\}$, $\tau = \{1235\} \in S_B = S$, and $B' = \{1234\}$ we find that condition (i) of the definition of consistency (or any of its equivalents (i') and (i'')) is not satisfied: $\tau \cap B' = \{123\}$ is not a face of any cell of $S_{B'} = S^{5\to4} = \{1234\}$.

The geometric idea behind this example is that $S_{B'}$ is not consistent with S_B because $S_{B'}$ can only be obtained by a lift in which 1, 2, 3, and 4 are coplanar and S_B by one in which they are not coplanar.

4. The Baues Poset of Subdivisions of $C(n, d)$ is Spherical

In this section we consider the poset of all subdivisions of a cyclic polytope $C(n, d)$. We are going to see that it is homotopy equivalent to the $(n - d - 2)$-sphere, thus proving Theorem 1.1.

The idea is to use induction on the number of vertices and to show that the poset $\omega(C(n, d))$ of subdivisions of $C(n, d)$ is homotopy equivalent to the suspension of the poset $\omega(C(n - 1, d))$ of subdivisions of $C(n - 1, d)$.

The crucial map that provides us with an inductive argument is the deletion for subdivisions at n, which was defined in the previous section. Throughout this section we will denote by $S\backslash n$ the subdivision $S^{n\to n-1}$ of $C(n - 1, d)$ obtained by sliding the vertex n to $n - 1$ in a subdivision S of $C(n, d)$.

Let $\widehat{\omega}(C(n - 1, d))$ be the poset of non-trivial subdivisions of $C(n - 1, d)$ augmented with two extra elements S_n and S_{n-1} which are incomparable and above every other element of $\omega(C(n - 1, d))$. Then we define the following order-preserving map of posets.

Definition 4.1.

$$\Pi : \begin{cases} \omega(C(n, d)) & \to \quad \widehat{\omega}(C(n - 1, d)), \\ S & \mapsto \quad \begin{cases} S_{n-1} & \text{if } [n - 1] \in S, \\ S_n & \text{if } ([n - 2] \cup \{n\}) \in S, \\ S\backslash n & \text{otherwise.} \end{cases} \end{cases}$$

Observe the following: $[n-1]$ and $[n-2] \cup \{n\}$ cannot be both cells in S because they intersect improperly unless $n \le d + 1$, in which case they are not spanning. If none of these cells is in S then $S\backslash n$ is non-trivial by construction. Thus, Π is

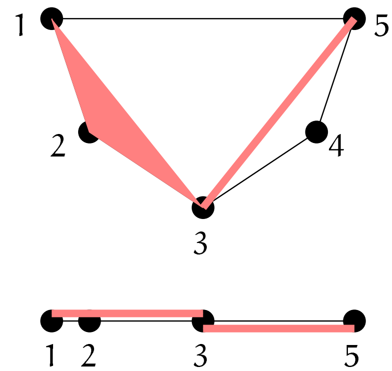

FIGURE 1. The subdivision $\{123, 35\}$ of $C(5, 1)$, induced by the cells 123 and 35 of the subdivision $\{123, 135, 345\}$ of $C(5, 2)$.

well-defined. Also, $\Pi^{-1}(S_n)$ and $\Pi^{-1}(S_{n-1})$ have a single element, namely the subdivisions obtained from the trivial one by pushing $n-1$ and n, respectively (see Definition 2.4). Since these two subdivisions are easily seen to be maximal elements in $\omega(C(n, d))$ and since the deletion operator is order-preserving, Π is order preserving.

Theorem 4.2. *The map $\Pi : \omega(C(n, d)) \to \widehat{\omega}(C(n-1, d))$ is a homotopy equivalence. In particular, $\omega(C(n, d))$ is homotopy equivalent to an $(n-d-2)$-sphere.*

Proof. Let us first show how to derive the second part from the first one. It is well known that $\omega(C(n, d))$ is homeomorphic to an $(n-d-2)$-sphere whenever $n \le d+3$ because then all subdivisions are regular (see, e.g., [13]) and $\omega(C(n, d))$ is the face poset of the secondary polytope. Then, if we fix d and apply induction on n, we can inductively assume that $\omega(C(n-1, d))$ is homotopically an $(n-d-3)$-sphere. Since $\widehat{\omega}(C(n-1, d))$ is the suspension of $\omega(C(n-1, d))$, it is homotopically an $(n-d-2)$-sphere. Thus, if Π is a homotopy equivalence, $\omega(C(n, d))$ is homotopically an $(n-d-2)$-sphere.

For proving that Π is a homotopy equivalence we will use the following Lemma from [3]. A proof of this lemma appears in [23].

Lemma 4.3 (Babson)**.** *Let $f : P \to Q$ be an order-preserving map of posets. If*

(i) $f^{-1}(x)$ *is contractible for every* $x \in Q$*, and*
(ii) $P_{\le y} \cap f^{-1}(x)$ *is contractible for every* $x \in Q$ *and* $y \in P$ *with* $f(y) > x$*,*

then f *induces a homotopy equivalence.* □

Let $S \in \widehat{\omega}(C(n-1, d))$. If $S \in \{S_n, S_{n-1}\}$ then the two conditions of Lemma 4.3 are trivial for S. Otherwise, we will prove in Lemma 4.7 that the posets $\Pi^{-1}(S)$ and $\omega(C(n, d))_{\le S'} \cap \Pi^{-1}(S)$ (where S' denotes a subdivision of $C(n, d)$ with $S' \setminus n$ coarser than S) are respectively isomorphic to certain subposets $\omega(\mathrm{lk}_S(n-1))$ and $\omega_{\le S_0}(\mathrm{lk}_S(n-1))$ of $C(n-2, d-2)$. These subposets are defined below and proved to be contractible in Theorem 4.5 and Corollary 4.6. □

In the remainder of this section, we provide the details referenced in the proof above.

Definition 4.4. Let $\overline{S}$ be a subdivision of the cyclic polytope $C(n, d+1)$ and S a subdivision of $C(n, d)$. S is *induced* by $\overline{S}$ if every cell $\sigma \in S$ is a face (perhaps a non-proper one) of a cell $\sigma' \in \overline{S}$.

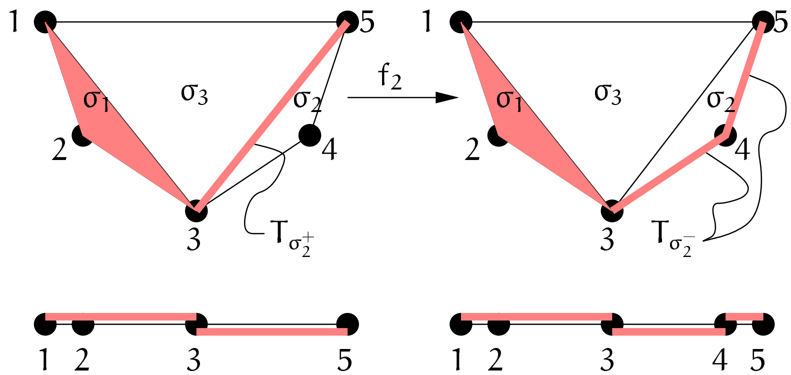

FIGURE 2. An example of the map f_i for $d = 1$ and $i = 2$: the map f_2 sends $\{123, 35\}$ to $\{123, 34, 45\}$. The inverse image $f_2^{-1}(\{123, 34, 45\})$ consists of the subdivisions $\{123, 35\}$, $\{123, 345\}$ and $\{123, 34, 45\}$. Also, $g_2(\{123, 34, 45\}) = \{123, 345\}$.

See Figure 1 for a sketch in dimension one. One can think of subdivisions of $C(n, d)$ as cellular sections of the natural projection $C(n, d+1) \to C(n, d)$. The subdivisions of $C(n, d)$ induced by a subdivision $\overline{S}$ of $C(n, d+1)$ are the sections "contained" in $\overline{S}$. In what follows we are interested in the refinement poset of all subdivisions of $C(n, d)$ which are induced by a certain subdivision $\overline{S}$ of $C(n, d+1)$. We will denote this poset by $\omega(\overline{S})$ and want to prove that it is homotopically equivalent to a single point (i.e., contractible).

Theorem 4.5. *The poset $\omega(\overline{S})$ of subdivisions of $C(n, d)$ which are induced by a subdivision $\overline{S}$ of $C(n, d+1)$ is contractible.*

Proof. Let k denote the number of cells in $\overline{S}$. By Corollary 2.16, there is a numbering $\sigma_1, \dots, \sigma_k$ of the cells of $\overline{S}$ such that if σ_i is above σ_j then $i > j$.

Let $S \in \omega(\overline{S})$ be a subdivision of $C(n, d)$. Let us regard S as a collection of (perhaps non-proper) faces of cells of $\overline{S}$. Then, for every cell σ_i of $\overline{S}$ we can tell whether σ_i is above, on, or below S. Let us call *height* of S the maximal index i of a cell σ_i on or below S. For each $i = 0, \dots, k$ we denote $\omega(\overline{S}; i)$ the subposet of $\omega(\overline{S})$ consisting of the subdivisions of height at most i. It is obvious that $\omega(\overline{S}) = \omega(\overline{S}; k)$ and that $\omega(\overline{S}; 0)$ has a single element: the lower envelope of $C(n, d+1)$. In what follows we will prove that $\omega(\overline{S}; i)$ and $\omega(\overline{S}; i-1)$ are homotopically equivalent, for every $i = 1, \dots, k$.

Consider first the following situation. Let $S \in \omega(\overline{S})$ with $\sigma_i \in S$. Then we can get two new elements S_{σ_i+} and S_{σ_i-} of $\omega(\overline{S})$ substituting σ_i in S for its upper and lower envelope, respectively.

We now construct the homotopy equivalence $f_i : \omega(\overline{S}; i) \to \omega(\overline{S}; i-1)$. We define f_i to be the identity on those $S \in \omega(\overline{S}; i)$ with height at most $i-1$. If S has height i then either S contains σ_i, in which case we take $f_i(S) = S_{\sigma_i-}$, or S contains the upper envelope of σ_i. In this case $S = T_{\sigma_i^+}$ for some $T \in \omega(\overline{S})$. We then define $f_i(T_{\sigma_i^+}) = T_{\sigma_i^-}$. See Figure 2 for a sketch in dimension one.

In this way, the inverse image of an element $S \in \omega(\overline{S}; i-1)$ is

(i) S itself if S does not contain the lower envelope of σ_i.

(ii) If S contains the lower envelope of σ_i, then $S = T_{\sigma_i-}$ for some $T \in \omega(\overline{S}; i)$ and $f^{-1}(S) = f^{-1}(T_{\sigma_i-}) = \{T, T_{\sigma_i-}, T_{\sigma_i+}\}$.

Define the following order-preserving map:

$$g_i : \begin{cases} \omega(\overline{S}; i-1) & \to \quad \omega(\overline{S}; i), \\ \qquad S & \mapsto \quad \begin{cases} S & \text{in case (i)}, \\ T & \text{in case (ii)}. \end{cases} \end{cases}$$

Then $f_i \circ g_i = \mathrm{id}_{\omega(\overline{S};i-1)}$ and $g_i \circ f_i \geq \mathrm{id}_{\omega(\overline{S};i)}$, which means that f_i and g_i are homotopy inverses to each other by Quillen's order homotopy theorem [7, 10.11]. Thus, $\omega(\overline{S}; i)$ is homotopy equivalent to $\omega(\overline{S}; i-1)$. □

We consider now the following situation. Let $\overline{S}$ be a subdivision of $C(n, d+1)$. Let S_0 be a subdivision of $C(n, d)$ such that $S_0 \in \omega(\overline{S_0})$ for some $\overline{S_0}$ coarser than $\overline{S}$ (in particular, for every cell σ in S_0 the collection $\{\ \sigma' \in \overline{S} : \sigma' \subset \sigma\ \}$ is a subdivision of σ). We denote $\omega_{\leq S_0}(\overline{S})$ the subposet of $\omega(\overline{S})$ consisting of subdivisions of $C(n, d)$ which refine S_0. Then:

Corollary 4.6. *$\omega_{\leq S_0}(\overline{S})$ is contractible.*

Proof. Let $S_0 = \{\tau_1, \ldots, \tau_k\}$. Since every subconfiguration of a cyclic polytope is a cyclic polytope as well, if we consider τ_i as a cell of $\overline{S_0}$, which is a subdivision of $C(n, d+1)$, then the refinement $\overline{S}$ of $\overline{S_0}$ induces a subdivision S_i of $C(\tau_i, d+1)$. It makes sense then to consider the poset $\omega(S_i)$, which is contractible by Theorem 4.5. Since in a generic configuration the different cells of a subdivision can be refined independently, one can easily prove that the poset $\omega_{\leq S_0}(\overline{S})$ is (isomorphic to) the direct product of the posets $\omega(S_i)$. Thus, it is contractible. □

We are now in position to prove the crucial statement of this section, which relates the fibers of the map Π of Definition 4.1 with the posets $\omega(\overline{S})$ and $\omega_{\leq S_0}(\overline{S})$.

Lemma 4.7. *Let $\Pi : \omega(C(n, d)) \to \widehat{\omega}(C(n-1, d))$ be the order preserving map of Definition 4.1. Let $S' \in \omega(C(n, d))$ be a non-trivial subdivision of $C(n, d)$ and let $S \in \omega(C(n-1, d))$ be a non-trivial subdivision of $C(n-1, d)$ with $S \leq \Pi(S')$ (i.e. S refines $S' \backslash n$).Then:*

1. *The poset $\Pi^{-1}(S)$ is isomorphic to the poset $\omega(\mathrm{lk}_S(n-1)) \subset \omega(C(n-2, d-2))$.*
2. *Let $S_0 = \mathrm{lk}_{S'}(\{n, n-1\})$, which is a subdivision of $C(n-2, d-2)$. Then the subdivision $\overline{S}_0 = \mathrm{lk}_{S' \backslash n}(n-1)$ of $C(n-2, d-1)$ is coarser than $\mathrm{lk}_S(n-1)$ and satisfies that $S_0 \in \omega(\overline{S_0})$. In particular, the poset $\omega_{\leq S_0}(\mathrm{lk}_S(n-1))$ is non-empty and well-defined.*
3. *$\omega(C(n, d))_{\leq S'} \cap \Pi^{-1}(S)$ is isomorphic to the poset $\omega_{\leq S_0}(\mathrm{lk}_S(n-1))$ for the subdivision S_0 of $C(n-2, d-2)$ defined above.*

Thus, $\Pi^{-1}(S)$ and $\omega(C(n, d))_{\leq S'} \cap \Pi^{-1}(S)$ are contractible.

Proof. (1) Observe first that $\mathrm{lk}_S(n-1)$ is a subdivision of $C(n-2, d-1)$. Thus, $\omega(\mathrm{lk}_S(n-1))$ is a collection of subdivisions of $C(n-2, d-2)$. We define the following order-preserving map of posets:

$$\pi : \Pi^{-1}(S) \to \omega(\mathrm{lk}_S(n-1))$$

by $\pi(T) = \mathrm{lk}_T(\{n, n-1\})$. This map is well-defined because $\Pi(T) = S$ implies that $\mathrm{lk}_T(\{n, n-1\}) \subset \mathrm{lk}_S(n-1)$, and is clearly order-preserving. For proving (i) we only need to show that π is bijective and π^{-1} order-preserving.

For this we observe the following: let $T \in \Pi^{-1}(S)$ and $\sigma \in \mathrm{lk}_S(n-1)$. We can say whether σ lies in, above or below $\pi(T) = \mathrm{lk}_T(\{n, n-1\})$, as we did in the proof of Theorem 4.5. Then, $\sigma \cup \{n\} \in T$ (resp. $\sigma \cup \{n-1\} \in T$ or $\sigma \cup \{n, n-1\} \in T$) if and only if σ is below $\pi(T)$ (resp. above $\pi(T)$, or in $\pi(T)$). Let us consider the map

$$\pi^{-1} : \omega(\mathrm{lk}_S(n-1)) \to \Pi^{-1}(S)$$

defined by

$$\begin{aligned}\pi^{-1}(T) := \{\, \sigma \in S \,:\, n-1 \notin \sigma \,\} \\ \cup \{\, \sigma \cup \{n\} \,:\, \sigma \in \mathrm{lk}_S(n-1),\ \sigma \text{ is below } T \,\} \\ \cup \{\, \sigma \cup \{n-1\} \,:\, \sigma \in \mathrm{lk}_S(n-1),\ \sigma \text{ is above } T \,\} \\ \cup \{\, \sigma \cup \{n, n-1\} \,:\, \sigma \in \mathrm{lk}_S(n-1),\ \sigma \in T \,\}.\end{aligned}$$

One can prove that $\pi^{-1}(T)$ is indeed a subdivision (e.g., by using Lemma 2.2), and it follows from the definition of π^{-1} that

$$\pi^{-1}(T)\backslash n = S \quad \text{and} \quad \pi \circ \pi^{-1}(T) = \mathrm{lk}_{\pi^{-1}(T)}(\{n, n-1\}) = T$$

(i.e., $\pi^{-1}(T) \in \Pi^{-1}(S)$, and π^{-1} is well-defined and π is surjective). Finally, the remark above proves that $\pi^{-1} \circ \pi$ is the identity map and thus π is injective.

(2) Since S refines $S'\backslash n$, the subdivision $\mathrm{lk}_S(n-1)$ refines $\overline{S_0} = \mathrm{lk}_{S'\backslash n}(n-1)$. This proves the first assertion. We now have to prove that $S_0 \in \omega(\overline{S_0})$. That is to say, that every cell of $S_0 = \mathrm{lk}_{S'}(\{n, n-1\})$ is a face of a cell of $\overline{S}_0 = \mathrm{lk}_{S'\backslash n}(n-1)$. Let σ be a cell of $\mathrm{lk}_{S'}(\{n, n-1\})$. By definition of link, $\sigma \cup \{n, n-1\}$ is a cell of S'. Then,

- If $\sigma \cup \{n-1\}$ is spanning in $C(n-1, d)$ then it is a cell of $S'\backslash n$ (by definition of $S'\backslash n = S'^{\,n \to n-1}$) and thus σ is a cell in $\mathrm{lk}_{S'\backslash n}(n-1)$.
- If $\sigma \cup \{n-1\}$ is not spanning in $C(n-1, d)$, then $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are facets of $\sigma \cup \{n-1, n\}$ in S'. In the sliding process $n \to n-1$ these two facets match to one another, so that $\sigma \cup \{n-1\}$ becomes a facet of some cell in $S'\backslash n$ and σ is a facet of some cell in $\mathrm{lk}_{S'\backslash n}(n-1)$.

(3) We just need to prove that the order-preserving bijection π of part (1) restricts to a bijection between $\omega(C(n, d))_{\le S'} \cap \Pi^{-1}(S) \subset \Pi^{-1}(S)$ and $\omega_{\le S_0}(\mathrm{lk}_S(n-1)) \subset \omega(\mathrm{lk}_S(n-1))$.

Let $T \in \omega(C(n, d))_{\le S'} \cap \Pi^{-1}(S)$. Since T refines S', $\pi(T) = \mathrm{lk}_T(\{n, n-1\})$ refines $\mathrm{lk}_{S'}(\{n, n-1\}) = S_0$ and thus $\pi(T) \in \omega_{\le S_0}(\mathrm{lk}_S(n-1))$.

Reciprocally, let $T \in \omega_{\le S_0}(\mathrm{lk}_S(n-1))$, so that T refines $\mathrm{lk}_{S'}(\{n, n-1\})$. We want to see that $\pi^{-1}(T)$ refines S'. We consider the four types of cells in $\pi^{-1}(T)$ and see that they are contained in cells of S':

- If $\sigma \in S$ with $n-1 \notin \sigma$, then $\sigma \subset \sigma'$ for some $\sigma' \in S'\backslash n$, since S refines $S'\backslash n$. The fact that $n-1 \notin \sigma$ implies that also $\sigma \subset \sigma''$ for some $\sigma'' \in S'$ (σ'' is the cell of S' whose slide produces the cell σ' of $S\backslash n$).
- If $\sigma' = \sigma \cup \{n, n-1\}$ with $\sigma \in T \subset \mathrm{lk}_S(n-1)$, then σ' is contained in a cell of S' since T refines $\mathrm{lk}_{S'}(\{n, n-1\})$.
- If $\sigma' = \sigma \cup \{n\}$ with $\sigma \in \mathrm{lk}_S(n-1)$ below T, then $\sigma \cup \{n-1\} \in S$ is contained in a cell of $S'\backslash n$ (because S refines $S'\backslash n$). Let $\sigma'' \cup \{n-1\}$ be that cell. The facts that T refines $\mathrm{lk}_{S'}(\{n, n-1\})$ and σ is below T imply that σ'' is in or below $\mathrm{lk}_{S'}(\{n, n-1\})$. Thus, either $\sigma'' \cup \{n\}$ or $\sigma'' \cup \{n, n-1\}$ are in S'. In particular, $\sigma' = \sigma \cup \{n\}$ is contained in a cell of S'.
- In a similar way, if $\sigma' = \sigma \cup \{n-1\}$ with $\sigma \in \mathrm{lk}_S(n-1)$ above T, then $\sigma \cup \{n-1\} \in S$ is contained in a cell of $S'\backslash n$ (because S refines $S'\backslash n$). Let $\sigma'' \cup \{n-1\}$ be that cell. The facts that T refines $\mathrm{lk}_{S'}(\{n, n-1\})$ and σ is above T imply that σ'' is in or above $\mathrm{lk}_{S'}(\{n, n-1\})$. Thus, either $\sigma'' \cup \{n-1\}$ or $\sigma'' \cup \{n, n-1\}$ are in S'. In particular, $\sigma' = \sigma \cup \{n-1\}$ is contained in a cell of S'.

$\square$

5. The Number of Bistellar Neighbors

We will now provide a triangulation of $C(11,5)$ with flip deficiency, i.e., fewer flips than the dimension of the corresponding secondary polytope. This example was found (together with the others mentioned in Theorem 1.2) while enumerating the set of all triangulations of $C(11,5)$ and $C(12,5)$ by a special C++ computer program. The algorithm makes full use of the fact that the set of triangulations of a cyclic polytope forms a bounded poset [10]. Modulo implementation details, the algorithm is straightforward; thus we do not discuss it here. Table 1 contains the resulting numbers of triangulations. This same table appears in [2].

number of points:	3	4	5	6	7	8	9	10	11	12
dimension 2	1	2	5	14	42	132	429	1,430	4,862	16,796
dimension 3		1	2	6	25	138	972	8,477	89,405	1,119,280
dimension 4			1	2	7	40	357	4,824	96,426	2,800,212
dimension 5				1	2	8	67	1,233	51,676	5,049,932
dimension 6					1	2	9	102	3,278	340,560
dimension 7						1	2	10	165	12,589
dimension 8							1	2	11	244
dimension 9								1	2	12
dimension 10									1	2

Table 1. The number of triangulations of $C(n,d)$ for $n \leq 12$.

Example 5.1. Throughout this section, T will be the following collection of 36 simplices in $C(11,5)$. We give it in five pieces which we call T_3, T_9, T_6, T_- and T_+, since the vertices 3, 6 and 9 play a special role in them. All the simplices in T contain either 3 or 9. The parts T_3, T_9 and T_6 consist respectively of those simplices not containing 3, not containing 9 and containing both 3 and 9 but not 6. Then, T_- and T_+ consist of the simplices containing 3, 6 and 9, divided into two groups according to whether they contain two elements in $\{1,2,4,5\}$ and one in $\{7,8,10,11\}$ or vice versa. T is symmetric under the reversal of the indices.

$$\begin{aligned}
T_3 := \{&\{1,2,6,7,8,9\},\{1,2,6,7,9,11\},\{1,2,7,8,9,11\},\\
&\{1,6,7,9,10,11\},\{1,7,8,9,10,11\},\{4,5,6,7,9,11\},\\
&\{4,5,6,9,10,11\},\{4,5,7,8,9,11\},\{5,6,7,8,9,11\}\}\\
T_9 := \{&\{3,4,5,6,10,11\},\{1,3,5,6,10,11\},\{1,3,4,5,10,11\},\\
&\{1,2,3,5,6,11\},\{1,2,3,4,5,11\},\{1,3,5,6,7,8\},\\
&\{1,2,3,6,7,8\},\{1,3,4,5,7,8\},\{1,3,4,5,6,7\}\}\\
T_6 := \{&\{1,2,3,9,10,11\},\{1,3,4,5,8,9\},\{1,3,4,5,9,10\},\\
&\{2,3,7,8,9,11\},\{3,4,5,7,8,9\},\{3,4,7,8,9,11\}\}\\
T_- := \{&\{1,2,3,6,8,9\},\{1,2,3,6,9,11\},\{1,3,5,6,8,9\},\\
&\{1,3,5,6,9,10\},\{3,4,5,6,9,10\},\{3,4,5,6,7,9\}\}\\
T_+ := \{&\{3,4,6,9,10,11\},\{1,3,6,9,10,11\},\{3,4,6,7,9,11\},\\
&\{2,3,6,7,9,11\},\{2,3,6,7,8,9\},\{3,5,6,7,8,9\}\}.
\end{aligned}$$

T is a triangulation and has only the following four bistellar flips: Two upward ones supported on

$$\{1,2,3,6,7,8,9\},\{3,4,5,6,9,10,11\},$$

and two downward ones, supported on

$$\{1,2,3,6,9,10,11\},\{3,4,5,6,7,8,9\}.$$

The fact that this example is a triangulation and has only the claimed flips is computationally straightforward once the example is in hand. It was checked once more by the maple program PUNTOS [14] which studies triangulations of arbitrary point configurations and by two other independent maple routines. Here we will only give a brief sketch of how to proof the second part.

Theorem 5.2. *The collection* T *of simplices of Example 5.1 is a triangulation of* $C(11, 5)$*.*

Theorem 5.3. *Let* $A = \{a_1, \dots, a_7\}$ *be a circuit of* $C(11, 5)$ *which supports a flip of* T*. Then,*

(i) A *contains 3 and 9.*
(ii) A *contains 6.*
(iii) A *contains exactly two elements among 1, 2, 4 and 5 and other two among 7, 8, 10 and 11.*
(iv) A *contains one of the pairs* $\{1, 2\}$, $\{4, 5\}$ *and one of* $\{7, 8\}$, $\{10, 11\}$*.*

Thus, T *has only the four bistellar flips mentioned in Example 5.1.*

Proof. To say that $A = \{a_1, \dots, a_7\}$ supports a flip of T means that T contains one of the two triangulations of A, which are

$$T^e_A := \{ A\backslash\{a_i\} : i = 2, 4, 6 \} \quad \text{and} \quad T^o_A := \{ A\backslash\{a_i\} : i = 1, 3, 5, 7 \},$$

where we assume $a_1 < \cdots < a_7$. Moreover, the flip supported on A is upward (in the poset structure on the collection of triangulations of $C(11, 5)$) if $T^o_A \subset T$ and downward if $T^e_A \subset T$.

If $A = \{a_1, \dots, a_7\}$ supports a flip, at least three simplices of T have to be contained in A and at least two of them must contain a_i, for each $i = 1, \dots, 7$. With this simple remark and the fact that T is symmetric under reversal of indices the interested reader can prove parts (i), (ii), (iii), and (iv) by a not too long case study. For example, if A does not contain 3 or 9 then one of the two triangulations T^e_A or T^o_A of A is contained in T_3, T_9 respectively. If it does not contain 6, then T^e_A or T^o_A is contained in the twelve simplices of T which do not use 6 (the six simplices of T_6 together with three from T_3 and three from T_9). This reduces considerably the search.

The conclusion of the Theorem follows from parts (i), (ii), (iii), and (iv) as follows: By (i), (ii), and (iii) A contains 3, 6 and 9 plus two vertices among 1, 2, 4 and 5 and other two among 7, 8, 10 and 11. Then (iv) implies that the only four possibilities for A are those in Example 5.1. That these four circuits actually support flips can be trivially checked by finding among the simplices in T one of the two triangulations T^o_A and T^e_A, for each case. Also, this check tells whether the flip is upwards or downwards. □

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