CONSTRUCTION OF REAL ALGEBRAIC PLANE NODAL CURVES WITH GIVEN TOPOLOGY AND GENERICALLY OPTIMAL DEGREE I: the orientable case

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Abstract

We study a constructive method to find an algebraic curve in the real projective plane with a (possibly singular) topological type given in advance. Our method works if the topological model T to be realized has only double singularities. In that case, it gives an algebraic curve of degree $2N+2K$, where N and K are the numbers of double points and connected components of T. This bound is generically optimal and the topological models T for which the degree is optimal have a combinatorial characterization.

The construction is based on a preliminar topological manipulation of the topological model followed by some perturbation techniques to obtain the polynomial defining the algebraic curve. This paper considers only the case in which T is orientable. The non-orientable case will appear in a separate paper.

1991 Mathematics Subject Classification: 14P25, 14Q05

1 Introduction

In a previous paper by the author [Santos1] it is shown that any real algebraic plane nodal curve with N singular (double) points and K connected components in the projective plane is isotopic to a real algebraic plane curve of degree at most $4N+2K$. Also, the conjecture is raised that the degree bound can be lowered to $2N+2K$.

In this paper we settle down the conjecture in the afirmative, for *orientable curves*. Moreover, we give a topological-combinatorial characterization of curves for which the degree bound is optimal. The conjecture also holds in the non-orientable case [Santos2]. The proof of the non-orientable case is more intrincate and will be detailed in a forthcoming paper [Santos3].

Let us fix some concepts and notation. Throughout this paper we will use the term *algebraic curve* as an abbreviation for *real projective algebraic plane curve*. By this we mean a non-zero real homogeneous polynomial $f \in \mathbb{R}[X,Y,Z]$ in three variables. Sometimes, by abuse of language, we will call algebraic curve the zero set

$V(f) \subset \mathbb{RP}^2$ of the polynomial f. We will normally assume that we have an affine chart given for the projective plane. This allows us to speak of the *line at infinity* and to say, for example that a certain conic is an ellipse, or a circle.

An algebraic curve f is called *orientable* if its zero set $V(f)$ has an open neighborhood which is orientable; equivalently, if it can be moved by an isotopy to the affine chart in the projective plane. An algebraic curve f is called *nodal* if all its singularities are order 2 singular points with two different tangents, real or complex. If the tangents are real the singular point is called a *node*. If they are complex, we will call it a *simple double isolated point*.

Two algebraic curves (in general, two subsets V and W of $\mathbb{RP}^2$) are said to have the same *topological type* if there exists a global homeomorphism of the plane into itself sending V to W, that is, if the pairs $(\mathbb{RP}^2, V)$ and $(\mathbb{RP}^2, W)$ are *topologically equivalent*. Note that this condition is equivalent to V and being isotopic to W, and stronger than V and W being homeomorphic. Our main result in this paper is the following, which is a re-writting of Theorem 4.3:

Theorem 1.1 *Let f be an orientable nodal algebraic curve with K connected components and N nodes. Then, f is topologically equivalent to a certain nodal algebraic curve f_ε of degree $2N + 2K$. Moreover, one can find such an f_ε as being a small perturbation of the form $f_\varepsilon := f + \varepsilon g$, where f is a product of $N + K$ ellipses and g is the product of $2N + 2K$ lines.*

Of course, for most curves the degree bound in our theorem can be significantly lowered. For example, the classical optimal bounds by Bezout and Harnack indicate that for every N and K there are algebraic curves with N double points and K connected components with degree, essentially, $\sqrt{2N + 2K}$. However, we can say that our bound is *generically optimal* in the following sense: for every N and every K, there are orientable, nodal algebraic curves with N double points and K connected components which have not the topological type of any algebraic curve of degree lower than $2N + 2K$. This is shown in Section 5. Moreover, in that section, we give a topological characterization of algebraic curves for which the degree in our main theorem cannot be lowered (see Theorem 5.2 and its Corollary 5.3 for the precise characterization)

The structure of the paper is as follows. In Section 2 we introuce the notion of a *topological model* for an algebraic curve and the basic notions and results needed in our topological construction. Section 3 shows the main construction of a complicated topological model from simple pieces, in which our construction of algebraic curves is based. The algebraic part of the construction consists in a perturbation technique, which is shown in Section 4. Finally, Section 5 shows under which conditions our construction produces optimal degree.

2 Topological preliminaries.

We want to find an algebraic curve whose zero set has the same topological type of a certain algebraic curve given in advance. Equivalently, we could say that we are given a certain subset $T \subseteq \mathbb{RP}^2$ in the projective plane and want to find an algebraic curve f such that $V(f)$ has the same topological type as T. The conditions that such a T must satisfy for this to be possible are contained in the following definition (cf. for exampe [Boch-Cos-Roy]).

Definicition 2.1 Let T be a subset of $\mathbb{RP}^2$. We say that T is a *topological model* for an algebraic curve if it is homeomorphic to a graph with an even (possibly zero) number of edges incident to each vertex. We say that an algebraic curve f *realizes* a topological model T if its zero set $V(f) \subset \mathbb{RP}^2$ has the same topological type as T.

By a *nodal (topological) model* we mean a topological model such that all of the vertices of the underlying graph G_T have degrees 0, 2 or 4. We say that a topological model is *orientable* if it can be isotopically moved to a position where it does not intersect the line at infinity (equivalently, if it has an orientable open neighborhood).

Let T be a nodal, orientable topological model in $\mathbb{RP}^2$. The points where T is locally homeomorphic to a line will be called *regular.* The rest of the points are the vertices of degrees 0 and 4 of the underlying graph G_T and will be called, respectively, *isolated points* and *double* points (or *vertices*) of T. A double point P will be called *disconnecting* if $T \setminus P$ has one connected component more than T.

Our basic topological operation on a topological model T will be the *desingularization* of some of its vertices. Let P a vertex (double point) of T. The desingualarization of T at P consists in considering a suitable small open neighborhood U of P and substituting $T \cap U$ for two disjoint open curves in such a way that we get a new model with one vertex less. This operation was called a 'flip' in [G.Corb.-Recio] and [G.Corb.-Santos]. There are exactly two ways, up to topological equivalence, of desingularizing a double point. These are shown in Figure 1. If the double point was disconnecting, one of the two desingularizations leaves the number of connected components unchanged and the other one increases it by one.

Whenever we perform a desingularization of a curve, we will mark the place where it has been done with a *bonding line* which joins the two branches which we have inserted. In all our figures, bonding lines will appear in greyish, dotted lines. The reason for including bonding lines is that topological models are considered modulo topological equivalence. Thus, we are allowed to transform them by global homeomorphisms. The transformed bonding lines will tell us what topological change is needed to recover the original topological type from the desingularized one.

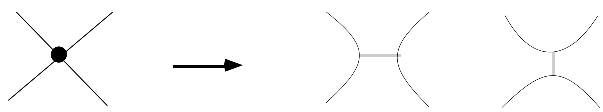

Figure 1: Desingularization of a double point P.

We call *faces* of T the connected components of $\mathbb{RP}^2 \setminus T$. Clearly T has a unique non-orientable face F_0. We will call *depth* of an arbitrary face F of T the minimal number of crossings with T needed to go from F_0 to F (a crossing at a double point of T counts twice). It is a well-known property that the parity of the intersection number with T of a path joining F to F_0 does not depend on the path (this is not true in general for non-orientable models). Thus, adjacent faces have depths which differ by 1. Figure 2 shows the depth diagram of a certain topological model.

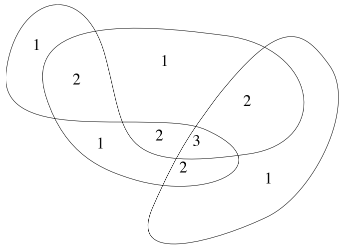

Figure 2: Depth of faces.

Let P be a vertex of T. Since the four faces around P have consecutive depths, we have two possibilities for the distribution of depths around P (see Figure 3):

Figure 3: Possible depth distributions around a vertex

- we will say that P is a vertex of Type I if the depths of faces around P are r, $r+1$, r and $r+1$, for some $r \geq 0$
- we will say that P is a vertex of Type II if the depths of faces around P are r, $r+1$, $r+2$ and $r+1$, for some $r \geq 0$.

We will say that a desingularization of T at some of its vertices is *depth-consistent* if the two faces which are joined by the desingularization of each vertex have the same depth. Figure 4 shows a depth-consistent desingularization of the curve appearing in Figure 2.

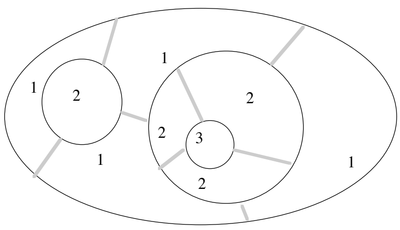

Figure 4: A depth-consistent desingularization of the curve in Figure 2.

Depth-consistency can be equivalently stated saying that each face of the desingularized model T' has the same depth as all the faces of T from which it has been obtained. Note that the different faces of the original model T which form a face of T' are still 'separated' by the bonding lines. If the desingularization is depth-consistent, then there is no ambiguity in considering the bonding lines or not for computing the depth of a face of T'. Both desingularizations of a vertex of type II are depth consistent, but only one for vertices of type I.

3 Main Construction

Figure 4 suggests the fact that for any topological model T, a non-singular model T' obtained from T by a depth-consistent dsingularization can be drawn as a union of disjoint ellipses with bonding lines being straight line segments. This property will be the basis for our algebraic construction. However, we will perform a partial desingularization of T. That is, some vertices of T will not be desingularized. In the following results T is supposed to be connected. Only in Theorem 3.5 we will deal with non-connected models.

Proposition 3.1 *Let T be a connected, nodal, orientable topological model in $\mathbb{RP}^2$. Then, there is a connected, nodal, orientable topological model T' obtained as a desingularization of T in some of its vertices, and with the following properties (see Figure 5):*

(i) The desingularization is depth-consistent.

(ii) Every vertex of T' disconnects T' and is of type II.

(iii) Let b be a bonding line of T'. Let r be the depth of the face in which b is. Then, at least one of the two faces adjacent to the extremal points of b has depth $r-1$.

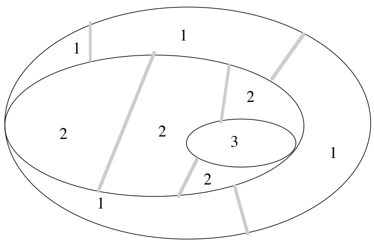

Figure 5: Desingularization of the model in Figure 2 which satisfies the conditions in Proposition 3.1.

Proof: Let P be a vertex of type I. Let f_0 and f_2 be the two faces of maximal depth $r+1$ around P and let f_1 and f_3 be the faces of depth r. We claim that the desingularization at P that joins f_0 to f_2 does not disconnect T. Indeed, if the desingularization disconnects T, then for going from one of f_1 or f_3 to the unbounded face it will be necessary to cross the new face obtained joining f_0 and f_2. This contradicts the fact that this face has a higher depth. The claim will still be true if we desingularize all vertices of type I in the way that joins faces of the maximal depth. Moreover, this way of desingularizing ensures condition (iii) for the bonding lines obtained.

After desingularizing all vertices of type I, we proceed to desingularize non-disconnecting vertices of type II one by one, until all the remaining vertices are disconnecting vertices. Condition (iii) is automatically satisfied for the unique depth-consistent desingularization of a vertex of type II. □

Proposition 3.2 *Let T be a connected, nodal, oriented topological model in $\mathbb{RP}^2$. Then, a partial desingularization T' of T satisfying the conditions of Proposition 3.1 can be transformed by a global homeomorphism of $\mathbb{RP}^2$ into the following form:*

- *T' is the union of a certain number of ellipses.*
- *if two of the ellipses intersect at a point P, then they do it tangentially and one is inside the other.*
- *the bonding lines are straight line segments.*

Moreover, the points of the unique outermost ellipse in T' which are intersections with other ellipses or extremes of bonding lines can be prescribed from the beginning (maintaining their circular order).

Proof: The proof will use induction on the maximal depth of faces in T'. If the maximal depth is 1 then T' has no vertices and consists on a unique oval with some bonding lines. Bonding lines must lie inside the oval, because of condition (iii) in Proposition 3.1. Thus, T' can be transformed into an ellipse with the bonding lines being straight line segments.

If the maximal depth of a face in T' is $r > 1$, we still have very particular properties for T': for a certain vertex P of T', the depth-consistent desingularization of T' at P is precisely the one that disconnects T'. Moreover, one of the connected components resulting is inside the other one, because P is a vertex of type II. Let us call the inner one the *ear* at vertex P.

Then, T' consists of an outer oval with some of these 'ears' attached to it in its inner side. Each ear itself is a topological model in the conditions of Proposition 3.1, but with maximal depth strictly less than r. Moreover, different ears are not connected to each other by bonding lines, because of condition (iii) in Proposition 3.1. However, an ear may have bonding lines connecting it to the outer oval, or there might be bonding lines connecting the outer oval to itself, through its inner face. Let us do the following:

First of all, draw the outer oval as an ellipse and prescribe along it the extremal points for bonding lines and the points where ears are to be attached, in a way that their circular order is preserved. Secondly, realize inner bonding lines of the outer oval as line segments joining the prescribed points. Then insert a small tangent ellipse at each point where an ear has to be attached (small enough for not intersecting other ears or bonding lines. Then, draw tha bonding lines joining the ears to the prescribed points in the outer ellipse. This can be done in a unique way modulo topological equivalence. Finally, prescribe in each inner ellipse the points where inner ears and bonding lines are to be attached, and draw them using inductive hypothesis.

The fact that the resulting topological model and bonding lines is topologically equivalent to T' follows automatically from the fact that each step in the 'drawing' process of T' is unique, modulo topological equivalence. □

Corollary 3.3 *Let T be a connected, nodal, orientable topological model with N vertices ($N > 0$). Then, there is a connected, nodal, orientable topological model $\mathcal{T}$ from which T can be obtained by desingularization of some vertices, in the following conditions:*

- *$\mathcal{T}$ is a union of ellipses, one of which has the others inside.*
- *any two ellipses of $\mathcal{T}$ which intersect, do it tangentially.*
- *$\mathcal{T}$ has $N + 1$ ellipses and at most $2N$ vertices (tangencies between ellipses)*

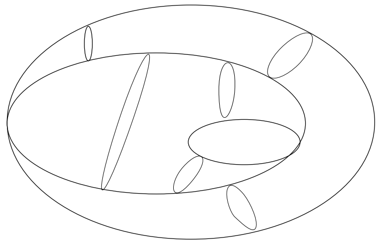

Figure 6: The topological model $\mathcal{T}$ of Corollary 3.3.

- *The singular points of $\mathcal{T}$ are in general position (no three of them on the same line).*

Proof: Consider the topological model T' obtained from T in Proposition 3.1, embedded in the form described in Propoition 3.2. T' has one ellipse more than it has double points. In other words, T has $N_1 + 1$ ellipses and N_2 bonding lines, with $N_1 + N_2 = N$. Substitute each bonding line of T' by a sufficiently narrow ellipse joining the two ends of the bonding line, and tangent to the ellipses at the ends and let $\mathcal{T}$ the topological model so obtained (see Figure 6).

Clearly, the topological model T can be recovered (modulo topological equivalence) by desingularizing one of the two tangency points of these new ellipses. This is exhibited in Figure 7 which shows the full sequence of topological manipulations done at a vertex.

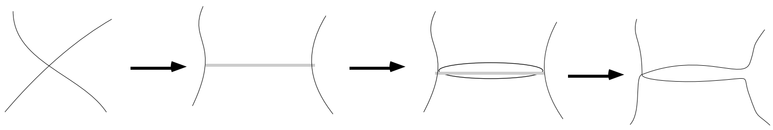

Figure 7: Topological changes at a vertex.

General position of the tangency points can be obtained thanks the freedom we have in Proposition 3.2 for choosing the extremal points of bonding lines and the tangency points of ellipses. □

Remark 3.4 Suppose that the original topological model T has a non-disconnecting vertex. We claim that, in these conditions, the numbers of ellipses and vertices of $\mathcal{T}$ in Corollary 3.3 can be decreased by one.

Indeed, if T has a non-disconnecting vertex, then the desingularized model T' has at least one bonding line connecting two nested ellipses. In this case, the insertion of the inner ellipse (the 'ear') in the proof of Proposition 3.2 can save one bonding line with the following trick: instead of inserting the ear as a small ellipse, insert it as an ellipse (as narrow as needed) joining the contact point of the ear to the extremal point of the bonding line. Then, add the other bonding lines if any (see Figure 8). The resulting model is not in the conditions of Proposition 3.1, but it still serves for the construction in Corollary 3.3.

Let us finally apply our last result to the non-connected case.

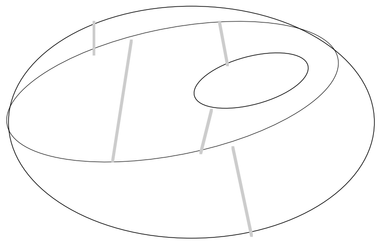

Figure 8: Saving one bonding line with a non-disconnenting vertex.

Theorem 3.5 *Let T be a nodal, orientable topological model with N_1 double points and N_2 isolated points. Then, there is a nodal, orientable topological model $\mathcal{T}$ from which T can be obtained by desingularization of some vertices, in the following conditions:*

- *$\mathcal{T}$ is a union of $N_1 + K$ ellipses and N_2 isolated points.*
- *any two ellipses of $\mathcal{T}$ which intersect, do it tangentially. There are at most $2N_1$ such tangency points.*
- *The singular (i.e., double or isolated) points of $\mathcal{T}$ are in general position (no three of them on the same line).*

Proof: Follows from Corollary 3.3. Let $T_1, \ldots, T_K$ be the connected components of T. Starting with the outermost ones, apply the Corollary to the connected components which have double points and realize the others by ellipses or isolated points. Place a copy of the resulting models $\mathcal{T}_\rangle$ in the appropriate part of $\mathbb{RP}^2$ (reducing them as needed) in order to get $\mathcal{T}$ in the required conditions. □

4 Perturbation of algebraic curves.

In order to obtain our main theorem 4.3 from Theorem 3.5, we only need to consider the topological model $\mathcal{T}$ obtained there as being an algebraic curve of degree $2N + 2K$ and algebraically perturb it in order to desingularize *some* singular points. One way to do this could be enlarge some of the ellipses in small amounts so that every tangency point becomes two transversal crossings (nodes). Then we could use the classical Brussotti's Theorem (cf. [Gudkov, p. 12]. This result says that a singular curve having only nodal points can be perturbed to a curve of the same degree where some of the singular points are desingularized in an arbitrary, prescribed, way.

Nevertheless, we will show an explicit way to perturb the curve $\mathcal{T}$ of Theorem 3.5 in the desired way, which makes our results more algorithmic. Let us first of all formalize the concept of a *perturbation* of an algebraic curve. Perturbation techniques are quite standard in the study of the topology of real algebraic curves (see [Gudkov, Viro]).

Let f_0 be an algebraic curve with finitely many singularities. Let $(f_\varepsilon)\varepsilon \in \mathbb{R}$ be a family of algebraic curves defined by polynomials f_ε of the same degree as f_0 and whose coefficients vary continuously with ε. Then, for ε sufficiently close to zero, the zero-sets $V(f_\varepsilon)$ are contained in an arbtrarily small neighborhood of

$V(f_0)$ and their topology coincides with the topology of $V(f_0)$ except, maybe, at small neighborhoods of the singular points of f_0. Moreover, the possible changes of topology at the singular points can be predicted, if the singularities of f_0 are sufficiently simple. Our perturbations will be explicitly given in the form $f_\varepsilon = f + \varepsilon g$, where g is a polynomial of the same degree as f and with a finite number of common zeroes with f.

The change in the topology of a curve in a neighborhood of a singular point will be called a *dissipation*. In our perturbations, the singular points appearing will be double points with two different analytic branches. If the two branches are complex the point appears as an isolated point in $V(f)$. If the two branches are real the topology of the curve in a neighborhood of the singular point is that of a topological double point, and we call it a *node*. These two types of singularities are classified as A_k^+ and A_k^- (with an odd k) in [Viro, p. 1098 ff.] and are diffeomorphic to the ones in $Y^2 + X^{k+1}$ and $Y^2 - X^{k+1}$, respectively (see also [Arn-Var-GusZ]). Finally, we will assume that a point P which is singular for f is either non-singular or nodal for g.

Lemma 4.1 *Let $f \in \mathbb{R}[X,Y,Z]$ be a homogeneous polynomial of a certain degree d, only having singularities in the A_k^+ or A_k^- series. Let $f_\varepsilon = f + \varepsilon g$ be a perturbation of f by a certain homogeneous plynomial $g \in \mathbb{R}[X,Y,Z]$ of the same degree d. Suppose that for every singular point P of f, either g is non-singular at P or P is nodal for g. Then, the only possible dissipations of f are those appearing in Figure 9.*

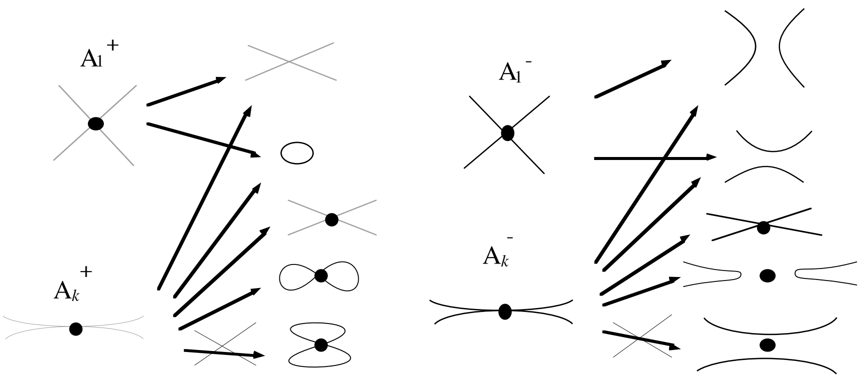

Figure 9: Dissipations of an A_k^+ or A_k^- singularity.

Proof: For the case of the A_1^- and the A_1^+ we are not saying anything new; the two possibilities shown are the only dissipations possible (except, of course, leaving the singular point unchanged). For the case of the A_k^+ and A_k^- some other things could happen but we can say the following things:

- no 'new ovals' can appear near a singular point P by a perturbation of the type $f + \varepsilon g$, except in the A_k^+ case. In this case only one such oval can appear and, if it appears, then the perturbed curve consists on only that oval (in a neighborhood of P): this claim follows from the fact that in a certain small neighborhood of a singular point P of f, the curves f_{ε_1} and f_{ε_2} have no common zeroes (except maybe the point P itself), for different values $\varepsilon_1 \neq \varepsilon_2$

of the parameter. This, together with the fact that any new oval should collapse to P, prevents new ovals from appearing except if that oval contains P and decreases its radius to zero as ε goes to zero. In this case no other parts of the perturbed curve can appear in the dissipation, because they should also isotopically move towards P as ε goes to zero.

- in a small neighborhood U of a singular point P of f, the perturbed curve has no singular point except, maybe, P itself: indeed, in $U \cap \{g = 0\} \setminus \{P\}$ there are no singular points because $f \neq 0$. In $U \cap \{g \neq 0\}$, a singular point of $f + \varepsilon g$ is the same thing as a critical point of the function $-f/g$, with critical value ε. Sard's Lemma tells us that the set of such critical values is discrete.

- a singular point P of f is a singular point of the perturbed curve if and only if it is a singular point of g; in this case, P is a nodal point of f_ε: this follows, for example, from a development of f and g as Taylor polynomials around P, in a suitable affine chart.

The above three properties only permit the five dissipations shown in Figure 9, for the A_k^+ and A_k^- cases. However, the last one is easily ruled out by counting intersection numbers with a suitable vertical line (the intersection numbers cannot increase by a small perturbation). All the others are possible. □

We can specify a bit more what dissipation will be produced in the following cases, which will be the only cases needed in our construction:

For the A_1^+ and A_1^- cases the singular point P (and the topology in a neighborhood of P) will change in a perturbation if and only if P is not a singular point of g. If this is the case, only one of the two possible dissipations is compatible with the signs of f and εg in a neighborhood of P (in particular, a change in the sign of ε changes the dissipation obtained).

For the A_k^- case, if g has a nodal singularity at P, than $f + \varepsilon g$ has a nodal singularity at P and we can say that:

- if one of the tangents to g coincides with the unique tangent to f, then the dissipation is the third one in the right column of Figure 9.

- if none of the two tangents to g coincide with the tangent to f, then the third and forth dissipations are obtained, depending on the signs of f and εg.

Theorem 4.2 *Let $C_0 \in \mathbb{R}\mathbb{P}^2$ be an algebraic curve defined by a homogeneous polynomial f of degree d. Suppose that all the singularities of f are a certain number of A_k^+ or A_k^- points $P_1, \dots, P_k$ and of A_k^- points $Q_1, \dots, Q_l$. Suppose that we want to perturb f preserving all the P_i, converting a number l_1 of the Q_i in nodes (with no change in the topology), and desingularizing the other $l_2 = l - l_1$ Q_i in a prescribed way.*

Suppose finally that the singular points of f are in general position, i.e. no three of them on the same line, and that we have $l_1 + k + l/2 \leq d$. Then, these dissipations can be simultaneously obtained in the form $f_\varepsilon = f + \varepsilon g$, where g is a product of d different lines.

Demostración: Assume $\varepsilon > 0$. According to what we said above, the following conditions on g are sufficient to guarantee the desired perturbation:

- For the k A_1 points to be preserved, that g has a singular (nodal) point at each of them.
- For the l_2 tangency points to be desingularized, that g does not vanish at them, and has the appropriate sign.
- For the l_1 tangency points to be converted in nodes, that g has a singular (nodal) point at each of them, and the appropiate distribution of signs in a neighborhood of them.

Let $r_1, \ldots, r_{k+l_1}$ be straight lines, each passing through two of the singular points to be preserved and such that each of the points lies in two of them. Then, the product g_1 of those $k + l_1$ straight lines has a nodal singular point at each of them, because of the general position assumption on the points.

Let $s_1, \ldots, s_{d-k-l_1}$ be lines, each passing through two of the points Q_i, not passing through the points P_i and so that each of the Q_i lies in exactly one of them. These lines exist, because of the condition $l/2 \leq d - k - l_1$. Then, the lines can be slightly moved (as shown in Figure 10) in such a way that the product g_2 of them has a prescribed sign at each point Q_i.

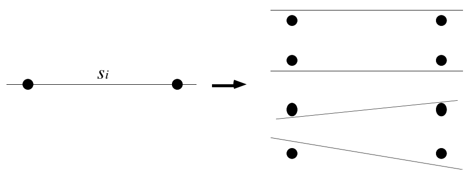

Figure 10: Obtention of the adequate sign at a point by moving s_i.

So, make the signs of g_2 at the points Q_i be the ones that we need in order to obtain $g = g_1 g_2$ with the appropriate signs, and take ε sufficiently small and positive. □

This, together with Theorem 3.5, gives our main theorem:

Theorem 4.3 *Let T be an orientable, nodal topological model with N double points and K connected components. Then, T can be algebraically realized by a curve $f_\varepsilon := f + \varepsilon g$ of degree $2N + 2K$, with f being a product of $N + K$ ellipses or degenerate conics and g being a product of $2N + 2K$ lines.*

Proof:

Let f be the product of the ellipses obtained in the model $\mathcal{T}$ of Theorem 3.5, and a factor of the form $(cX - aZ)^2 + (bX - aY)^2 + (cY - bZ)^2$ for each isolated point $(a, b, c) \in \mathbb{RP}^2$ of $\mathcal{T}$. Each connected component T_i of T contributes $N_i + 1$ ellipses or degenerate conics to f, where N_i is the number of double points in T_i. Thus, f is as in the statement.

Now, $l_1 + k = N$ and $l = l_1 + l_2 \leq 2N$. Thus, $l_1 + k + l/2 \leq 2N < d$. Then, Theorem 4.2 proves the assertion. □

5 Optimality of the construction

The degree produced by our construction is *generically* optimal, for any given K and N. Indeed, let K and N be given positive integers. Consider a topological model generalizing the one in Figure 11, for $N = 3$. Insert inside it $K - 1$ additional ellipses inside the innermost face, one inside another. This produces a nodal, orientable topological model with N vertices and K connected components. By our Theorem 4.3, this model can be realized with a nodal algebraic curve of degree $2N + 2K$. In the other hand, it cannot be realized by any algebraic curve of degree lower than that because, in any realization of the model, any straight line passing through the innermost face intersects the curve (at least) $2N + 2K$ times, counted with multiplicity.

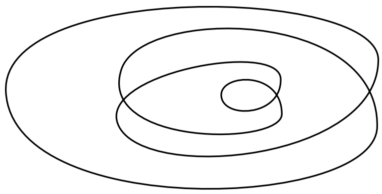

Figure 11: A simple model, not realizable with degree lower than 8.

Nevertheless, for most topological models our construction does not yield an optimal degree. Let us remark that a construction producing the optimal degree for any given model will provide a constructive answer to Hilbert's XVI problem for nodal curves, while a general answer for the simpler case of non-singular curves is only known up to degree 7 [Gudkov, Viro, Wilson]. Thus, there is no hope in obtaining such an optimal construction. The purpose of this section is to show in what cases our construction is really optimal. This will give us the somehow surprising result that the only obstructions to lowering the degree in the construction are those which are obvious (as the one in the example above).

As a first result, in remark 3.4 we mentioned that if the topological model T has a non-disconnecting vertex, then the degree of the construction can be lowered, at least, by two. Thus, we only need to consider the case of topological models with only disconnecting vertices. This condition is necessary but not sufficient: for example, the seven topological models in Figure 12 can easily be constructed with degree 4.

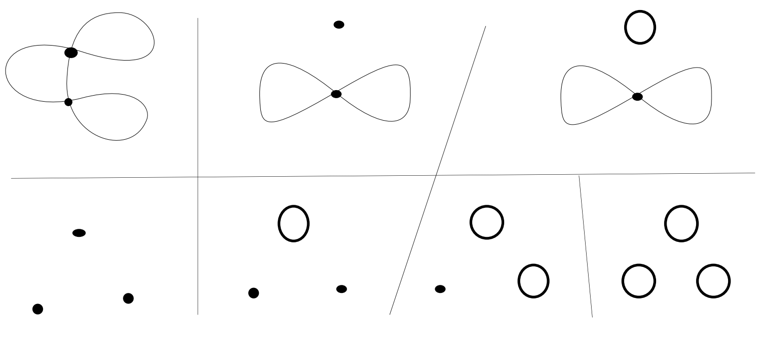

Figure 12: Some models which can be realized with degree 4.

Let T be a nodal, orientable topological model, all of whose double points disconnect it. Let N be the number of double points of T and K the number of connected components. If we desingularize every vertex of T in the way that disconnects T we get a non-singular topological model T_0 with $N + K$ connected components. The topological structure of T_0 can be represented in a rooted tree, with a node for each connected component of T_0 and an extra node (the root of the tree) 'at infinity'. A component C_1 is a son of a second component C_2 in the tree if and only if C_1 is inmediately inside C_2. The sons of the root node are the outermost components.

This tree structure of T_0 easily suggests a different construction procedure for an algebraic realization ofthe topological model T, which produces the same degree $2N + 2K$: starting with the outermost components, realize each of them independently, either with an ellipse or with a degenerate conic. Then, attach one by one the other components in the place indicated by their bonding line if they are connected to an exterior one, or in the appropriate face if they are not. After all components have been inserted, perturb the curve so obtained, using Theorem 4.2, in order to obtain a nodal algebraic curve.

The interesting point is that now a necessary condition for the model not being realizable with degree lower than $2N+2K$ is that the tree of connected components of T_0 has at most two *leaves* (innermost connected components): if this is the case, then, for any algebraic realization of T_0 any line intersecting the two innermost components will cut every connected component of T_0 at least twice (counted with multiplicities). If T was realizable with degree lower than $2N + 2K$, then T_0 would also be, by means of a small perturbation (via, e.g., Brussotti's Theorem). Thus, T itself cannot be realized with degree lower than $2N + 2K$. This necessary condition turns out to be also sufficient:

Lemma 5.1 *In the above conditions, if $N + K \geq 3$ and the tree of connected components of T_0 has at least three leaves, then T can be realized with degree $2N + 2K - 2$.*

Demostración: For $N + K = 3$ all the possibilities are shown in Figure 12. The first one is obtained by perturbing the product of two ellipses. The second one, by the curve $Y^2Z^2 = X^2Z^2 - X^4$. The first in the second row by $X^2Y^2 + X^2Z^2 + Y^2Z^2$. All the others, by perturbations of the above. Moreover, in any of the cases, different connected components can be realized as small as one wants and passing through prescribed points in the projective plane.

In the case $N + K > 3$, consider the construction procedure described above, by means of the tree of connnected components of T_0. As the tree has at least three leaves, we can consider these three leaves as the three last components and insert them at the same time by adding degree 4 to the construction, instead of 6. This is possible using the curves in Figure 12 and having into account that we don't really need the connected components to be attached being tangent to the previous ones. We can place them with two nodal intersections and then perturb (using, e.g., Brussotti's Theorem) in order to desingularize one of them in the appropriate way. □

Theorem 5.2 *Let T be a nodal, orientable topological model such that all its double points disconnect it. Let T_0 the desingularized model obtained from T disconnecting at every vertex. Then:*

(i) If the tree of connected components of T_0 has at most two leaves, then T cannot be algebraically realized with degree lower than $2N + 2K$.

(ii) If the tree of connected components of T_0 has at least three leaves, then T can be algebraically realized with degree $2N + 2K - 2$. □

Finally, we can rewrite Theorem 5.2 in the following way. The statement tells us that the only topological models which cannot be algebraically realized with degree lower than $2N + 2K$ are those for which this is evident, because such a realization has $2N + 2K$ intersection points with a certain line.

Corollary 5.3 *Let T be a nodal, orientable topological model in the projective plane with K connected components and N double points. Then, the following conditions are equivalent:*

(i) T is not topologically equivalent to any algebraic curve of degree lower than $2N + 2K$.

(ii) there are two points in the projective plane such that any pseudoline passing through them intersects T in at least $2N+2K$ points, counted with multiplicity.

Proof: That the first statement implies the second follows from Theorem 5.2. That the second implies the first is obvious. □

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List of Figures

List of Tables