The Baues Conjecture in corank 3. *

Miguel Azaola
Universidad de Cantabria
Departamento de Matemáticas, Estadística y Computación
Av. de los Castros s/n, 39071 Santander, SPAIN
E-mail: azaola@matesco.unican.es

March 16, 2000

Abstract

We prove that the space of polyhedral subdivisions of a configuration of $r+3$ vectors in r-space is spherical or contractible depending on whether the configuration is acyclic or not, thereby proving a special case of the Generalized Baues Conjecture.

Introduction

The Baues problem concerns the study of the space of all the *polyhedral subdivisions* of an *acyclic vector configuration* [19].

A vector configuration $\mathcal{A}$ in $\mathbb{R}^r$ is a finite spanning set of labelled vectors (we allow repetitions) in the linear space $\mathbb{R}^r$. If there exists a linear hyperplane which leaves all the elements of $\mathcal{A}$ on the the same open half-space, then $\mathcal{A}$ is said to be acyclic or *pointed*. The number r is called the *rank* of $\mathcal{A}$, while the *corank* of $\mathcal{A}$ is $\#(\mathcal{A}) - r$.

Following [3] and [11] we introduce the following definitions (see section 1 for precise definitions of the terms involved):

Definition 1

- A *polyhedral subdivision* (or *subdivision*, for short) of $\mathcal{A}$ is a *covering* collection of *cells* of $\mathcal{A}$ which pairwise *intersect properly*.
- A *triangulation* of $\mathcal{A}$ is a subdivision whose cells are *simplices* of $\mathcal{A}$.

We will sometimes regard a triangulation not as a mere set of full-dimensional simplices, but as a set of full-dimensional simplices and their faces (i.e. as a simplicial complex). We will often switch between these two points of view.

*Supported by project PB97-0358 of the Spanish "Dirección General de Investigación Científica y Técnica".

Given subdivisions S_1 and S_2 of $\mathcal{A}$, we say that S_1 *refines* S_2 (and denote it by $S_1 \leq S_2$) if every cell of S_1 is contained in some cell of S_2. The refinement relation is a partial ordering in the set of all subdivisions of $\mathcal{A}$. The *trivial* subdivision (consisting of only one cell $\mathcal{A}$ and denoted by $\hat{1}$) is clearly the unique maximal element.

Definition 2 The *Baues poset* of $\mathcal{A}$ is the set

$$\mathrm{Baues}(\mathcal{A}) := \{S \ : \ S \ subdivision \ of \ \mathcal{A}, \ S \neq \hat{1}\}$$

partially ordered by refinement.

Every partially ordered finite set (or *poset*) P has naturally associated a simplicial complex, known as the *order complex* of P [6]. Its vertices are the elements of P and its simplices are the chains in P. The order complex associated to Baues($\mathcal{A}$) is known as the *Baues complex* of $\mathcal{A}$. When talking about topological properties of a poset, we refer to the associated order complex (as it is usually done in the literature). Thus, for example, the homotopy type of a poset means the homotopy type of its order complex, and a contractible poset is one whose order complex has the homotopy type of a single point.

The *Generalized Baues Conjecture for Triangulations* (or GBCT, for short) claims that the Baues poset of a corank k acyclic vector configuration has the homotopy type of a $(k-1)$-dimensional sphere. See [19] for an overview on the matter. In rank or corank at most 2 the Baues poset is known to be not only homotopy spherical, but homeomorphic to the $(k-1)$-sphere: In these cases every subdivision is *regular* and the poset of regular subdivisions is isomorphic to that of proper faces of a k-dimensional polytope known as the *secondary polytope* of $\mathcal{A}$ (see [3], [4] or [11]). It has also been shown in [10] that the GBCT is true in rank 3. There are other particular cases in which the question has been answered affirmatively, as the case in which $\mathcal{A}$ is the set of vertices of a cyclic polytope (see [17]). There are no results which strictly disprove the GBCT, but it is a particular case of the *Generalized Baues Conjecture* (or GBC), which has been disproved in the general case by Rambau and Ziegler [18]. Also, it is remarkable that the natural generalization of the GBCT to *oriented matroids* (which are briefly introduced in section 1) is false, as a consequence of the results in [14] and the *Cayley trick* [12].

An important feature related to the GBCT is the *flip* connectivity between triangulations. Without going into detail, we will just say that regular triangulations are represented by the vertices of the secondary polytope, while flips between them are represented by the edges of the same polytope. This implies that regular triangulations are connected by flips. Moreover, according to Balinski's theorem [24, Theorem 3.14], regular triangulations and their flips define a k-connected graph (k being the corank of $\mathcal{A}$): The

1-skeleton of the secondary polytope. This leads to the question of flip connectivity between triangulations, regular or not. The graph defined by triangulations and flips happens to be homeomorphic to certain subcomplex of Baues($\mathcal{A}$), and this fact suggests some relation between the pretended $(k-1)$-sphericity of Baues($\mathcal{A}$) and the k-connectivity of the graph. A family of examples are shown in [22] in which the corank of $\mathcal{A}$ grows to infinity and the connectivity of the graph remains bounded. Moreover, Santos in [20] exhibits an example in corank 317 which has a disconnected graph of triangulations (in fact, the graph in his example has an isolated vertex). Although this does not disprove the GBCT, it introduces serious suspicions about its veracity.

It was shown in [2] that the graph of triangulations of an acyclic corank 3 vector configuration is 3-connected. On the other hand, corank 4 examples with connectivity number less than 4 exist. Thus corank 3 is the case just in the border between good and bad behaviour (at least for the graph of triangulations). This motivates the study of the GBCT in corank 3. In this paper we prove that:

Theorem 3 *The Baues poset of a corank 3 vector configuration $\mathcal{A}$ is:*

1. *Homotopy equivalent to the 2-dimensional sphere S^2 if $\mathcal{A}$ is acyclic (GBCT in corank 3).*
2. *Contractible if $\mathcal{A}$ is not acyclic.*

We conjecture that the space of triangulations of a corank 3 oriented matroid (realizable or not) as defined in [7] is also spherical in the acyclic case and contractible in the non-acyclic one. We believe that the techniques we present here can be applied to the non-realizable case. (See [7], [5], [21], [1] and [2] for a deeper insight in triangulations and subdivisions of oriented matroids.)

The structure of the article is as follows. In section 1 we define the terms involved in Definition 1 and we recall some basic notions of oriented matroid theory and Gale duality as well as some facts concerning triangulations of circuits. In section 2 we reduce the proof of Theorem 3 to a problem of contractibility of certain subposets of Baues($\mathcal{A}$) by means of Gale duality and the Quillen's fibers Lemma. In section 3 we develop a technique of "coarsening and refining" subdivisions that allow us to perform successive retractions of the mentioned subposets, which we finally prove to be contractible in section 4 modulo Lemma 4.10. The mentioned lemma has a rather long proof, so we devote the whole section 5 to it.

1 Preliminaries

Throughout this section and section 2 $\mathcal{A}$ will denote a rank r vector configuration with n elements, and the results we present are valid in arbitrary rank or corank.

The terms involved in Definition 1

A cell (or *full-dimensional* subset) of $\mathcal{A}$ is any spanning subset of $\mathcal{A}$. The simplices of $\mathcal{A}$ are its independent subsets. The elements of $\mathcal{A}$ will be often called *vertices* of $\mathcal{A}$.

For any subset $C \subset \mathcal{A}$ the *positive span* of C is the polyhedral cone $\mathrm{conv}(C)$ of all non-negative linear combinations of the elements of C. The *relative interior* $\mathrm{relconv}(C)$ is the set of strictly positive linear combinations of the elements of C. The linear span of C will be denoted by $\mathrm{span}(C)$. Later on we will identify the cones with their intersections with the unit sphere, which are spherical polytopes. Hence the notations $\mathrm{conv}(C)$ and $\mathrm{relconv}(C)$, which stand for "convex hull" and "relative interior of the convex hull" of C, respectively.

We say that two subsets C_1 and C_2 of $\mathcal{A}$ intersect properly if $\mathrm{conv}(C_1 \cap C_2) = \mathrm{conv}(C_1) \cap \mathrm{conv}(C_2)$ and $\mathrm{span}(C_1 \cap C_2) \cap C_1 = \mathrm{span}(C_1 \cap C_2) \cap C_2$. A collection S of subsets of $\mathcal{A}$ is said to be covering if the positive span of $\mathcal{A}$ is contained in (and hence, is equal to) the union of the positive spans of the subsets in S. Given $C \subset \mathcal{A}$, a subset F of C is said to be a *face* of C if it is the intersection of C and some face of the cone $\mathrm{conv}(C)$ (which, therefore, must be the positive span $\mathrm{conv}(F)$ of F). From this point of view, two subsets C_1 and C_2 of $\mathcal{A}$ intersect properly if and only if their intersection is a common face F and their positive spans $\mathrm{conv}(C_1)$ and $\mathrm{conv}(C_2)$ intersect in $\mathrm{conv}(F)$. It is easy to check that, for any $C \subset \mathcal{A}$, a face of a face of C is a face of C. If $C \subset \mathcal{A}$ is a simplex, then the faces of C are all the subsets of C, and two simplices C_1 and C_2 of $\mathcal{A}$ intersect properly if and only if $\mathrm{conv}(C_1 \cap C_2) = \mathrm{conv}(C_1) \cap \mathrm{conv}(C_2)$.

Oriented Matroids

Most of our techniques in this paper come implicitly or explicitly from oriented matroid theory, which we now introduce. A general reference on the topic is [7].

Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^r$. A linear dependence of the elements of $\mathcal{A}$ can be regarded as an element of $\mathbb{R}^n$ defined by the ordered sequence of coefficients of the dependence. If we consider the signs (plus, minus or zero) of the coefficients rather than their values, what we get is known as a *signed vector* (or *vector*, for short, if there is no ambiguity) of $\mathcal{A}$. If the support of a signed vector (i.e. the set of non-zero coordinates) is minimal with respect to inclusion, then it is said to be a *circuit* of $\mathcal{A}$. Analogously, the values that a linear form $f \in (\mathbb{R}^r)^*$ takes on the elements of $\mathcal{A}$ define an element of $\mathbb{R}^n$ which, if we consider the signs rather than the actual values, is known as a *signed covector* (or *covector*, for short) of $\mathcal{A}$. If the support of a signed covector is minimal, then it is said to be a *cocircuit* of $\mathcal{A}$. The sets of circuits, cocircuits vectors and covectors of $\mathcal{A}$ contain the same amount of information since either of them can be recovered from any other. This information is known as the *oriented matroid* of $\mathcal{A}$.

More precisely, oriented matroids are defined axiomatically in terms of a ground set (which in the case of vector configurations is the mere set of elements) and a set of circuits, cocircuits, vectors or covectors satisfying certain properties. Not every oriented matroid in this axiomatic system can be obtained from a vector configuration as above. The ones which can are said to be *realizable*.

An oriented matroid is *acyclic* if it has a positive covector (i.e. a covector all whose entries are positive) or, equivalently, if all its circuits have both positive and negative entries. Observe that this agrees with our definition of acyclic vector configuration. Such circuits will be called *acyclic circuits*. On the other hand, an oriented matroid is said to be *totally cyclic* if it has a positive vector or, equivalently, if all its cocircuits have both positive and negative entries.

Every oriented matroid $\mathcal{M}$ has a *dual* $\mathcal{M}^*$, whose ground set is the same and whose vectors (resp. circuits) are the covectors (resp. cocircuits) of $\mathcal{M}$ and vice versa. Clearly, the operation of passing to the dual is involutive. An oriented matroid is acyclic if and only if its dual is totally cyclic. An oriented matroid may neither be acyclic nor totally cyclic.

Given an oriented matroid $\mathcal{M}$ on a ground set E and given $p \in E$, the *deletion* of p in $\mathcal{M}$ is defined to be the oriented matroid $\mathcal{M} \setminus p$ whose ground set is $E \setminus \{p\}$ and whose circuits are the circuits of $\mathcal{M}$ in which the coefficient of p is zero. The *contraction* of p in $\mathcal{M}$ is the oriented matroid $\mathcal{M}/p$ whose ground set is $E \setminus \{p\}$ and whose cocircuits are the cocircuits of $\mathcal{M}$ in which the coefficient of p is zero. If $\mathcal{M}^*$ is the dual oriented matroid of $\mathcal{M}$, then $\mathcal{M}^* \setminus p$ is the dual of $\mathcal{M}/p$ and vice versa. That is, the operations of deletion and contraction are dual to each other. When dealing with a vector configuration $\mathcal{A}$, the deletion of an element $p \in \mathcal{A}$ is denoted by $\mathcal{A} \setminus p$ since it can be realized by removing the element p from the vector configuration $\mathcal{A}$. Analogously, the contraction of p is denoted by $\mathcal{A}/p$ and it can be realized by projecting the elements of $\mathcal{A}$ along the direction of p to a hyperplane not containing p.

Gale duality

We recall that $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^r$. A *Gale transform* of $\mathcal{A}$ (see [24]) is a vector configuration $\mathcal{B} = \{b_1, \dots, b_n\} \subset \mathbb{R}^{n-r}$ such that $\sum_{i=1}^n a_i \otimes b_i = 0$

in $\mathbb{R}^r \otimes \mathbb{R}^{n-r}$.

Observe that the definition produces an implicit 1-1 correspondence between $\mathcal{A}$ and $\mathcal{B}$. This correspondence allows us to abuse notation without ambiguity in the following way: For $C = \{a_{i_1}, \dots, a_{i_k}\} \subset \mathcal{A}$, $\mathcal{B} \setminus C$ will denote the set $\{b_{i_{k+1}}, \dots, b_{i_n}\}$ obtained by complementation of indices (and the same for $D \subset \mathcal{B}$ and $\mathcal{A} \setminus D$). From now on, $\mathcal{B}$ will denote a Gale transform of $\mathcal{A}$, and hence, $\mathcal{B}$ will be a vector configuration with n elements in $\mathbb{R}^{n-r}$.

Remarks 1.1 *The following are some straightforward properties of the Gale transform:*

1. *$\mathcal{B}$ is unique up to linear automorphism of $\mathbb{R}^{n-r}$. In particular, the oriented matroid of $\mathcal{B}$ is unique.*
2. *$\mathcal{A}$ is also a Gale transform of $\mathcal{B}$.*
3. *The signed vectors of $\mathcal{A}$ are the signed covectors of $\mathcal{B}$. Therefore, the circuits of $\mathcal{A}$ are the cocircuits of $\mathcal{B}$. I.e. the oriented matroids of $\mathcal{A}$ and $\mathcal{B}$ are dual to each other.*
4. *$\mathcal{A}$ is acyclic if and only if $\mathcal{B}$ is totally cyclic.*
5. *The operations of deletion and contraction are dual to each other (in the Gale sense too): $\mathcal{B} \setminus p$ is a Gale transform of $\mathcal{A}/p$.*
6. *A subset $C \subset \mathcal{A}$ is spanning in $\mathbb{R}^r$ if and only if $\mathcal{B} \setminus C$ is independent in $\mathbb{R}^{n-r}$ and vice versa. In particular, $C \subset \mathcal{A}$ is a basis of $\mathbb{R}^r$ if and only if $\mathcal{B} \setminus C$ is a basis of $\mathbb{R}^{n-r}$.*

Proposition 1.2 *Let $p, q \in \mathcal{A}$. If either $(\{p\}, \emptyset)$ or $(\{p\}, \{q\})$ is a covector of $\mathcal{A}$, then Baues($\mathcal{A}$) is poset-isomorphic to Baues($\mathcal{A}/p$).*

Proof: We consider the map which sends each subdivision S of $\mathcal{A}$ to the link of p in S, which is a subdivision of $\mathcal{A}/p$. This map can always be defined and is order preserving. It is routine to check that, in the conditions of the lemma, this map is also bijective, and thus an isomorphism between Baues($\mathcal{A}$) and Baues($\mathcal{A}/p$). Let us just describe its inverse in the less trivial case; when $(\{p\}, \{q\})$ is a covector of $\mathcal{A}$. Given a subdivision S of $\mathcal{A}/p$ we construct the corresponding subdivision of $\mathcal{A}$ as follows: Let B be a cell of S. If $q \in B$ we consider the cell $B \cup \{p, q\}$ of $\mathcal{A}$. If $q \notin B$ we consider the cells $B \cup \{p\}$ and $B \cup \{q\}$ of $\mathcal{A}$. The set of cells so obtained define the desired subdivision of $\mathcal{A}$. □

By Proposition 1.2, there is no loss of generality in assuming that $\mathcal{B}$ has neither zero elements (which give circuits $(\{p\}, \emptyset)$) nor positive multiples of one another (which give circuits $(\{p\},\{q\})$). Then, by normalisation of its elements, $\mathcal{B}$ can be identified with a set of distinct points in the sphere S^{n-r-1}. By this identification, simplices of $\mathcal{B}$ identify with spherical simplices: Independent subsets of $\mathcal{B}$ with 1, 2, 3 and $n-r-1$ elements identify with sets of points in S^{n-r-1} which will be called *vertices* (or often simply elements), *edges*, *triangles* and *covertices* of $\mathcal{B}$ respectively. The positive span of any subset $C \subset \mathcal{B}$ intersects S^{n-r-1} in a closed region which will be also denoted by $\mathrm{conv}(C)$ but which we will call the *convex hull* of C. In the same way, the relative interior of C intersects S^{n-r-1} in a region which we will keep calling relative interior of C and denoting by $\mathrm{relconv}(C)$. Summing up, in the sequel, when referring to $\mathcal{B}$, we will always be thinking of a spherical situation in which there are no multiple points. From this point of view, a triangulation of $\mathcal{B}$ can be regarded as a geometric triangulation of $\mathrm{conv}(\mathcal{B})$ whose vertices are elements of $\mathcal{B}$.

Triangulations of a circuit

Given any circuit (or vector, or cocircuit or covector) Z it is customary to denote Z^+ and Z^- the subsets of $\mathcal{A}$ consisting of elements with positive and negative entry, respectively. The *support* $Z^+ \cup Z^-$ of Z is denoted $\overline{Z}$.

An acyclic circuit Z can be triangulated in exactly the following two ways:

$$\mathcal{T}^\epsilon(Z) := \{\overline{Z} \setminus \{p\} \ : \ p \in Z^\epsilon\},\ \epsilon \in \{+,-\}$$

where the simplices $\overline{Z}\setminus\{p\}$ might not be full-dimensional. The triangulations $\mathcal{T}^+(Z)$ and $\mathcal{T}^-(Z)$ are known as the positive and the negative triangulations of Z respectively.

Analogously, every non-acyclic circuit can be triangulated in exactly one way:

$$\mathcal{T}^+(Z) := \{\overline{Z} \setminus \{p\} \ : \ p \in \overline{Z}\}$$

Let C be a subset of $\mathcal{A}$ containing the support of precisely one circuit Z (for example a spanning subset with $r+1$ elements). Then C can be triangulated in exactly as many ways as Z. If, for instance Z is an acyclic circuit, those triangulations would be:

$$\mathcal{T}^\epsilon(C) := \{C \setminus \{p\} \ : \ p \in Z^\epsilon\},\ \epsilon \in \{+,-\}$$

2 Subdivisions of $\mathcal{A}$, simplices of $\mathcal{B}$ and Quillen's fibers Lemma.

The aim of this section is to reduce the problem of determining the homotopy type of $\mathrm{Baues}(\mathcal{A})$ to that of showing that certain subposets of $\mathrm{Baues}(\mathcal{A})$ are contractible, by means of Quillen's Lemma. A key idea is to codify the subdivisions of $\mathcal{A}$ in terms of set of simplices of the Gale transform $\mathcal{B}$. This is done for triangulations in [8] and [2] via de notion of *virtual chamber* of $\mathcal{B}$, on which our ideas are inspired although we will not introduce it explicitly. We recall that we are assuming $\mathcal{A}$ to have arbitrary corank.

Definition 2.1 (i) If S is a subdivision of $\mathcal{A}$ and σ is a simplex of $\mathcal{B}$, we say that:

- S *lies on* σ if $\mathcal{A} \setminus \sigma$ is a cell of S.
- S *lies on* $\overline{\sigma}$ if S lies on some face of σ (possibly σ itself).

(ii) A simplex σ of $\mathcal{B}$ is *empty* if $\mathcal{B} \cap \text{conv}(\sigma) = \sigma$.

(iii) Two subsets of $\mathcal{B}$ *overlap* if their relative interiors intersect in a non-empty set.

Lemma 2.2 *Let S be a subdivision of A and let σ_1 and σ_2 be two simplices of $\mathcal{B}$. If S lies on both σ_1 and σ_2, then σ_1 and σ_2 overlap.*

Proof: The cells $B_1 := \mathcal{A} \setminus \sigma_1$ and $B_2 := \mathcal{A} \setminus \sigma_2$ of S intersect properly. This implies there is a linear hyperplane H in $\mathbb{R}^r$ which weakly separates B_1 and B_2 with $H \cap (B_1 \cup B_2) = B_1 \cap B_2$. That is, there is a covector Z of $\mathcal{A}$ such that $B_1 \cap Z^+ = \emptyset$, $B_2 \cap Z^- = \emptyset$ and $Z^0 \cap (B_1 \cup B_2) = B_1 \cap B_2$. Thus, there is a vector Z of $\mathcal{B}$ such that $\sigma_1 \setminus (\sigma_1 \cap \sigma_2) \subset Z^+ \subset \sigma_1$ and $\sigma_2 \setminus (\sigma_1 \cap \sigma_2) \subset Z^- \subset \sigma_2$. It is not hard to see that this is equivalent for σ_1 and σ_2 to overlap. □

Every covertex l of $\mathcal{B}$ spans a linear hyperplane which divides $\mathbb{R}^{n-r}$ into two open half-spaces or, equivalently, defines a great sphere which divides S^{n-r-1} into two open hemispheres called *sides* (or sometimes *open sides*) of l. The closed hemispheres defined by l are called *closed sides* of l. An *orientation* of l is a choice of one side l^+ as the positive and the other one l^- as the negative. An *oriented covertex* of $\mathcal{B}$ is a covertex l of $\mathcal{B}$ together with an orientation of l. The positive and negative closed sides defined by an oriented covertex l are denoted by $\overline{l^+}$ and $\overline{l^-}$ respectively. The closed sides of l (oriented or not) intersect in the great sphere defined by l which we denote l^0.

Definition 2.3 Let l be an oriented covertex of $\mathcal{B}$. We say that:

(i) S lies on l^+ if there is a simplex σ of $\mathcal{B}$ in which S lies such that $\text{relconv}(\sigma) \subset l^+$.

(ii) S lies on $\overline{l^+}$ if there is a simplex σ of $\mathcal{B}$ in which S lies such that $\text{relconv}(\sigma) \subset \overline{l^+}$

(iii) S lies on l^0 if there is a simplex σ of $\mathcal{B}$ in which S lies such that $\text{relconv}(\sigma) \subset l^0$.

Remarks 2.4 *The following properties are straightforward for a subdivision S of $\mathcal{A}$ and an oriented covertex l of $\mathcal{B}$:*

- *S lies on l^+ if and only if there is a simplex σ of $\mathcal{B}$ in which S lies such that conv(σ) $\subset \overline{l^+}$ (i.e. $\sigma \subset \overline{l^+}$) and $l^+ \cap \sigma \neq \emptyset$.*
- *S lies on $\overline{l^+}$ if there is a simplex σ of $\mathcal{B}$ in which S lies such that conv(σ) $\subset \overline{l^+}$. Equivalently, if there is a simplex σ of $\mathcal{B}$ in which S lies such that $\sigma \subset \overline{l^+}$.*
- *S lies on l^0 if and only if there is a simplex σ of $\mathcal{B}$ in which S lies such that conv(σ) $\subset l^0$. Equivalently, if there is a simplex σ of $\mathcal{B}$ in which S lies such that $\sigma \subset l^0$.*
- *S lies on l^0 if and only if S lies on both $\overline{l^+}$ and $\overline{l^-}$.*
- *If σ is a simplex of $\mathcal{B}$ and S lies on both σ and l^+, then $\sigma \cap l^+ \neq \emptyset$.*

Lemma 2.5 *Let S be a subdivision of $\mathcal{A}$ and let l be a covertex of $\mathcal{B}$. Then S lies on at most one of the sets l^+, l^- and l^0.*

Proof: If S lies on both l^+ and l^0, then S lies on simplices σ^+ and σ^0 of $\mathcal{B}$ with relconv(σ^+) $\subset l^+$ and relconv(σ^0) $\subset l^0$. Hence, σ^+ and σ^0 do not overlap, which is impossible by Lemma 2.2. The same argument proves that S can neither lie on both l^- and l^0 nor on both l^+ and l^-. □

The following example shows that S might lie on neither l^+, l^- nor l^0: Let $\mathcal{A}$ be the configuration of column vectors in the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

with Gale transform $\mathcal{B}$ represented by the matrix

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Let $S = \{\{a_1, a_4\}\}$, which lies only on the simplex $\{b_2, b_3\}$, and let $l = \{b_1\}$ or $l = \{b_4\}$.

Proposition 2.6 *Let $\mathcal{T}$ be a triangulation of $\mathcal{B}$. Then, for every subdivision $S \in$ Baues($\mathcal{A}$), there is a unique simplex σ of $\mathcal{T}$ (not necessarily a full dimensional one) such that S lies on σ.*

Proof: First we prove existence. Let $\mathcal{T}'$ be a triangulation of $\mathcal{A}$ which refines S (such a triangulation can always be constructed using, for example, the *pulling* technique introduced in [13]). In [8] it was proved that there is a unique (full dimensional) simplex ρ of $\mathcal{T}'$ whose complementary set $\mathcal{B} \setminus \rho$ is a (full dimensional) simplex of $\mathcal{T}$. On the other hand, there is a cell B of S with $\rho \subset B$. Hence, $\sigma := \mathcal{B} \setminus B$ is contained in $\mathcal{B} \setminus \rho$ and S lies on σ. Since σ is a face of the simplex $\mathcal{B} \setminus \rho$ (and is a nonempty one, since S is not the trivial subdivision), σ is a simplex of $\mathcal{T}$.

For uniqueness, if there is another simplex of $\mathcal{T}$ in which S lies, then this simplex must overlap σ (by Lemma 2.2), which is impossible since both are simplices of the same triangulation $\mathcal{T}$. □

The set of subdivisions of $\mathcal{A}$ which lie on a simplex τ of $\mathcal{B}$ (together with the refinement relation) is a subposet of Baues($\mathcal{A}$) which we will denote $\text{Baues}_\tau(\mathcal{A})$, while $\text{Baues}_{\overline{\tau}}(\mathcal{A})$ will denote the subposet of those subdivisions which lie on $\overline{\tau}$ (i.e. on some face of τ which could be τ itself).

Lemma 2.7 *Let $\mathcal{T}$ be a triangulation of a subset $C \subset \mathcal{B}$ and let τ be a simplex of $\mathcal{B}$ with conv(τ) $\subset$ conv(C). If a subdivision S lies on τ, then S lies on some simplex of $\mathcal{T}$.*

Proof: Since $\mathcal{T}$ is a triangulation of C, the union of the convex hulls of the simplices of $\mathcal{T}$ is conv(C), which is a convex set. Therefore, $\mathcal{T}$ can be extended to a triangulation $\mathcal{T}'$ of $\mathcal{B}$ (using, for example, the placing technique introduced in [13]). By Proposition 2.6, S lies on some simplex σ of $\mathcal{T}'$. By Lemma 2.2, σ and τ overlap, but since conv(τ) $\subset$ conv(C), the only simplices of $\mathcal{T}'$ which overlap τ are the simplices of $\mathcal{T}$. Therefore, σ is a simplex of $\mathcal{T}$. □

In Figure 1 we depict two rank 3 configurations which are Gale transforms of each other. The first one is acyclic and we show it projected to the affine plane. One can see that the cells of the triangulation in (a) correspond by complementation precisely to those simplices which contain in their convex hulls the shaded region in (b). It is easy to see that exactly one of such simplices belongs to each triangulation of the configuration in (b), since the shaded region is a cell of the cell complex defined as the coarsest common refinement of all the triangulations of (b) (i.e. the *chamber complex* of (b)). Despite of Proposition 2.6 and Lemma 2.7, a subdivision of a vector configuration do not always correspond to a cell of the chamber complex in this fashion.

The remaining of the section is devoted to show how the problem of the homotopy type of Baues($\mathcal{A}$) can be reduced to that of the contractibility of certain subposets of Baues($\mathcal{A}$). Let $\mathcal{T}$ be a triangulation of $\mathcal{B}$. We consider the reverse incidence relation in $\mathcal{T}$: $\sigma_1 \leq \sigma_2$ if and only if $\sigma_2 \subset \sigma_1$. This is a partial ordering relation whose associated poset is the opposite to the usual face poset. In particular, their order complexes are isomorphic. Since the order complex of the face poset of a simplicial complex Σ is isomorphic to the

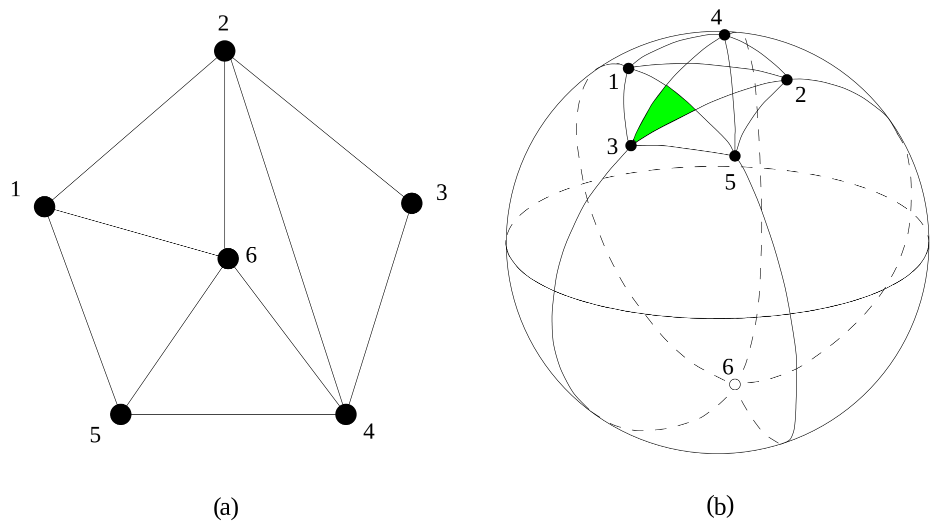

Figure 1: Two rank 3 configurations which are Gale transforms of each other. The triangulation in (a) corresponds by complementation to the set of simplices containing the shaded region in (b).

first barycentric subdivision of Σ (which is homeomorphic to Σ itself), we conclude that the order complex of $(\mathcal{T}, \leq)$ (where "$\leq$" is the relation we have defined above) is homeomorphic to $\mathcal{T}$. But $\mathcal{T}$ is homeomorphic to $\text{conv}(\mathcal{B})$, and hence, to S^{n-r-1} if $\mathcal{B}$ is totally cyclic and to the $(n-r-1)$-dimensional ball, B^{n-r-1} otherwise. In poset topology terms,

Remark 2.8 *$(\mathcal{T}, \leq)$ is homeomorphic to S^{n-r-1} if $\mathcal{A}$ is acyclic and to B^{n-r-1} otherwise.*

For any triangulation $\mathcal{T}$ of $\mathcal{B}$ we define the *Quillen map over $\mathcal{T}$*

$$F_{\mathcal{T}} : \text{Baues}(\mathcal{A}) \longrightarrow \mathcal{T}$$

as follows. For any $S \in \text{Baues}(\mathcal{A})$, $F_{\mathcal{T}}(S)$ is the (unique) simplex of $\mathcal{T}$ in which S lies.

Our purpose is to show that if $\mathcal{A}$ is a corank 3 vector configuration, then, for a certain triangulation $\mathcal{T}$ of $\mathcal{B}$, $F_{\mathcal{T}}$ induces a homotopy equivalence between $\text{Baues}(\mathcal{A})$ (with the refinement ordering) and $(\mathcal{T}, \leq)$. We will thus have that if $\mathcal{A}$ is acyclic, then $\text{Baues}(\mathcal{A})$ is homotopy equivalent to S^2, and if $\mathcal{A}$ is not acyclic, then $\mathcal{A}$ is homotopy equivalent to B^2, and therefore contractible. We will make use of the following result.

Lemma 2.9 (Quillen's Lemma) *Let P and Q be two posets. Let $F : P \longrightarrow Q$ be an order-preserving surjection (i.e. a poset epimorphism) and suppose that for every $y \in Q$, the fiber $F^{-1}(Q_{\geq y})$ of y is contractible. Then, F induces a homotopy equivalence between P and Q.*

For a proof of this lemma see, for example, [6].

Lemma 2.10 *For any triangulation $\mathcal{T}$ of $\mathcal{B}$, the map $F_{\mathcal{T}}$ is surjective and order preserving.*

Proof: Let σ be a simplex of $\mathcal{T}$. Then $B := \mathcal{A} \setminus \sigma$ is a cell of $\mathcal{A}$. B is not $\mathcal{A}$ itself, since we are not considering the empty set as a simplex of $\mathcal{T}$. By extending $\{B\}$ to a subdivision of $\mathcal{A}$ (this can be done, for example, in a lexicographic fashion), we obtain a nontrivial subdivision S of $\mathcal{A}$ which has B as a cell, that is, which lies on σ. This gives surjectiveness.

Now let us show that $F_{\mathcal{T}}$ is order preserving. Let $S_1, S_2 \in \mathrm{Baues}(\mathcal{A})$ with $S_1 < S_2$. Let σ_1 and σ_2 be the simplices of $\mathcal{T}$ such that S_i lies on σ_i, $i = 1, 2$. We have to show that $\sigma_2 \subset \sigma_1$, that is, if we define $B_i := \mathcal{A} \setminus \sigma_i$ for $i \in \{1, 2\}$, we have to show that $B_1 \subset B_2$. But since S_i lies on σ_i for $i \in \{1, 2\}$, we have that B_i is a cell of S_i for $i \in \{1, 2\}$, and since S_1 refines S_2, we conclude that B_1 is contained in some cell B_2' of S_2. If $B_2' \neq B_2$, then B_1 and B_2 do not overlap. Hence, there exists a linear hyperplane H of $\mathbb{R}^r$ which weakly separates B_1 and B_2, that is, there is a covector Z of $\mathcal{A}$ such that $B_1 \cap Z^+ = \emptyset$ and $B_2 \cap Z^- = \emptyset$. Therefore, there is a vector Z of $\mathcal{B}$ such that $Z^+ \subset \sigma_1$ and $Z^- \subset \sigma_2$. As can be seen in, for example, [16, Proposition 2.2], this implies that σ_1 and σ_2 do not intersect properly, which is not possible since both are simplices of the same triangulation $\mathcal{T}$ of $\mathcal{B}$. $\square$

Remark 2.11 *Taking into account Remark 2.8 and Lemmas 2.9 and 2.10, in order to prove Theorem 3, it suffices to exhibit a triangulation $\mathcal{T}$ of $\mathcal{B}$ whose Quillen fibers by $F_{\mathcal{T}}$ are contractible, that is, such that the subposet of $\mathrm{Baues}(\mathcal{A})$ of those subdivisions which lie on some face of σ (including σ itself) is contractible for every simplex $\sigma \in \mathcal{T}$.*

In section 4 we will show that, if $\mathcal{A}$ has corank 3, such a subposet is contractible for any empty simplex σ of $\mathcal{B}$ (Proposition 4.1 and Theorems 4.6 and 4.9). Thus, any triangulation $\mathcal{T}$ of $\mathcal{B}$ which uses all the elements of $\mathcal{B}$ would satisfy our requirements.

3 Some useful homotopy equivalences

Throughout the remaining of this paper, $\mathcal{A}$ will be assumed to have corank 3.

In this section we introduce a technique of coarsening and refining subdivisions of $\mathcal{A}$ along a particular oriented covertex (i.e. edge) of $\mathcal{B}$. This will induce homotopy equivalences for appropriate subspaces of the Baues complex. We do not know whether the analogous statements hold in higher corank.

Definition 3.1 We say that two edges l_1 and l_2 *cross* each other (or that one crosses the other) if they overlap and $l_1 \cup l_2$ has rank 3 (i.e. spans a 3-dimensional vector subspace). Equivalently, l_1 and l_2 cross each other if and only if (l_1, l_2) is a circuit.

Remarks 3.2 *Let l, l', τ and τ' be an edge, an empty edge, a triangle and an empty triangle of $\mathcal{B}$ respectively. The following assertions are straightforward:*

- *Not all the (three) edges of τ can overlap l.*
- *The simplices l and τ' overlap if and only if l crosses some edge of τ'.*
- *If l overlaps both τ and some edge m of τ, then l crosses m.*
- *If p, q and r are the vertices of τ' and $\{p,q\}$ is the only edge of τ' which l' crosses, then r is a vertex of l'.*
- *If l and l' overlap but do not cross each other, then $conv(l') \subset conv(l)$.*

Lemma 3.3 *Let l be an edge of $\mathcal{B}$ which overlaps an empty triangle τ of $\mathcal{B}$. If a subdivision S of $\mathcal{A}$ lies on $\overline{\tau}$, then S lies on some closed side of l.*

Proof: First suppose that l is empty. Let $\tau = \{p, q, r\}$ and assume, without loss of generality, that S crosses $\{p, q\}$. Since l and τ are empty, either $r \in l$ or l crosses some other edge of τ, which we can assume to be $\{p, r\}$. In either case, the triangle $l \cup \{p\}$ can be extended to a triangulation $\mathcal{T}$ of $\tau \cup l$ in which every simplex is contained in some closed side of l. By Lemma 2.7, S lies on some simplex of $\mathcal{T}$. If l is not empty, then it contains an empty edge m in its convex hull which overlaps τ, thus S lies on some closed side of m. But clearly, S lies on some closed side of m if and only if S lies on some closed side of l. □

Definition 3.4 Given an oriented empty edge m of $\mathcal{B}$ and subdivisions S_1 and S_2 of $\mathcal{A}$, we say that:

(i) S_1 is *incident* to m^+ if S lies on every triangle $m \cup \{t\}$ with $t \in m^+$.

(ii) S_1 and S_2 are *incident* (to each other) *along* m^+ if one of them is incident to m^+, the other one lies on m and for every simplex σ of $\mathcal{B}$ which does not have an edge contained in m^0, S_1 lies on σ if and only if S_2 lies on σ.

Remarks 3.5 *Let m be an oriented empty edge of $\mathcal{B}$ and let S be a subdivision of $\mathcal{A}$.*

1. *S is incident to m^+ if and only if S lies on every triangle of the form $l \cup \{t\}$ where l is an edge of $\mathcal{B}$ such that $m \subset conv(l)$ and $t \in m^+$.*

   Suppose S is incident to m^+, let l be an edge of $\mathcal{B}$ with $m \subset conv(l)$ and $t \in m^+$. Extend $\{l \cup \{t\}\}$ to a triangulation $\mathcal{T}$ of $\mathcal{B}$. Since S lies on $m \cup \{t\}$, by Lemmas 2.7 and 2.2 there is a unique face σ of $l \cup \{t\}$ such that S lies on σ and such a face σ must overlap $m \cup \{t\}$. Since $relconv(m \cup \{t\}) \subset relconv(l \cup \{t\})$, $l \cup \{t\}$ is the only simplex of $\mathcal{T}$ which overlaps $m \cup \{t\}$. Therefore, $\sigma = l \cup \{t\}$, thus S lies on $l \cup \{t\}$. The converse is trivial.

2. *Analogously, S lies on m if and only if S lies on every edge l of $\mathcal{B}$ such that $m \subset conv(l)$.*

   The proof is the same as in previous remark, except that one has to choose $t \notin m^0$ arbitrarily in order to take $l \cup \{t\}$ as a starting triangle for $\mathcal{T}$.

3. *If S_1 and S_2 are incident along m^+, the one which is incident to m^+ refines the one which lies on m.*

   Say S_1 lies on m. For every edge l with $m \subset conv(l)$ and for every $t \in m^+$, S_1 lies on l while S_2 lies on $l \cup \{t\}$, and for any other simplex of $\mathcal{B}$ either both S_1 and S_2 lie on it or none of them lie on it. Passing to complements in $\mathcal{A}$ we have that every cell of S_2 is contained in some cell of S_1.

4. *Of course, we can reverse the orientation of m in the previous definition and, hence, define the same notions with respect to m^-.*

Theorem 3.6 *Let S be a subdivision of $\mathcal{A}$ and let m be an oriented empty edge of $\mathcal{B}$ with $m^+ \neq \emptyset$. If S either lies on m or is incident to m^+, then there exists a unique subdivision S' of $\mathcal{A}$ which is incident to S along m^+.*

Proof: Suppose that S lies on m. The subdivision S' (in case it exists) is determined by the simplices of $\mathcal{B}$ in which it lies, since their complements in $\mathcal{A}$ are the cells of S'. Assume that there exists S' incident to S along m^+. In particular, S' is incident to m^+, so by Remarks 3.5, S' must lie on every triangle $l \cup \{t\}$, where $t \in m^+$ and l is an edge of $\mathcal{B}$ with $m \subset \text{conv}(l)$. We claim that these triangles, together with the simplices in which S lies which do not have an edge contained in m^0, define (by taking complements in $\mathcal{A}$) a subdivision of $\mathcal{A}$. The claim proves not only existence, but also uniqueness of S'. This follows from the obvious fact that we cannot obtain a new subdivision of $\mathcal{A}$ by adding cells to a given one.

Note that, by Remark 3.5, S lies on every edge l of $\mathcal{B}$ such that $m \subset \text{conv}(l)$. Moreover, since any two simplices of $\mathcal{B}$ on which S lies must overlap, the simplices on which S lies and having an edge contained in m^0 are all edges overlapping m. Since m is empty, they must contain m in their convex hulls.

Now we prove the claim. Let S' be the collection of cells of $\mathcal{A}$ obtained by complementation of the simplices of described above (i.e. those of the form $l \cup \{t\}$, being l an edge with $m \subset \operatorname{conv}(l)$ and $t \in m^+$, and those on which S lies having no edge contained in m^0). Let Z be the cocircuit of $\mathcal{B}$ defined by the oriented edge m, which we identify with the corresponding circuit of $\mathcal{A}$. Let l be an edge of $\mathcal{B}$ such that $m \subset \operatorname{conv}(l)$. Since $l \subset Z^0$, the cell $\mathcal{A} \setminus l$ of S contains $\overline{Z}$. Moreover, every cell B of S which contains $\overline{Z}$ is obtained this way: $\mathcal{B} \setminus B$ must overlap m (which is an empty edge) and $\mathcal{B} \setminus B \subset Z^0$, thus $\mathcal{B} \setminus B = l$ for some edge l of $\mathcal{B}$ such that $m \subset \operatorname{conv}(l)$. For a fixed edge l in these conditions, the set $\{\{l \cup \{t\}\} \ : \ t \in m^+\}$ defines (by complementation) the positive triangulation of $\mathcal{A} \setminus l$ (which is a spanning subset of $\mathcal{A}$ with $r+1$ elements). Thus, S' is obtained from S by substituting every cell B of S which contains $\overline{Z}$ by its positive triangulation. Hence, S' covers $\operatorname{conv}(\mathcal{A})$. It remains to show that every new cell (meaning every cell of S' which is not a cell of S) intersects properly with any other cell of S'.

Let B_1 and B_2 be two cells of S' and suppose B_1 is a new cell. Then, B_1 is a full-dimensional simplex of $\mathcal{A}$ which is contained in a unique cell B_1' of S (and hence, B_1 is in the positive triangulation of B_1' by construction). First suppose B_2 is not a new cell and let F be the common (possibly empty) face of B_1' and B_2. Since F is a face of B_1', it is triangulated by any triangulation of B_1'. Since B_2 is not new, it does not contain $\overline{Z}$, thus $\overline{Z} \not\subset F$. Therefore, F is a simplex, and thus it is a face of some element in the positive triangulation of B_1', hence F and B_1 intersect properly in a simplex $G = F \cap B_1$. Now, $B_1 \cap B_2 = B_1 \cap B_1' \cap B_2 = B_1 \cap F = G$, and since B_1 is a simplex and $G \subset B_1$, G is a face of B_1. Also, since F is a simplex and $G \subset F$, G is a face of F, and since F is a face of B_2, G is a face of B_2. Therefore, B_1 and B_2 intersect in a common face. Moreover, $\operatorname{conv}(B_1) \cap \operatorname{conv}(B_2) = \operatorname{conv}(B_1) \cap \operatorname{conv}(B_1') \cap \operatorname{conv}(B_2) = \operatorname{conv}(B_1) \cap \operatorname{conv}(F) = \operatorname{conv}(G)$, and hence, B_1 and B_2 intersect properly.

On the other hand, if B_2 is new, then it is in the positive triangulation of some cell B_2' of S. In this case, the common face F of B_1' and B_2' contains $\overline{Z}$. $G_1 := B_1 \cap F$ is a simplex of the positive triangulation of $\overline{Z}$ and so is $G_2 := B_2 \cap F$. Hence G_1 and G_2 intersect properly. B_1 is obtained from B_1' by dropping an element of $\overline{Z}$, thus G_1 is obtained from F by dropping the same element of $\overline{Z}$. Hence, G_1 has the same rank as F, thus $\operatorname{conv}(B_1) \cap \operatorname{conv}(F) \subset \operatorname{conv}(B_1) \cap \operatorname{span}(G_1) = \operatorname{conv}(G_1)$, so $\operatorname{conv}(B_1) \cap \operatorname{conv}(F) = \operatorname{conv}(G_1)$. Analogously, $\operatorname{conv}(B_2) \cap \operatorname{conv}(F) = \operatorname{conv}(G_2)$. On the one hand, $B_1 \cap B_2 = B_1 \cap B_2 \cap B_1' \cap B_2' = B_1 \cap B_2 \cap F = G_1 \cap G_2$, thus B_1 and B_2 intersect in a common face. On the other hand, $\operatorname{conv}(B_1) \cap \operatorname{conv}(B_2) = \operatorname{conv}(B_1) \cap \operatorname{conv}(B_2) \cap \operatorname{conv}(B_1') \cap \operatorname{conv}(B_2') = \operatorname{conv}(B_1) \cap \operatorname{conv}(B_2) \cap \operatorname{conv}(F) = \operatorname{conv}(G_1) \cap \operatorname{conv}(G_2) = \operatorname{conv}(G_1 \cap G_2) = \operatorname{conv}(B_1 \cap B_2)$, so B_1 and B_2 intersect properly.

If S is incident to m^+, the proof follows the same lines. For uniqueness, the argument is word by word the same as above and, for existence, the idea is to substitute the cells of the form $\mathcal{A} \setminus (l \cup \{t\})$ by those of the form $\mathcal{A} \setminus l$ (where, as usual, $t \in m^+$ and l is an edge of $\mathcal{B}$ with $m \subset \text{conv}(l)$). One has to prove that the collection of cells of $\mathcal{A}$ so obtained is a subdivision of $\mathcal{A}$, and this can be done using an essentially identical argument. □

Lemma 3.7 (Lemma 3.3 in [23]) *Let $f : P \longrightarrow P$ be a poset endomorphism such that*

$$f(f(x)) = f(x) \leq x, \forall x \in P$$

Then the surjection $f : P \longrightarrow f(P)$ is a homotopy equivalence.

Lemma 3.8 *Let m be an oriented empty edge of $\mathcal{B}$ and let P be a subposet of Baues($\mathcal{A}$) such that:*

1. *Every element of P lies on $\overline{m^+}$.*
2. *For every $S \in P$ which lies on m^0, S lies on m and the subdivision S^+ which is incident to S along m^+ is in P.*

Let $P_m^+ := \{S \in P \ : \ S \text{ lies on } m^+\}$. The map $R : P \longrightarrow P$, defined by $R(S) := S^+$ if S is in the conditions of part 2 and $R(S) := S$ otherwise, induces a homotopy equivalence between P and P_m^+.

Proof: We claim that R satisfies the hypotheses of Lemma 3.7, and hence, it induces a homotopy equivalence between P and $R(P)$. Clearly, $R(P) = P_m^+$ and $R(R(S)) = R(S) \leq S, \forall S \in P$, and hence it remains to prove that R is order preserving.

Let $S_1, S_2 \in P$ with $S_1 < S_2$. If S_1 and S_2 are both in P_m^+, there is nothing to prove. On the other hand, if $S_2 \in P_m^+$ then $S_1 \in P_m^+$: Otherwise S_1 lies on m, and hence, S_2 lies on some face ρ of m, but since $S_2 \in P_m^+$, S_2 lies on some simplex σ of $\mathcal{B}$ with relconv$(\sigma) \subset m^+$, and thus, σ and ρ do not overlap, which is impossible. Therefore, we can assume $S_2 \in P \setminus P_m^+$, that is, S_2 lies on m. If $S_1 \in P_m^+$, we want to show that $S_1 \leq S_2^+$. But if B_1 is a cell of S_1, then $B_1 \subset B_2$ for some cell B_2 of S_2. If B_2 is a cell of S_2^+, then there is nothing to prove. If B_2 is not a cell of S_2^+, then B_2 is a spanning set with $r+1$ and it can be refined in exactly two ways (both triangulations of B_2). Since S_1 lies on m^+, B_1 must be a simplex of the positive triangulation of B_2, and hence, B_1 is a cell of S_2^+ (i.e. S_2^+ lies on $\mathcal{A} \setminus B_2 = l \cup \{t\}$ for some edge l with $m \subset \text{conv}(l)$ and some $t \in m^+$). It remains the case in which both S_1 and S_2 lie on m. We want to show that $S_1^+ \leq S_2^+$, but since $S_1^+ \leq S_1 \leq S_2$, S_1^+ and S_2 are in the conditions of the previous case, and hence $S_1^+ \leq S_2^+$. □

Lemma 3.9 *Let m be an oriented empty edge of $\mathcal{B}$ and let P be a subposet of Baues($\mathcal{A}$) such that:*

1. *Every element of P lies on some closed side of m.*

2. *For every $S \in P$ which lies on m^-, S is incident to m^- and the subdivision S^0 which is incident to S along m^- is in P.*

Let $\overline{P_m^+} := \{S \in P \ : \ S \text{ lies on } \overline{m^+}\}$. The map $R : P \longrightarrow P$, defined by $R(S) := S^0$ if S is in the conditions of part 2 and $R(S) := S$ otherwise, induces a homotopy equivalence between P and $\overline{P_m^+}$.

Proof: We claim that R satisfies the hypotheses of Lemma 3.7, and hence, it induces a homotopy equivalence between P and $R(P)$. Clearly, $R(P) = \overline{P_m^+}$ and $R(R(S)) = R(S) \geq S \ \forall S \in P$. Thus, it remains to show that R is order-preserving.

Let S_1 and S_2 be elements of P with $S_1 < S_2$. If $S_1 \in \overline{P_m^+}$ then there is a simplex $\sigma \subset \overline{m^+}$ such that S_1 lies on sigma. Since $S_1 < S_2$, S_2 lies on some face of σ, and hence, S_2 lies on $\overline{m^+}$. But, in this case, $R(S_1) = S_1 < S_2 = R(S_2)$. Thus, we can assume $S_1 \in P \setminus \overline{P_m^+}$. If $S_2 \in \overline{P_m^+}$, we want to show that $S_1^0 \leq S_2$. If $B_1 \in S_1^0$ is not a new cell there is nothing to prove. Let us assume that B_1 is a new cell. Then B_1 is of the form $\mathcal{A} \setminus l$. Let σ be one of the full-dimensional simplices of the negative triangulation of B_1. The simplex σ is a cell of S_1, so there exists $B_2 \in S_2$ with $\sigma \subset B_2$. Hence, S_2 lies on a face $\rho = \mathcal{B} \setminus B_2$ of $\mathcal{B} \setminus \sigma = l \cup \{t\}$ (for some $t \in m^-$). Since $S_2 \in \overline{P_m^+}$, S_2 lies on $\overline{m^+}$, thus $\rho \subset l$, which implies that $\mathcal{A} \setminus l \subset \mathcal{A} \setminus \rho$, that is, $B_1 \subset B_2$. It remains the case in which both S_1 and S_2 are in $P \setminus \overline{P_m^+}$. We want to prove that $S_1^0 \leq S_2^0$, but since $S_1 \leq S_2 \leq S_2^0$, S_1 and S_2^0 are in the conditions of the previous case, thus $S_1^0 \leq S_2^0$. □

Lemma 3.10 *Let m be an oriented empty edge of $\mathcal{B}$ and let P be a subposet of Baues($\mathcal{A}$) such that:*

1. *Every element of P lies on some closed side of m.*

2. *For every $S \in P$ which lies on m^-, S is incident to m^- and the subdivision S^0 which is incident to S along m^- is in P.*

3. *For every $S \in P$ which lies on m^0, S lies on m and the subdivision S^+ which is incident to S along m^+ is in P.*

Then P and $P_m^+ := \{S \in P \ : \ S \text{ lies on } m^+\}$ are homotopy equivalent.

Proof: By Lemma 3.9, P and $\overline{P_m^+}$ are homotopy equivalent. By hypothesis, for every $S \in P$ which lies on m^0, S lies on m and the subdivision S^+ which is incident to S along m^+ is in P. That is to say that for every $S \in \overline{P_m^+}$ which lies on m^0, S lies on m and S^+ is in P. Hence, $Q := \overline{P_m^+}$ is in the hypotheses of Lemma 3.8 and, therefore, Q is homotopy equivalent to $Q_m^+ := \{S \in Q \ : \ S \textit{ lies on } m^+\} = \{S \in P \ : \ S \textit{ lies on } m^+\} = P_m^+$. Hence, P and P_m^+ are homotopy equivalent. □

Remark 3.11 *Of course, Theorem 3.6 and Lemmas 3.8, 3.9 and 3.10 remain true if we substitute every plus sign by a minus sign and vice versa.*

4 Empty simplices induce contractible subposets

In this section we prove Theorem 3. We recall (see Remark 2.11) that taking into account Lemmas 2.9 and 2.10, it suffices to show that the subposet of Baues($\mathcal{A}$) induced by an empty simplex of $\mathcal{B}$ and its faces is contractible. We also recall that we are assuming $\mathcal{A}$ to have corank 3.

4.1 Vertices

Proposition 4.1 *Let $p \in \mathcal{B}$. $Baues_{\{p\}}(\mathcal{A}) = Baues_{\overline{\{p\}}}(\mathcal{A})$ consists of a single element S_p, and hence it is contractible.*

Proof: We will call S_p the subdivision of $\mathcal{A}$ whose cells are $\mathcal{A} \setminus \{p\}$ together with all those cells obtained by joining p to a facet of $\mathcal{A} \setminus \{p\}$ which is *visible* from p (if any). By a facet of $\mathcal{A} \setminus \{p\}$ which is visible from p we mean a facet of $\mathcal{A} \setminus \{p\}$ which joined to p gives a cell of $\mathcal{A}$ (i.e. a spanning subset of $\mathcal{A}$) which intersects properly with $\mathcal{A} \setminus \{p\}$. It is well known that S_p so defined is a subdivision of $\mathcal{A}$, and it lies on $\{p\}$. Let S be any subdivision of $\mathcal{A}$ which lies on $\{p\}$. Then, $\mathcal{A} \setminus \{p\}$ is a cell of S. Any cell B of S different from $\mathcal{A} \setminus \{p\}$ must contain p as an element, and hence, $B \setminus \{p\} \subset \mathcal{A} \setminus \{p\}$ must be a facet of B and, therefore, a facet of $\mathcal{A} \setminus \{p\}$ visible from p. Thus, $S \subset S_p$, and since no subdivision of $\mathcal{A}$ can be properly contained in any other, $S = S_p$. □

4.2 Edges

The main goal of this subsection is to prove Theorem 4.6. The successive steps we take to do so (Lemmas 4.3, 4.4 and 4.5) are sketched in Figure 2, which does not truly represent the geometric situation and is intended to serve only as a quick guide of the proof. Note that the first step depicted in Figure 2 corresponds to the last of the three lemmas.

Throughout this subsection, $m = \{p, q\}$ will be an empty edge of $\mathcal{B}$ and $\Omega(m)$ will denote the set of empty edges of $\mathcal{B}$ which cross m. We define

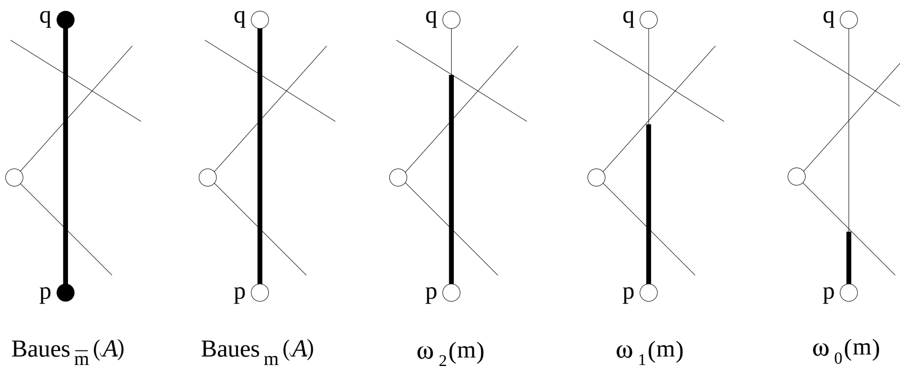

Figure 2: Visual guide of section 4.2: The successive retractions are represented from left to right. In each case, the thick segment represents schematically the subposet under consideration. The final subposet $\omega_0(m)$ has a single element.

a binary relation in $\Omega(m)$ as follows. If $l, l' \in \Omega(m)$ are distinct and do not cross each other, then they must not overlap (since both are empty). Moreover, their relative interiors must intersect relconv(m) in exactly two distinct points x and x' respectively. In these conditions we say that l' is *closer to q than l*, and denote it by $l \prec_q l'$, if $x' \in \text{relconv}(\{x, q\})$. This is a partial ordering relation in $\Omega(m)$. We extend it arbitrarily to a linear ordering in $\Omega(m)$ and denote the extended ordering by $<_q$. Let $\Omega(m) = \{l_1, \ldots, l_k\}$, with $l_i <_q l_j$ for $i < j$. We consider every l_i oriented so that $p \in l_i^+$. For every $i = 0, 1, \ldots, k$ we define $\omega_i(m) := \{S \in \text{Baues}_m(\mathcal{A}) \ : \ S \textit{ lies on } l_j^+, \ \forall\, j > i\}$. Note that $\omega_k(m) = \text{Baues}_m(\mathcal{A})$.

Lemma 4.2 *Let $i \in \{1, \ldots, k\}$ and let $S \in \omega_i(m) \setminus \omega_{i-1}(m)$. Then S either lies on l_i or is incident to l_i^-.*

Proof: Let $i \in \{1, \ldots, k\}$ and let $S \in \omega_i(m) \setminus \omega_{i-1}(m)$. Suppose S does not lie on l_i. The triangles $l_i \cup \{p\}$ and $l_i \cup \{q\}$ (and their faces) form a triangulation $\mathcal{T}$ of $l_i \cup m$. By Lemma 2.7, there is exactly one simplex σ of $\mathcal{T}$ in which S lies. Since S lies on m, σ overlaps m. Thus, σ is either $l_i \cup \{p\}$, $l_i \cup \{q\}$ or l_i. Since S does not lie on l_i, $\sigma \neq l_i$. Since $S \in \omega_i(m) \setminus \omega_{i-1}(m)$, S does not lie on l_i^+. Hence, $\sigma \neq l_i \cup \{p\}$. Therefore, S lies on $\rho := l_i \cup \{q\}$.

Let $t \in l_i^-$, $t \neq q$. First suppose that $t \notin \text{conv}(\rho)$. Then, either $\rho \subset \text{conv}(l_i \cup \{t\})$ (and then, $l_i \cup \{t\}$ defines a triangulation of $\rho \cup \{t\}$ in which the unique simplex which overlaps ρ is $l_i \cup \{t\}$, so S lies on $l_i \cup \{t\}$) or some edge l of $l_i \cup \{t\}$ crosses an edge of ρ. In the last case, l crosses m and $l_i <_q l$, thus $l = l_j$ for some $j > i$. Hence, S lies on l_j^+. Say $l_i = \{r, s\}$ and $l_j = l = \{s, t\}$. The triangles $l_j \cup \{q\}$ and $l_j \cup \{r\}$ define a triangulation $\mathcal{T}'$ of $\rho \cup \{t\}$ and S lies on some simplex of $\mathcal{T}'$, which must overlap ρ. The only simplices of $\mathcal{T}'$ which overlap ρ are $l_j \cup \{q\}$, $l_j \cup \{r\}$ and l_j. Since S lies on l_j^+, S lies on $l_j \cup \{r\} = l_i \cup \{t\}$.

Finally suppose that $t \in \text{conv}(\rho)$. Then some edge of $l_i \cup \{t\}$ different from l_i (say $\{s,t\}$) crosses m. Thus, $l_j = \{s,t\}$ for some $j > i$, and therefore, S lies on l_j^+. The triangles $l_j \cup \{q\}$, $l_j \cup \{r\}$ and $\{q,r,t\}$ (in case it is actually a triangle) define a triangulation of $\rho \cup \{t\}$ in one of whose simplices S lies. Such a simplex cannot be a face of $\{q,r,t\}$, because no face of $\{q,r,t\}$ overlaps m, since r and t are on the same open side of m (and q is a vertex of m). The only simplices of the simplicial complex defined by $l_j \cup \{q\}$ and $l_j \cup \{r\}$ which overlap m are $l_j \cup \{q\}$, $l_j \cup \{r\}$ and l_j. Since S lies on l_j^+, S lies on $l_j \cup \{r\} = l_i \cup \{t\}$. □

Lemma 4.3 *For each $i \in \{0, \ldots, k\}$, the poset $\omega_i(m)$ is homotopy equivalent to $\omega_0(m)$. In particular, $Baues_m(\mathcal{A}) = \omega_k(m)$ is homotopy equivalent to $\omega_0(m)$.*

Proof: If $i = 0$ there is nothing to prove. Let $i \in \{1, \ldots, k\}$. We want to apply Lemma 3.10 to $\omega_i(m)$ and the edge l_i to show that $\omega_i(m)$ and $\omega_{i-1}(m)$ are homotopy equivalent. The triangles $l_i \cup \{p\}$ and $l_i \cup \{q\}$ define a triangulation of $m \cup l_i$ in which the simplices which overlap m are $l_i \cup \{p\}$, $l_i \cup \{q\}$ and l_i. Hence, every element of $\omega_i(m)$ lies on some of those three simplices and, in particular, on some closed side of l_i. Let $S \in \omega_i(m)$. If S lies on l_i^0, by the previous argument, S lies on l_i. Moreover, the subdivision S^+ which is incident to S along l_i^+ is in $\omega_i(m)$: On one hand, S lies on m and m is not one of the simplices on which S lies we remove when passing to S^+. On the other hand, if σ is a simplex of $\mathcal{B}$ on which S lies with $\text{relconv}(\sigma) \subset l_j^+$ for some $j > i$, then, either $\text{relconv}(l_i) \subset \text{relconv}(\sigma)$ and S^+ lies on $\sigma \cup \{p\} = l_i \cup \{p\}$ (and since $\text{relconv}(l_i \cup \{p\}) \subset l_j^+$, S^+ lies on l_j^+) or S^+ lies on σ and, therefore, on l_j^+. We conclude that $S^+ \in \omega_i(m)$

Now suppose $S \in \omega_i(m)$ lies on l_i^-. Then, $S \in \omega_i(m) \setminus \omega_{i-1}(m)$ and S does not lie on l_i. By Lemma 4.2, S is incident to l_i^-, and since m is not one of the simplices on which S lies we remove when passing to S^0, S^0 lies on m. If S lies on a simplex σ of $\mathcal{B}$ with $\text{relconv}(\sigma) \subset l_j^+$ for some $j > i$, then, either σ is one of the triangles removed when passing to S^0 (in this case $\text{relconv}(l_i) \subset l_j^+$, and since S^0 lies on l_i, S^0 lies on l_j^+) or S^0 lies on σ and, therefore, on j^+. We conclude that $S^0 \in \omega_i(m)$

Now we can apply Lemma 3.10 to conclude that $\omega_i(m)$ is homotopy equivalent to $\omega_i(m)^+_{l_i} = \omega_{i-1}(m)$. By induction on i we are done. □

For the next result we will introduce the following notation. For a vertex v of a simplicial complex $\mathcal{S}$, we denote the star, the link and the anti-star of v in $\mathcal{S}$ by $\text{str}_v(\mathcal{S})$, $\text{lnk}_v(\mathcal{S})$ and $\text{astr}_v(\mathcal{S})$ respectively. The star and the anti-star of v in $\mathcal{S}$ are subcomplexes whose union is the whole $\mathcal{S}$ and whose intersection is the link of v in $\mathcal{S}$. In the following statement and in the sequel S_p denotes the unique subdivision which (according to Proposition 4.1) lies on $\{p\}$.

Lemma 4.4 *The subposets $\omega_0(m)$ and $lnk_{S_p}(Baues_{\overline{m}}(\mathcal{A}))$ coincide and consist of exactly one element. In particular, they are contractible.*

Proof: It is clear that S_p is a maximal element of Baues$(\mathcal{A})$, since the only subdivision of $\mathcal{A}$ which is properly refined by S_p is the trivial one (i.e. that whose unique cell is $\mathcal{A}$ itself), which is not an element of Baues$(\mathcal{A})$. Therefore, S_p is a maximal element of $\text{Baues}_{\overline{m}}(\mathcal{A})$ too, so $\text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$ consists of those elements of $\text{Baues}_{\overline{m}}(\mathcal{A})$ which properly refine S_p. Let $S \in \text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$. Clearly, S_q does not refine S_p, thus S lies on m. That is, $\mathcal{A} \setminus m$ is a cell of S. On the other hand S refines S_p, which implies that S induces a subdivision of the deletion $\mathcal{A} \setminus p$, one of whose cells must be $\mathcal{A} \setminus m \subset \mathcal{A} \setminus p$. But $\mathcal{A} \setminus m = (\mathcal{A} \setminus p) \setminus \{q\}$, that is, the whole $\mathcal{A} \setminus p$ except for the point q. In Proposition 4.1 we saw that there is only one subdivision (let us call it S_q^p) of $\mathcal{A} \setminus p$ which lies on $\{q\}$ (i.e. having $(\mathcal{A} \setminus p) \setminus \{q\}$ as a cell). Hence, $S_q^p \subset S$. The same argument as in Proposition 4.1 proves that there is a unique way to extend S_q^p to a subdivision of $\mathcal{A}$, and thus, S is determined. We conclude that $\text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$ has at most one element. Since the extension of S_q^p considered lies on m and refines S_p, it is in $\text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$, and thus, $\text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$ has exactly one element.

Let us show that $\omega_0(m) \subset \text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$. Let $S \in \omega_0(m)$. S lies on m and m is empty. Hence, no vertex of $\mathcal{B}$ overlaps m. By Lemma 2.2, S cannot lie on any vertex of $\mathcal{B}$, that is, for any cell $B \subset \mathcal{A}$ of S, $\mathcal{A} \setminus B$ has at least two elements. Let B be a cell of S with $p \in B$. Then $\sigma := \mathcal{B} \setminus B$ is an edge or a triangle of $\mathcal{B}$ which by Lemma 2.2 overlaps m. Since $S \in \omega_0(m)$, S lies on l^+ for every empty edge l which crosses m (were we consider l oriented so that $p \in l^+$), and hence so occurs for every edge which crosses m, empty or not. This implies that if σ is an edge, then σ does not cross m. Since σ overlaps m and m is empty, σ must contain m in its convex hull. Thus $p \in \text{conv}(\sigma)$. If σ is a triangle, either it contains m in its convex hull (and therefore $p \in \text{conv}(\sigma)$) or some edge l of σ crosses m. No other edge e of σ can cross m, since S would lie on the wrong side of either l or e, meaning the side which contains q. Therefore, some vertex of m is in $\text{conv}(\sigma)$. Since S lies on the side of l which contains p, $q \notin \text{conv}(\sigma)$. Hence, $p \in \text{conv}(\sigma)$. So, no matter which the case is, $p \in \text{conv}(\sigma)$, and since $p \in B$, we have that $p \notin \sigma$. We conclude that there is a circuit Z of $\mathcal{B}$ supported on $\sigma \cup \{p\}$ with $Z^+ = \{p\}$. That is, there is a cocircuit Z of $\mathcal{A}$ supported on $(\mathcal{A} \setminus B) \cup \{p\}$ such that $Z^+ = \{p\}$. Thus, $B \subset Z^0 \cup Z^+$ and $\mathcal{A} \setminus \{p\} \subset Z^- \cup Z^0$. We conclude that B does not overlap $\mathcal{A} \setminus \{p\}$, that is, every cell of S which overlaps $\mathcal{A} \setminus \{p\}$ is contained in $\mathcal{A} \setminus \{p\}$, and hence, S induces a subdivision of $\mathcal{A} \setminus p$. We conclude that S refines S_p. Since S lies on m, $S \in \text{lnk}_{S_p}(\text{Baues}_{\overline{m}}(\mathcal{A}))$.

It remains to show that $\omega_0(m) \neq \emptyset$, but this follows from the fact that it is homotopy equivalent to $\mathrm{Baues}_m(\mathcal{A})$, which is nonempty since it contains $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$. □

Lemma 4.5 *$\mathrm{Baues}_m(\mathcal{A})$ is a deformation retract of $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$.*

Proof: Since $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ is a singleton and $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ is the cone over $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ with apex in S_p, $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ is homeomorphic to an edge having $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ as a vertex. Therefore, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ is a deformation retract of $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$. This retraction can be extended to $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$ by defining it as the identity on $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$.

The corresponding homotopy (which is relative to $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ can be extended to $[0,1] \times \mathrm{Baues}_{\overline{m}}(\mathcal{A})$ (valuated in $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$) by defining it as relative to $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$. We conclude that $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$ is a deformation retract of $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$.

Now we observe that

$$\mathrm{str}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))) = \mathrm{str}_{S_q}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})) \tag{1}$$

since S_p and S_q do not refine each other, and hence, the stars of S_p and S_q intersect at most at their links. In particular,

$$\mathrm{lnk}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))) = \mathrm{lnk}_{S_q}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$$

Joining these two facts, we conclude that $\mathrm{lnk}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})))$ is a deformation retract of $\mathrm{str}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})))$. Repeating the argument presented above, we conclude that $\mathrm{astr}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})))$ is a deformation retract of $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$. But our observation (1) also implies that

$$\mathrm{astr}_{S_q}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))) = \mathrm{astr}_{S_q}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})) \cap \mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$$

and, on the other hand, it is clear that

$$\mathrm{astr}_{S_q}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})) \cap \mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A})) = \mathrm{Baues}_m(\mathcal{A})$$

thus $\mathrm{Baues}_m(\mathcal{A})$ is a deformation retract of $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{m}}(\mathcal{A}))$, and hence, of $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$. □

Theorem 4.6 *$\mathrm{Baues}_{\overline{m}}(\mathcal{A})$ is contractible for each empty edge m of $\mathcal{B}$.*

Proof: By Lemma 4.5, $\mathrm{Baues}_m(\mathcal{A})$ is a deformation retract of $\mathrm{Baues}_{\overline{m}}(\mathcal{A})$. By Lemma 4.3, $\mathrm{Baues}_m(\mathcal{A})$ and $\omega_0(m)$ are homotopy equivalent. By Lemma 4.4, $\omega_0(m)$ is contractible. □

The following is a result we will use later on. Before stating it, let us fix some notation. Let l be an empty edge of $\mathcal{B}$ which crosses m and consider it oriented so that $p \in l^+$. We define $\text{Baues}_{m,\overline{l^+}}(\mathcal{A}) := \{S \in \text{Baues}_m(\mathcal{A}) \ : \ S \textit{ lies on } \overline{l^+}\}$, and for each $i \in \{0, \ldots, k\}$, $\omega_i(m, \overline{l^+}) := \{S \in \omega_i(m) \ : \ S \textit{ lies on } \overline{l^+}\}$. Observe that $\omega_k(m, \overline{l^+}) = \text{Baues}_{m,\overline{l^+}}(\mathcal{A})$. Also we define $\text{Baues}_{\overline{m},\overline{l^+}}(\mathcal{A}) := \{S \in \text{Baues}_{\overline{m}}(\mathcal{A}) \ : \ S \textit{ lies on } \overline{l^+}\}$.

Lemma 4.7 *$Baues_{m,\overline{l^+}}(\mathcal{A})$ and $Baues_{\overline{m},\overline{l^+}}(\mathcal{A})$ are contractible.*

Proof: Let us show that $\text{Baues}_{m,\overline{l^+}}(\mathcal{A})$ is contractible. Since l crosses m, $l = l_j$ for some $j \in \{1, \ldots, k\}$. Let $i > j$. We have shown in the proof of Lemma 4.3 that $\omega_i(m)$ and l_i satisfy the hypotheses of Lemma 3.10. Since $\omega_i(m, \overline{l^+}) \subset \omega_i(m)$, in order to prove that $\omega_i(m, \overline{l^+})$ and l_i satisfy the hypotheses of Lemma 3.10, we only have to show that the following two conditions are satisfied for every $S \in \omega_i(m, \overline{l^+})$:

- If S is incident to l_i^-, then the subdivision S^0 which is incident to S along l_i^- lies on $\overline{l^+}$.
- If S lies on l_i, then the subdivision S^+ which is incident to S along l_i^+ lies on $\overline{l^+}$.

If a simplex σ is contained in $\overline{l^+}$, then every face of σ is contained in $\overline{l^+}$. Therefore, if S is incident to l_i^- and lies on $\overline{l^+}$, then S^0 lies on $\overline{l^+}$, which proves the first item.

On the other hand, if an edge σ is contained in $\overline{l^+}$, then, since $p \in l^+$, $\sigma \cup \{p\}$ is contained in $\overline{l^+}$, which proves the second item.

Therefore, by applying Lemma 3.10, we conclude that $\omega_i(m, \overline{l^+})$ and $\omega_{i-1}(m, \overline{l^+})$ are homotopy equivalent. By composition of the homotopy equivalences so obtained (where i ranges over $\{j+1, \ldots, k\}$) we conclude that $\text{Baues}_{m,\overline{l^+}}(\mathcal{A}) = \omega_k(m, \overline{l^+})$ is homotopy equivalent to $\omega_j(m, \overline{l^+}) = \{S \in \omega_j(m) \ : \ S \textit{ lies on } \overline{l_j^+}\}$. It can be shown as in Lemma 4.3 that this poset satisfy, with respect to l_j, the conditions of Lemma 3.8, and hence, it is homotopy equivalent to $\{S \in \omega_j(m) \ : \ S \textit{ lies on } l_j^+\} = \omega_{j-1}(m)$, which, as we already know, is contractible.

The proof for $\text{Baues}_{\overline{m},\overline{l^+}}(\mathcal{A})$ is essentially the same. □

4.3 Triangles

Throughout this subsection, $\tau = \{p, q, r\}$ will denote an empty triangle of $\mathcal{B}$.

Let $\{e_1, \ldots, e_h\}$ be the set of empty edges of $\mathcal{B}$ which cross $\{q, r\}$ and have p as a vertex, with $e_i <_r e_j$, $\forall i < j$, considering them all oriented so that r is in their positive sides. Set $e_0 = \{p,q\}$ and $e_{h+1} = \{p,r\}$. The triangles of the form $e_i \cup e_{i+1}$ for $0 \leq i \leq h$ define a simplicial complex Δ. Consider $\{q,r\}$, e_{h+1} and e_0 oriented so that p, q and r are in their respective positive sides.

Note that $\{e_1, \ldots, e_h\}$ could be empty (i.e. $e_1 = \{p,r\}$). If this is the case, then either there is another vertex of τ for which there are edges overlapping τ having it as a vertex, and in this case we make $\{e_1, \ldots, e_h\}$ nonempty by relabelling the vertices of τ, or there is no edge of $\mathcal{B}$ overlapping τ and sharing a vertex with τ. It is easy to check that in this latter case there is no edge at all which overlaps τ. This case we want to discuss separately.

Suppose that no edge of $\mathcal{B}$ overlaps τ. We already know that exactly one subdivision of $\mathcal{A}$ lies on each vertex of τ. Let S lie on, say, $\{p,q\}$, and suppose S lies on some other simplex σ of $\mathcal{B}$. The simplices $\{p,q\}$ and σ must overlap, so σ must contain $\{p,q\}$ in its relative interior. On the other hand, if some simplex σ overlaps $\{p,q\}$, it overlaps every simplex which overlaps $\{p,q\}$ and, in particular, every simplex on which S lies. This means that $\mathcal{A} \setminus \sigma$ intersects properly with every cell of S, and hence it is in S. Thus, S lies precisely on those simplices of $\mathcal{B}$ which overlap $\{p,q\}$. In particular S is unique. So, exactly one subdivision of $\mathcal{A}$ lies on each edge of τ. The same argument shows that exactly one subdivision of $\mathcal{A}$ lies on τ. Hence, there is a natural isomorphism between $\text{Baues}_{\overline{\tau}}(\mathcal{A})$ and the barycentric subdivision of τ, which is contractible.

So, without loss of generality, we assume that $\{e_1, \ldots, e_h\} \neq \emptyset$.

Lemma 4.8 *$Baues_{\overline{\tau}}(\mathcal{A})$ equals the set of subdivisions $S \in Baues(\mathcal{A})$ such that S lies on some simplex of Δ and S lies on $\overline{\{q,r\}^+}$.*

Proof: Let S lie on $\overline{\tau}$. Recall that S lies on $\overline{e_0^+}$, $\overline{e_0^+}$ and $\overline{\{q,r\}^+}$, and on some closed side of e_i for every $1 \leq i \leq h$ (since $e_1, \ldots, e_h$ cross τ). Let $j = max\{i \ : \ S \ lies \ on \ \overline{e_{i^+}}\}$ and let $k = min\{i \ : \ S \ lies \ on \ \overline{e_{i^-}}\}$. By triangulating $\tau \cup e_j \cup e_k$ in such a way that $e_j \cup e_k$ is in the triangulation, it is easy to show that $j \leq k$ and that S lies on $\overline{\rho}$, where ρ is the simplex $e_j \cup e_k$. If $k = j$, then $\rho = e_j = e_k \in \Delta$. Clearly, if $j < k$, then $k = j + 1$ by the definitions of j and k, and hence $\rho \in \Delta$.

Conversely, suppose S lies on some simplex of Δ and on $\overline{\{q,r\}^+}$. Let $\rho = e_j \cup e_{j+1}$, $1 \leq j \leq h-1$, such that S lies on $\overline{\rho}$. We triangulate $\rho \cup \tau$ in such a way that τ is in the triangulation. Every simplex of such a triangulation which is contained in $\overline{\{q,r\}^+}$ is a face of τ, and hence, S must lie on $\overline{\tau}$. □

We define $f_\tau \ : \ \text{Baues}_{\overline{\tau}}(\mathcal{A}) \longrightarrow \Delta$ as the map which sends every subdivision in $\text{Baues}_{\overline{\tau}}(\mathcal{A})$ to the unique simplex of Δ on which it lies. This map is order preserving when we consider Δ ordered by the reverse incidence relation. The proof is the same as the one we provided for the map $F_{\mathcal{T}}$ in Lemma 2.10.

We want to apply Quillen's Lemma to f_τ. We shall prove that the fibers of f_τ : $\text{Baues}_{\overline{\mathcal{T}}}(\mathcal{A}) \longrightarrow Im(f_\tau)$ are contractible, so f_τ is a homotopy equivalence between $\text{Baues}_{\overline{\mathcal{T}}}(\mathcal{A})$ and $Im(f_\tau)$ (the image of f_τ). But

$$Im(f_\tau) = \Delta \setminus \{\textit{edges and vertices of } \Delta \textit{ which do not belong to } \tau\}$$

which is obviously contractible. Therefore, the contractibility of the fibers of f_τ : $\text{Baues}_{\overline{\mathcal{T}}}(\mathcal{A}) \longrightarrow Im(f_\tau)$ is all we need to finish the proof of Theorem 3.

Let ρ be a simplex in the image of f_τ. If ρ is a proper face of τ, then the fiber over ρ by f_τ is $\text{Baues}_{\overline{\rho}}(\mathcal{A})$, which is contractible. If ρ is not a proper face of τ, then either $\rho = e_i$ for some $i = 1, \dots, h$ or $\rho = e_i \cup e_{i+1}$ for some $i = 0, \dots, h$. If $\rho = e_i$, then the fiber over ρ by f_τ is $\text{Baues}_{\overline{e_i}, \overline{\{q,r\}^+}}(\mathcal{A})$ (considering $\{q, r\}$ oriented so that $p \in \{q, r\}^+$), which is contractible by Lemma 4.7. It remains to show that the fiber of a triangle $\rho = e_i \cup e_{i+1}$ is contractible. Such a fibre is

$$\{S \in \text{Baues}_{\overline{\rho}}(\mathcal{A}) \ : \ S \textit{ lies on } \overline{\{q,r\}^+}\}$$

and the triangle ρ has the special property that no edge of $\mathcal{B}$ having p (which is a vertex of ρ) as a vertex overlaps the edge of ρ opposite to p.

The following theorem is the goal of this subsection and all we need to finish the proof of our main result, Theorem 3.

Theorem 4.9 *$\text{Baues}_{\overline{\mathcal{T}}}(\mathcal{A})$ is contractible for every empty triangle τ of $\mathcal{B}$.*

Proof: We want to apply Quillen's Lemma to f_τ. The fact that f_τ is order preserving can be shown in a similar way as for $F_{\mathcal{T}}$ in Lemma 2.10. Let ρ be a simplex in the image of f_τ. If ρ is a proper face of τ, then the fiber of ρ by f_τ is $\text{Baues}_{\overline{\rho}}(\mathcal{A})$, and thus contractible. If ρ is not a proper face of τ, then ρ is either one of the edges e_i for $i \in \{1, \dots, h\}$ or one of the triangles $e_i \cup e_{i+1}$ for $i \in \{0, \dots, h\}$. In the first case, the fiber of ρ is $\text{Baues}_{\overline{e_i}, \overline{\{q,r\}^+}}(\mathcal{A})$, which is contractible by Lemma 4.7. In the second, the fiber of ρ is

$$\{S \in \text{Baues}_{\overline{\rho}}(\mathcal{A}) \ : \ S \textit{ lies on } \overline{\{q,r\}^+}\}$$

which we claim is contractible. If we prove so, then, by Quillen's Lemma, f_τ is a homotopy equivalence over its image. It is clear that the image of f_τ is

$$\Delta \setminus \{\textit{edges and vertices of } \Delta \textit{ which do not belong to } \tau\}$$

which is obviously contractible. Therefore, it suffices to prove the claim. Since the proof is rather technical, we prefer to state this assertion as the next lemma and work it out in a separate section. □

Note that the triangle ρ involved in our claim above is an empty triangle, has p as a vertex and satisfies that no edge of $\mathcal{B}$ having p as a vertex overlaps ρ. Note also that $\{q,r\}$ is an empty edge of $\mathcal{B}$ which overlaps ρ but does not cross $\rho \setminus \{p\}$ (the edge of ρ opposite to p).

Lemma 4.10 *Let $\tau = \{p,q,r\}$ be an empty triangle of $\mathcal{B}$ such that that no edge of $\mathcal{B}$ having p as a vertex overlaps τ. Let l be an empty edge of $\mathcal{B}$ which overlaps τ but does not cross $\{q,r\}$, oriented so that $p \in l^+$. Then*

$$Baues_{\overline{\tau},\overline{l^+}}(\mathcal{A}) := \{S \in Baues_{\overline{\tau}}(\mathcal{A}) \ : \ S \ lies \ on \ \overline{l^+}\}$$

is contractible.

5 Proof of Lemma 4.10

From now on τ and l will be an empty triangle and an empty edge of $\mathcal{B}$, respectively, in the conditions of Lemma 4.10. Since l overlaps τ, l crosses some edge of τ which cannot be $\{q,r\}$. Without loss of generality we assume that l crosses $\{p,q\}$.

For a guide of this section see Figure 3, which is quite a sketch and does not necessarily represent the actual geometric situation. The four arrows represent lemmas 5.1, 5.2, 5.5 and 5.6 in that order.

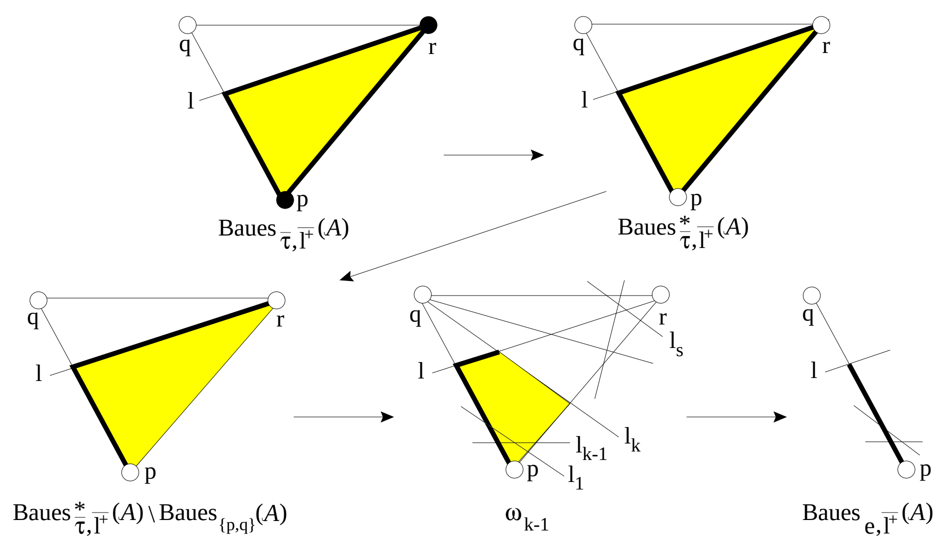

Figure 3: Visual guide of section 5: The arrows show the successive retractions. In each case, the shaded region together with the thick segments and black dots represent schematically the subposet under consideration.

First we proceed with some homotopy equivalences. We recall that the notations $\mathrm{str}_v(\mathcal{S})$, $\mathrm{lnk}_v(\mathcal{S})$ and $\mathrm{astr}_v(\mathcal{S})$ (for a vertex v of a simplicial complex $\mathcal{S}$) were introduced for Lemma 4.4. Let $\mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ denote the set of subdivisions in $\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ which do not lie on any vertex of τ.

Lemma 5.1 *$\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ and $\mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ are homotopy equivalent.*

Proof: First we want to show that $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ is a deformation retract of $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$.

Since S_p is the only subdivision of $\mathcal{A}$ which has $\mathcal{A}\setminus\{p\}$ as a cell, any subdivision S which is refined by S_p must have a cell which properly contains $\mathcal{A}\setminus\{p\}$, and hence S must be the trivial subdivision of $\mathcal{A}$, which is not an element of $\mathrm{Baues}(\mathcal{A})$. Therefore, S_p is a maximal element of $\mathrm{Baues}(\mathcal{A})$, and hence, it is a maximal element of $\mathrm{Baues}_{\overline{\tau}}(\mathcal{A})$. Thus the link of S_p in $\mathrm{Baues}_{\overline{\tau}}(\mathcal{A})$ is the subposet induced by the elements of $\mathrm{Baues}_{\overline{\tau}}(\mathcal{A})$ which properly refine S_p. Any of such subdivisions induces a subdivision of the deletion $\mathcal{A}\setminus p$, since $\mathcal{A}\setminus p$ is a cell of S_p. On the other hand, given a subdivision S of $\mathcal{A}\setminus p$, S can be extended to a subdivision of $\mathcal{A}$ in a unique way, namely, by joining p to every facet of S which is "visible" from p. Uniqueness follows straightforward from the fact that if S' extends S and B is a cell of S' with $p\in B$, then $B\setminus\{p\}\subset\mathcal{A}\setminus\{p\}$, and hence, $B\setminus\{p\}$ is a facet of some cell of S. Therefore, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ is isomorphic to the subposet of $\mathrm{Baues}(\mathcal{A}\setminus p)$ of those subdivisions of $\mathcal{A}\setminus p$ one of whose cells contains $(\mathcal{A}\setminus p)\setminus\tau=\mathcal{A}\setminus\tau=(\mathcal{A}\setminus p)\setminus\{q,r\}$. That is, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))\cong \mathrm{Baues}_{\overline{\{q,r\}}}(\mathcal{A}\setminus p)$. But $\mathcal{A}\setminus p$ is a corank 2 vector configuration, and hence, all its subdivisions are regular. Therefore, according to [3] , $\mathrm{Baues}(\mathcal{A}\setminus p)$ is canonically isomorphic to the incidence poset of the chamber complex of its Gale transform $\mathcal{B}/p$, which has rank 2. Moreover, according to this isomorphism, $\mathrm{Baues}_{\overline{\{q,r\}}}(\mathcal{A}\setminus p)$ is isomorphic to the subposet of those cells of the chamber complex of $\mathcal{B}/p$ which are contained in $\mathrm{conv}(\{q,r\})$, which is homeomorphic to $\mathrm{conv}(\{q,r\})$ itself, and hence, to a closed interval. We conclude that $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ is homeomorphic to a closed interval. Thus, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ is a deformation retract of $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$, which is a cone over this interval.

Now observe that $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))=\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$; every subdivision $S\in\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ lies on some closed side of l, and we want to show that S does not lie on l^-. If that was the case, there would be a simplex $\sigma\subset\overline{l^-}$ such that S lies on σ. Since S refines S_p we would then have that S_p lies on some face of σa and, in particular, on $\overline{l^-}$. This is absurd since S_p lies on $\{p\}$ and hence on l^+. Thus, $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))=\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$ and, in particular, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))=\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau}}(\mathcal{A}))$. Therefore, $\mathrm{lnk}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ is a deformation retract of $\mathrm{str}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$.

This deformation retraction can be naturally extended to a deformation retraction of $\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ onto $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$. We conclude that $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ is a deformation retract of $\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})$.

The subdivision S_q is not in $\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ since $q \in l^-$. This could be also the case of S_r, but it could also happen that r is a vertex of l (as in Figure 3). In this case l would be of the form $\{r,s\}$ and now it makes sense for a subdivision of $\mathcal{A} \setminus r$ to lie on one side of s, which is a covertex of $\mathcal{B}/r$. With similar arguments as above, one concludes that $\mathrm{lnk}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})) \cong \mathrm{Baues}_{\overline{\{p,s\}}}(\mathcal{A})$, and hence, that $\mathrm{lnk}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ is a deformation retract of $\mathrm{str}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$.

Since S_p and S_r do not refine each other, their stars (in any subposet) intersect at most at their links, and hence,

$$\mathrm{str}_{S_r}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))) = \mathrm{str}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$$

and

$$\mathrm{astr}_{S_r}(\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))) = \mathrm{astr}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})) \cap \mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$$

Thus, the deformation retraction which maps $\mathrm{str}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ onto $\mathrm{lnk}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ can be naturally extended to a deformation retraction of $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$ onto $\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})) \cap \mathrm{astr}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A}))$.

It is clear that

$$\mathrm{astr}_{S_p}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})) \cap \mathrm{astr}_{S_r}(\mathrm{Baues}_{\overline{\tau},\overline{l^+}}(\mathcal{A})) = \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$$

and thus we are done. □

Lemma 5.2 *The posets* $\mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ *and* $\mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A})$ *are homotopy equivalent.*

Proof: Let us consider $\{p,r\}$ oriented so that q is in its positive side. It is clear that every $S \in \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ lies on $\overline{\{p,r\}^+}$. Let $S \in \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ lie on $\{p,r\}^0$. Since S lies on $\overline{\tau}$, S lies on some face σ of τ. Since S lies on $\{p,r\}^0$, S lies on some simplex ρ of $\mathcal{B}$ which is contained in $\{p,r\}^0$. Since σ and ρ must overlap, σ has to be either $\{p\}$, $\{r\}$ or $\{p,r\}$. But S lies on none of the vertices of τ, thus S lies on $\{p,r\}$. Since $q \in \{p,r\}^+$, S^+ lies on τ. Clearly, S^+ does not lie on any vertex of τ. On the other hand, S lies on some simplex $\rho \subset \overline{l^+}$. Since l overlaps τ, some vertex t of l is in $\{p,r\}^+$. If ρ is an edge containing $\{p,r\}$ in its convex hull, then $\rho \cup \{t\} \subset \overline{l^+}$. Note that S^+ lies on $\rho \cup \{t\}$. If ρ is not an edge containing $\{p,r\}$ in its convex hull, then S^+ lies on ρ. Either way S^+ lies on $\overline{l^+}$, and hence, $S^+ \in \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$. By Lemma 3.8, $\mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})$ is homotopy equivalent to $\{S \in \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A})\ :\ S \textit{ lies on } \{p,r\}^+\} = \mathrm{Baues}^*_{\overline{\tau},\overline{l^+}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A})$, as we wanted to prove. □

Now we need to introduce some more notation. $\Omega(\{p,r\})$ will denote the set of empty edges which cross $\{p,r\}$, Ω_1 will be the set of empty edges which cross both $\{p,r\}$ and $\{p,q\}$, Ω_3 the set of empty edges which cross both $\{p,r\}$ and $\{q,r\}$, and $\Omega_2 := \Omega(\{p,r\}) \setminus (\Omega_1 \cup \Omega_3)$. Thus, Ω_2 is the set of empty edges which cross $\{p,r\}$ and have q as a vertex.

We consider the order relation of "being closer to r" defined in $\Omega(\{p,r\})$ and denoted by $\prec_r$, analogous to the one introduced in section 4.2.

Lemma 5.3 *There is a linear ordering $<_r$ in $\Omega(\{p,r\})$ that extends the partial ordering $\prec_r$ and such that the edges in Ω_2 are greater than those in Ω_1, and the edges in Ω_3 are greater than those in Ω_2 (for the ordering $<_r$).*

Proof: Clearly, given two edges $l_1 \in \Omega_1$ and $l_2 \in \Omega_2$, then either l_1 and l_2 are not comparable for the order relation $\prec_r$ or $l_1 \prec_r l_2$. The same situation holds for edges $l_2 \in \Omega_2$ and $l_3 \in \Omega_3$; either $l_2 \prec_r l_3$ or they are not comparable. We extend $\prec_r$ to an ordering relation in $\Omega(\{p,r\})$ which we denote by $<_r$ by defining $l_i <_r l_j$ for every $l_i \in \Omega_i$, $l_j \in \Omega_j$ with $i < j$. It is easy to check that this is an ordering relation. Finally, the relation $<_r$ so obtained can be extended to a linear ordering (which we still denote $<_r$) in $\Omega(\{p,r\})$ (possibly in several ways. We just choose one). □

We fix a linear ordering $<_r$ in $\Omega(\{p,r\})$ that extends the partial ordering $\prec_r$ and such that the edges in $\Omega_2 \cup \Omega_3$ are greater than those in Ω_1 for the ordering $<_r$ (provided by the previous lemma).

Let $\Omega_1 = \{l_1, \ldots, l_{k-1}\}$ and $\Omega_2 \cup \Omega_3 = \{l_k, \ldots, l_s\}$ with $l_i <_r l_j$, $\forall 1 \leq i < j \leq s$. For every $l_j \in \Omega(\{p,r\})$ (that is, for every $1 \leq j \leq s$), we define l_j^+ to be the side of l_j on which p lies. For every $i = k-1, \ldots, s$ we define

$$\omega_i := \{S \in \mathrm{Baues}^*_{\overline{\tau, l^+}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A}) \ : \ S \textit{ lies on } l_j^+, \ \forall\, j > i\}$$

Note that $\omega_s = \mathrm{Baues}^*_{\overline{\tau, l^+}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A})$ contains all other ω_i's. Also observe that every subdivision in ω_s lies on $\{q,r\}^+$; this is easy to check by taking a triangulation of $\tau \cup l$ which uses the triangle $l \cup p$.

Lemma 5.4 *Let $i \in \{k, \ldots, s\}$. If $S \in \omega_i \setminus \omega_{i-1}$, then S either lies on l_i or is incident to l_i^-.*

Proof: Suppose S does not lie on l_i. Since $S \in \omega_i \setminus \omega_{i-1}$, S does not lie on l_i^+. Therefore, by Lemma 3.3, S lies on $\overline{l_i^-}$. Since l_i is empty and does not cross $\{p,q\}$, either $q \in l_i$ or l_i crosses $\{q,r\}$. Thus, $\{l_i \cup \{r\}\}$ can be extended to a triangulation $\mathcal{T}$ of $\tau \cup l_i$ such that any triangle of $\mathcal{T}$ other than $l_i \cup \{r\}$ is contained in $\overline{l_i^+}$. We know that there is a unique simplex ρ of $\mathcal{T}$ on which

S lies. Since S lies on $\overline{l_i^-}$, ρ must be a face of $l_i \cup \{r\}$. Moreover, ρ is either $l_i \cup \{r\}$, l_i, $\{r\}$, $\{q\}$, or $\{q, r\}$ (the last two, only if $q \in l_i$), since no other face of $l_i \cup \{r\}$ overlaps a face of τ. Since S lies on $\{q, r\}^+$ and we are assuming that S does not lie on l_i we have that $\rho = l_i \cup \{r\}$.

Now let $t \in l_i^-$. If $\mathrm{conv}(\rho) \subset \mathrm{conv}(l_i \cup \{t\})$, by Lemmas 2.7 and 2.2, S lies on some face of $l_i \cup \{t\}$ which overlaps ρ. But $l_i \cup \{t\}$ itself is the only face of $l_i \cup \{t\}$ which overlaps ρ, so S lies on $l_i \cup \{t\}$. On the other hand, if $\mathrm{conv}(\rho) \not\subset \mathrm{conv}(l_i \cup \{t\})$, then some edge m of $l_i \cup \{t\}$ is in $\Omega_2 \cup \Omega_3$ with $l_i <_r m$. Thus $m = l_j$ for some $j > i$ and, since $S \in \omega_i$, S lies on $l_j^+ = m^+$. We extend $\{l_i \cup \{t\}\}$ to a triangulation $\mathcal{T}$ of $l_i \cup \{t, r\}$. S lies on a simplex σ of $\mathcal{T}$ which must intersect m^+ and overlap a face of τ. It is clear that the only simplices which satisfy these conditions are l_i, $l_i \cup \{t\}$ and $\{q\}$ (the last one, only if $q \in l_i$). Since S lies on $\{q, r\}^+$ and we are assuming that S does not lie on l_i, we have that S lies on $l_i \cup \{t\}$. □

Lemma 5.5 *The subposets ω_i and ω_{k-1} are homotopy equivalent for every $i > k - 1$ (that is, for every i such that $l_i \in \Omega_2 \cup \Omega_3$).*

Proof: For each $i \geq k$, we want to apply Lemma 3.10 to ω_i to show it is homotopy equivalent to ω_{i-1}. The proof will finish with the composition of the resulting homotopy equivalences.

Let $i \geq k$. By Lemma 3.3, since $\omega_i \subset \mathrm{Baues}_{\mathcal{T}}(\mathcal{A})$ and l_i overlaps τ, every element of ω_i lies on some closed side of l_i. If $S \in \omega_i$ lies on l_i^-, then $S \in \omega_i \setminus \omega_{i-1}$. By Lemma 5.4, S is incident to l_i^-. None of τ $\{p, q\}$ and $\{q, r\}$ are simplices we remove when passing from S to S^0, so $S^0 \in \mathrm{Baues}^*_{\mathcal{T}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A})$. Let $j > i$ and let ρ be a simplex in which S lies with $\mathrm{relconv}(\rho) \subset l_j^+$. If ρ is a triangle of the form $m \cup \{t\}$ with $t \in l_i^-$ and $l_i \subset \mathrm{conv}(m)$, then $l_i \subset \overline{l_j^+}$. Since both l_i and l_j are empty and cross τ, some vertex of l_i is in l_j^+. Therefore, $\mathrm{relconv}(l_i) \subset l_j^+$ and, since S^0 lies on l_i, S^0 lies on l_j^+. If ρ is not of the form $m \cup \{t\}$ with $t \in l_i^-$ and $l_i \subset \mathrm{conv}(m)$, then ρ is not one of the simplices we remove when passing from S to S^0, and hence, S^0 lies on ρ. Either way, S^0 lies on l_j^+ (for every $j > i$). On the other hand, S lies on a simplex which is contained in $\overline{l^+}$, every face of such a simplex is also contained in $\overline{l^+}$, so S^0 lies on $\overline{l^+}$. Hence, $S^0 \in \omega_i$.

If $S \in \omega_i$ lies on l_i^0, we take a simplex $\rho \subset l_i^0$ in which S lies, which must overlap a face of τ. Since $S \in \mathrm{Baues}^*_{\mathcal{T}}(\mathcal{A})$, such a face must be either τ or one of its edges. Therefore, ρ must be an edge containing l_i in its convex hull, and hence S lies on l_i too. None of τ, $\{p, q\}$ and $\{q, r\}$ are simplices we remove when passing from S to S^+. Hence, since S is an element of $\mathrm{Baues}^*_{\mathcal{T}}(\mathcal{A}) \setminus \mathrm{Baues}_{\{p,r\}}(\mathcal{A})$, so does S^+. Let $j > i$ and let ρ be a simplex in which S lies with $\mathrm{relconv}(\rho) \subset l_j^+$. If ρ is an edge of $\mathcal{B}$ containing l_i in its convex hull, then S^+ lies on $\rho \cup \{p\}$ and $\mathrm{relconv}(\rho \cup \{p\}) \subset l_j^+$. If ρ is not an edge of $\mathcal{B}$ containing l_i in its convex hull, then S^+ lies on ρ. Either way, S^+ lies on l_j^+ (for every $j > i$). On the other hand, if S lies on an edge contained in $\overline{l^+}$, then such an edge together with p is a triangle contained in $\overline{l^+}$, so S^+ lies on $\overline{l^+}$. Thus, $S^+ \in \omega_i$.

We have shown so far that l_i and ω_i satisfy the hypotheses of Lemma 3.10, thus ω_i is homotopy equivalent to $Q := \{S \in \omega_i \ : \ S \textit{ lies on } l_i^+\}$. But it follows straightforward from the definitions that $Q = \omega_{i-1}$. □

Lemma 5.6 *Let e denote the edge $\{p, q\}$ of τ. Then, the subposet ω_{k-1} is homotopy equivalent to $Baues_{e,\overline{l^+}}(\mathcal{A})$ (and thus, contractible).*

Proof: We want to apply Lemma 3.9 to the poset ω_{k-1} and the edge e (which we consider oriented so that $r \in e^+$). Clearly, every element of ω_{k-1} lies on $\overline{e^+}$. In particular, condition 1 of Lemma 3.9 is satisfied.

For condition 2, let $S \in \omega_{k-1}$ lie on e^+. Since $S \in \text{Baues}^*_{\overline{\tau}}(\mathcal{A}) \setminus \text{Baues}_{\{p,r\}}(\mathcal{A})$, S lies either on τ, on $\{q, r\}$ or on $\{p, q\} = e$. Since, moreover, S lies on $\overline{l^+}$, S lies on $\{q, r\}^+$. We are assuming that S lies on e^+, thus S lies on τ. Now let $t \in e^+$. Since τ is empty, either $\tau \subset \text{conv}(\{p, q, t\})$ or some edge of $\{p, q, t\} = e \cup \{t\}$ crosses an edge of τ which must be either $\{p, r\}$ or $\{q, r\}$. If $\tau \subset \text{conv}(\{p, q, t\})$, then $\{p, q, t\}$ defines a triangulation of $\tau \cup \{t\}$ and S must lie on one of its simplices. Such a simplex must overlap τ, and hence it must be $\{p, q, t\}$ itself. Thus, S lies on $\{p, q, t\}$. On the other hand, if some edge of $\{p, q, t\}$ crosses an edge of τ (since no edge of $\mathcal{B}$ having p as a vertex crosses $\{q, r\}$) such an edge of τ must be $\{p, r\}$, and hence, the edge of $\{p, q, t\}$ which crosses $\{p, r\}$ must be $\{q, t\}$. But this implies that $\{q, t\} \in \Omega_2$, so S lies on $\{q, t\}^+$. By extending $\{\{p, q, t\}\}$ to a triangulation $\mathcal{T}$ of $\tau \cup \{t\}$ and then extending $\mathcal{T}$ to a triangulation of $\mathcal{B}$, we conclude that S lies on a face ρ of $\{p, q, t\}$. Since ρ must overlap τ, ρ must be either $\{q, t\}$ or $\{p, q, t\}$. Since S lies on $\{q, t\}^+$, $\rho = \{p, q, t\}$. Therefore, S is incident to e^+.

Now, since S^0 lies on e, in order to show that S^0 is an element of ω_{k-1}, it remains to show it lies on $\overline{l^+}$ and on l_j^+, for each $j \geq k$. Let $j \geq k$ and let σ be a simplex in which S lies with $\text{relconv}(\sigma) \subset l_j^+$. In particular, $\sigma \subset \overline{l_j^+}$. If S^0 lies on σ, then S^0 lies on l_j^+. If S^0 does not lie on σ, then S^0 lies on a face of σ which must be an edge ρ containing e in its relative interior. Since $p \in l_j^+$, some point of $\text{relconv}(\rho)$ is in l_j^+, thus $\text{relconv}(\rho) \subset l_j^+$ and S^0 lies on l_j^+. On the other hand, S lies on some simplex contained in $\overline{l^+}$, and every face of such a simplex is contained in $\overline{l^+}$ as well, so S^0 lies on $\overline{l^+}$.

We can apply Lemma 3.9 to conclude that ω_{k-1} is homotopy equivalent to $\overline{(\omega_{k-1})_e^-} := \{S \in \omega_{k-1} \ : \ S \textit{ lies on } \overline{e^-}\} = \{S \in \omega_{k-1} \ : \ S \textit{ lies on } e^0\} = \{S \in \omega_{k-1} \ : \ S \textit{ lies on } e\}$. Since $e \subset \overline{l_j^+}$ for each $j > k-1$, every subdivision S of $\mathcal{A}$ which lies on e, lies on l_j^+, for every $j > k-1$. Therefore,

$\{S \in \omega_{k-1} \ : \ S \textit{ lies on } e\} = \mathrm{Baues}_{e,\overline{l^+}}(\mathcal{A}).$ □

References

[1] Anderson, L., Topology of combinatorial differential manifolds. *Topology*, **38** (1999), 197–221.

[2] Azaola, M. and Santos, F., The graph of triangulations of a point configuration with $d+4$ vertices is 3-connected. *Discrete Comput. Geom.*, to appear.

[3] Billera, L.J., Filliman, P. and Sturmfels, B., Constructions and complexity of secondary polytopes. *Adv. Math.*, **83** (1990), 155–179.

[4] Billera, L.J., Gel'fand, I.M. and Sturmfels, B., Duality and minors of secondary polyhedra. *J. Combinatorial Theory*, Ser. B, **57** (1993), 258–268.

[5] Billera, L.J. and Munson, B.S., Triangulations of oriented matroids and and convex polytopes. *SIAM Journal on Algebraic and Discrete Methods*, **5** (1984), 515–525.

[6] Björner, A., Topological methods, in: *Handbook of Combinatorics*, Elsevier, Amsterdam 1995, pp. 1819–1872.

[7] Björner, A., Las Vergnas, M., Sturmfels, B., White, N. and Ziegler, G.M., *Oriented Matroids.* Cambridge University Press, Cambridge 1992.

[8] De Loera, J.A., Hoşten, S., Santos, F. and Sturmfels, B., The polytope of all triangulations of a point configuration. *Doc. Math. J. DMV.*, **1** (1996), 103–119.

[9] De Loera, J.A., Santos, F. and Urrutia, J., The number of geometric bistellar neighbours of a triangulation. *Discrete Comput. Geom.*, **21** (1999) 1, 131–142.

[10] Edelman, P.H. and Reiner, V., Visibility complexes and the Baues problem for triangulations in the plane. *Discrete Comput. Geom.*, **20** (1998), 35–59.

[11] Gel'fand, I.M., Kapranov, M.M. and Zelevinsky, A.V., *Multidimensional Determinants, Discriminants and Resultants.* Birkhäuser, Boston 1994.

[12] Huber, B., Rambau, J. and Santos, F., The Cayley trick, lifting subdivisions and the Bohne-Dress Theorem on zonotopal tilings. *J. Eur. Math. Soc. (JEMS)*, to appear.

[13] Lee, C.W., Regular triangulations of convex polytopes. *Applied Geometry and Discrete Mathematics- The Victor Klee Festschrift (P. Gritzmann and B. Sturmfels eds.) DIMACS series in Discrete Math. and Theoretical Comp. Science*, **4** (1991), 443–456.

[14] Mnëv, N.E. and Richter-Gebert, J., Two constructions of oriented matroids with disconnected extension space. *Discrete Comput. Geom.*, **10** (1993), 241–250.

[15] Quillen, D., Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. Math.*, **28** (1978), 101–128.

[16] Rambau, J., Triangulations of cyclic polytopes and higher Bruhat orders. *Mathematika*, **44** (1997), 162–194.

[17] Rambau, J. and Santos, F., The generalized Baues problem for cyclic polytopes I, in *Combinatorics of Convex Polytopes (K. Fukuda and G.M. Ziegler, eds.), European J. Combin.*, **21** (2000), 65–83..

[18] Rambau, J. and Ziegler, G.M., Projections of polytopes and the generalized Baues conjecture. *Discrete Comput. Geom.*, **16** (1996), 215–237.

[19] Reiner, V., The generalized Baues problem, in *New Perspectives in Algebraic Combinatorics*, MSRI publications, **38** (1999), Cambridge University Press.

[20] Santos, F., A point configuration whose space of triangulations is disconnected. *J. Am. Math. Soc.*, to appear.

[21] Santos, F., Triangulations of oriented matroids. *Mem. Am. Math. Soc.*, to appear.

[22] Santos, F., Triangulations with very few geometric bistellar neighbors. *Discrete Comput. Geom.*, **23** (2000), 15–33.

[23] Sturmfels, B. and Ziegler, G.M., Extension spaces of oriented matroids. *Discrete Comput. Geom.*, **10** (1993), 23–45.

[24] Ziegler, G.M., *Lectures on Polytopes.* Springer-Verlag, New York 1994.