ON THE TOPOLOGICAL SHAPE OF PLANAR VORONOI DIAGRAMS *

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Abstract

Voronoi diagrams in the plane for strictly convex distances have been studied in [3], [5] and [7]. These distances induce the usual topology in the plane and, moreover, the Voronoi diagrams they produce enjoy many of the good properties of Euclidean Voronoi diagrams. Nevertheless, we show (Th.1) that it is not possible to transform, by means of a bijection from the plane into itself, the computation of such Voronoi diagrams to the computation of Euclidean Voronoi diagrams (except in the trivial case of the distance being affinely equivalent to the Euclidean distance). The same applies if we want to compute just the topological shape of a Voronoi diagram of at least four points (Th. 2).

Moreover, for any strictly convex distance not affinely equivalent to the Euclidean distance, new, non Euclidean shapes appear for Voronoi diagrams, and we show a general construction of a nine-point Voronoi diagram with non Euclidean shape (Th.3).

1. Introduction and Statement of Results

Given a partition V of the plane into finitely many regions, Ash and Bolker [1] have studied the problem of deciding if V is an Euclidean Voronoi diagram for some set of points (see also [2] and [5]). We can relax the conditions and ask if the given partition V has at least the same topological shape of an Euclidean Voronoi diagram of some finite set of points. Here and in what follows we say that two cellular decompositions of the plane, each with a finite number of cells, have the *same topological shape* if there is an homeomorphism of the plane onto itself sending cells to cells.

This question is theoretically quite easy, because one can construct an algorithm to decide it as follows: taking the coordinates of the points for the Voronoi diagram as indeterminates, the fact that the Voronoi diagram for these points has the shape of V can be expressed as a finite set of conditions on these indeterminates, in such a way that there exists an Euclidean Voronoi diagram with the shape of V if and only if the conditions are satisfied for some values of the indeterminates. Now, the conditions appearing are always about the position of the circle passing by three of the points respect to a fourth one, i.e. they are polynomial equalities or inequalities and real quantifier elimination gives the answer to the problem whether they have a solution or not.

More interesting is to study the question about having the same topological shape of a planar Euclidean Voronoi diagram for the entire collection of partitions V arising as Voronoi diagrams for a non Euclidean distance. Does the changing of the distance imply a drastic change on the shape of Voronoi diagrams? Concretely we will consider the class of normed distances verifying the strong triangle inequality (i.e. the triangle equality holds only for collinear points, cf. [6]). Voronoi diagrams for these distances (that we shall call in what follows *strictly convex distances*) have been first considered by Chew and Drysdale [3] and then studied by Klein [5], Mazón [8], exhibiting an algorithm for their computation. Moreover, these distances are in many other respects quite close to the Euclidean distance; for instance they yield the usual topology on the plane and the Voronoi diagrams for them induce the same kind of cellular decomposition of the plane as the Euclidean Voronoi diagrams do. Thus the problem we posed about the conservation of the topological shape of Voronoi diagrams is quite natural in this situation.

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The main results of this paper are:

Theorem 1. Let d and δ be two strictly convex distances in the plane.

(i) If $d = \delta \circ f$ with f a bijection of the plane onto itself (i.e. $d(P,Q) = \delta(f(P), f(Q)), \forall P, Q$), then f is an affine mapping and for every finite set $S \subset \mathbb{R}^2$:

$$f(Vor_d(S)) = Vor_\delta(f(S)).$$

(ii) If there exists a bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving the bisectors of every two points, i.e. such that:

$$\forall P, Q \in \mathbb{R}^2 \qquad f(Bi_d(P,Q)) = Bi_\delta(f(P), f(Q)),$$

then f is affine, $d = k \cdot \delta \circ f = \delta \circ (k \cdot f)$, for some constant $k > 0$, and thus we are in the conditions of (i).

Two distances such that $d = \delta \circ f$, with f an affine bijection will be called *affinely equivalent*. Part (i) of Theorem 1 says that if one knows how to compute Voronoi diagrams for a given distance δ, then one can also compute them for any other affinely equivalent distance d. For instance, the problem of computing Voronoi diagrams with respect to a strictly convex distance d whose unit ball is an ellipse can be reduced to compute Euclidean Voronoi diagrams.

Part (ii) of Theorem 1 establishes that, for two given strictly convex distances d and δ to be affinely equivalent, it suffices that a bijection f from the plane onto the plane exists such that it preserves bisectors (which are two-point Voronoi diagrams). In this case, (i) implies that the Voronoi diagram of any finite set of points will be also preserved. In other words, (ii) is a strong reciprocal of (i): the only transformations which allow to reduce the computation of the Voronoi diagram for one strictly convex distance to another one are bijective affinities. Note also that if we take the bijection f as being the identity, (i) and (ii) say that two distances produce identical Voronoi diagrams for every finite collection of points if and only if they have the same bisectors and that, in this case, they are related by $d = k\delta$ and so they have the same circles (a circle for a distance d is the set of points with equal distance to a fixed center; if it is convenient to specify the distance we shall call them d-circles).

After this, in some sense, negative result, we are interested in knowing when this procedure of reduction permits to obtain, if not the exact diagrams, at least their topological shape, as this is the hardest part in the computation of a Voronoi diagram ([4]). We find the next negative result, with the additional hypothesis of the distances being smooth (i.e. with smooth circles).

Theorem 2. Let d and δ be two strictly convex and smooth distances in the plane. If there exists a bijection f from the plane onto itself such that for every finite set S, $Vor_d(S)$ has the same topological shape as $Vor_\delta(f(S))$, then f is affine, $d = k\delta \circ f$ for some constant $k > 0$ and we are in the conditions of Th.1(i). Moreover, the hypothesis is only needed for sets S of four or less points and it is not sufficient to have it for sets of three points.

As a corollary to Theorems 1 and 2, to look for homeomorphism between Voronoi diagrams of strictly convex smooth distances is the same as to look for equality. We want to remark that the additional hypothesis of the distances being smooth is used in our proof, but possibly Theorem 2 would be still true without it.

Finally we ask if strictly convex distances induce, for all Voronoi diagrams, the topological shape of an Euclidean one. The answer is given in the following Theorem which states that it is so only for distances affinely equivalent to the Euclidean distance.

Theorem 3. If d is a strictly convex distance, not affinely equivalent to the Euclidean distance, then there exists some collection S of nine points whose Voronoi diagram with respect to distance d, $Vor_d(S)$, has not Euclidean shape.

2. Voronoi Diagrams for Strictly Convex Distances

Strictly convex distances include all the L_p distances for $1 < p < \infty$ for which Lee [7] has generalized the standard divide and conquer algorithm to compute the Voronoi diagram. Also Chew and Drysdale [3] proposed a further generalization of the divide and conquer algorithm to convex distance functions.

A *strictly convex distance* in the plane is any distance induced by a norm and such that the boundary of its unit ball contains no three collinear points. The closed unit ball of a strictly convex distance is a compact and strictly convex subset K of the plane that contains the origin in its interior and is symmetrical with respect to it. Conversely, any set K with these properties is the closed unit ball of a certain strictly convex distance [6]. The *distance* $d_K(P,Q)$ *induced by* K between two points P and Q, is measured as follows: translate K so that it is centered at P and call it K_P. Let Z be the unique point of intersection of the half line from P through Q with the boundary of K_P. The distance between P and Q is, by definition, the quotient of the Euclidean distances between P and Q and P and Z.

If the convex K has smooth boundary (i.e. if it has only one supporting line through each point of its boundary) we will say that the corresponding distance is *smooth*.

Let d be a strictly convex distance on the plane and P and Q any two distinct points. The *bisector* $Bi_d(P,Q)$ of P and Q with respect to the distance d is defined as $Bi_d(P,Q) = \{X \in \mathbb{R}^2 : d(P,X) = d(Q,X)\}$ and the *d-circle of centre P and radius r*, $C_d(P,r)$ is defined as $C_d(P,r) = \{X \in \mathbb{R}^2 : d(P,X) = r\}$. Three given points are said to be *d-cocircular* if they belong to some d-circle.

Let S be a finite collection of points. Let $H(P,Q) = \{X \in \mathbb{R}^2 : d(X,P) - d(X,Q) < 0\}$. Then:

$$R_{S,d}(P) = \bigcap_{Q \in S - \{P\}} H(P,Q)$$

is *the Voronoi region of P with respect to S* and:

$$Vor_d(S) = \bigcup_{P \in S} Bd\, R_{S,d}(P)$$

is *the Voronoi diagram of S with respect to the distance d*, where $Bd\, R_{S,d}(P)$ denotes the topological boundary of the Voronoi region $R_{S,d}(P)$.

Interested readers may consult [8] for general properties of strictly convex distances. Here we just recall the more important ones for our purposes:

(i) $d(X,Y) = d(X,P) + d(P,Y)$ if and only if P belongs to the closed segment $[X,Y]$ (strong triangle inequality).
(ii) Three given collinear points cannot be d-cocircular. Conversely, if d is smooth, three non collinear points are always d-cocircular. This converse is no longer true if d is not smooth.
(iii) Any two d-circles intersect at most in two points (i.e. there is at most one d-circle containing three given points).
(iv) Bisectors are simple curves that divide the plane in two unbounded regions.

Strictly convex distances produce Voronoi diagrams with very good properties, as stated in [8]. Summarizing, if S is a finite collection of points then:

(i) $R_{S,d}(P)$ is an open and not empty subset of the plane and $R_{S,d}(P) = \{X \in \mathbb{R}^2 : d(X,P) < d(X,Q), \forall Q \in S - \{P\}\}$.
(ii) If $X \in R_{S,d}(P)$, then the whole closed segment $[P,X]$ is contained in $R_{S,d}(P)$.
(iii) $Cl\, R_{S,d}(P) = \{X \in \mathbb{R}^2 : d(X,P) \leq d(X,Q)$, for every $Q \in S - \{P\}\}$, where $Cl\, R_{S,d}(P)$ denotes the topological closure of $R_{S,d}(P)$.
(iv) $\bigcup_{P \in S} Cl\, R_{S,d}(P) = \mathbb{R}^2$.

As a consequence the Voronoi diagram for any finite collection S of points induces a finite cellular decomposition of the plane in which the 2-dimensional cells are the Voronoi regions, and the 1-dimensional and 0-dimensional cells are, respectively, the edges and vertices of the diagram.

Given a Voronoi diagram $Vor_d(S)$, with d a strictly convex distance, its dual is called the *Delaunay diagram of S*, $Del_d(S)$. The Delaunay diagram is the imbedded graph whose vertices are the given collection S of points and having an edge between every two points whose corresponding Voronoi regions share an edge.

The polygons that appear as *regions* in the Delaunay diagram are characterized by the fact that there exists a certain d-circle passing through all the vertices of the polygon and having the rest of points of S outside. This implies that the polygons are *strictly convex* (they are convex and do not have three collinear vertices). In a similar way, an edge appears joining two points P and Q of the Delaunay diagram if and only if there exists a d-circle passing through P and Q and with all the other points of S outside. If S does not contain four d-cocircular points, the Delaunay diagram is a triangulation, known as the *Delaunay triangulation*.

The topological information of a Voronoi diagram is not lost in passing to the dual Delaunay diagram and Delaunay diagrams are much easier to handle, specially when our distance is not the Euclidean one, as the edges are now segments instead of bisectors. For this reason in the following examples we shall work with Delaunay diagrams.

In order to study the topological shapes of Voronoi diagrams coming from a strictly convex distance d, we can simply construct all the possible shapes of imbedded graphs with strictly convex regions and then find out, for each of these graphs, whether it is *realizable* as a Delaunay diagram. Figure 1 shows all the shapes of imbedded graphs with convex faces up to four vertices, with the corresponding shapes for their dual Voronoi diagrams, which are shown in soft lines.

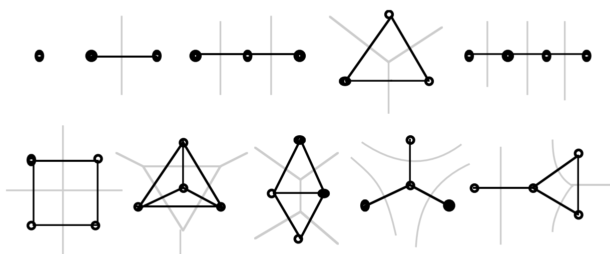

Figure 1

The first eight diagrams in Figure 1 are all of them easily realizable by some Euclidean Delaunay diagrams and, in fact, by Delaunay diagrams with any strictly convex distance. On the contrary the two last ones, with four points each, are not realizable by Euclidean diagrams. The reason is that Euclidean Delaunay diagrams have convex contour, i.e. the edges in the convex hull of S are edges of the Delaunay diagram. This is so because for every two consecutive points in the convex hull, a sufficiently large Euclidean circle passing through them can be found not containing any other point of S.

Moreover, this happens not only for the Euclidean distance, but for every *smooth* strictly convex distance. We conclude that no Delaunay diagram for a smooth distance has the shape of the two last diagrams in Figure 1.

Nevertheless, this is no longer true for non smooth distances. Figure 2 shows two unit balls which realize, respectively, the two last diagrams in Figure 1. In the two cases the realization is possible because of the existence of groups of three points which are neither collinear nor cocircular, for the distance we consider.

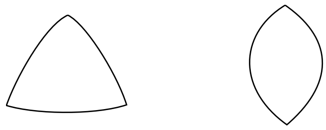

Figure 2

One can then think that only non smooth distances produce topological shapes different than the Euclidean ones for the Delaunay and Voronoi diagrams. In fact, up to six points, all possible shapes of imbedded graphs with convex contour and strictly convex polygons have been exhaustively explored resulting that all of them have the shape of some Euclidean Delaunay diagram. With seven points several shapes exist which cannot be realized by any Euclidean Delaunay diagram; two of them are shown in Figure 3.

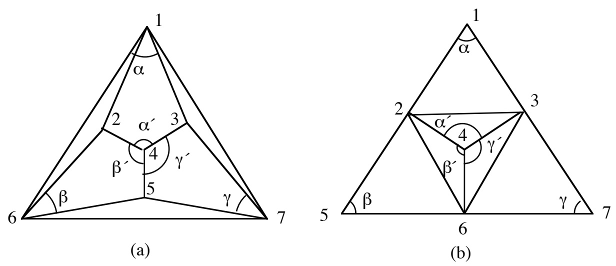

Figure 3

If the diagram in Figure 3(a) was an Euclidean Delaunay diagram, then the points 1, 2, 3 and 4 would lie in an Euclidean circle and thus the sum of the angles α and α' would equal to 180^o. The same thing would happen to polygons (**2456**) and (**3457**), and we would have that $\alpha+\beta+\gamma+\alpha'+\beta'+\gamma' = 3\times 180^o$. But $\alpha'+\beta'+\gamma' = 360^o$ and so $\alpha+\beta+\gamma = 180^o$. This is not possible, because the sum of the three angles of the triangle (**167**) is also 180^o, and $\alpha+\beta+\gamma$ is clearly smaller.

In Figure 3(b) we must have $\alpha+\alpha' < 180^o$, as point 4 must be outside the Euclidean circle passing through 1, 2 and 3. With similar arguments we conclude $\alpha+\beta+\gamma+\alpha'+\beta'+\gamma' < 3\times 180^o$ and then $\alpha+\beta+\gamma < 180^o$. This is again impossible as the contour of the diagram (the hexagon (**125673**)) is convex.

The question is now whether these shapes can be realized by Delaunay diagrams for some smooth strictly convex distance or not. The answer is affirmative as indicated in Figure 4 for the diagram in Figure 3(a). The sixteen points in the left part of Figure 4 form a strictly convex, symmetrical polygon, and thus there exists a symmetrical, smooth and convex closed curve passing through all of them. If we take as unit d-circle any such curve, then the right part of the figure is actually a Delaunay diagram for the induced distance d: to see this, note that polygons (**1234**), (**2456**) and (**3457**) have their vertices in the d-circles with centers $(0,9)$, $(-8,3)$ and $(8,3)$ and radii $8/17$, 1 and 1, respectively. A similar construction can be made for the diagram in Figure 3(b).

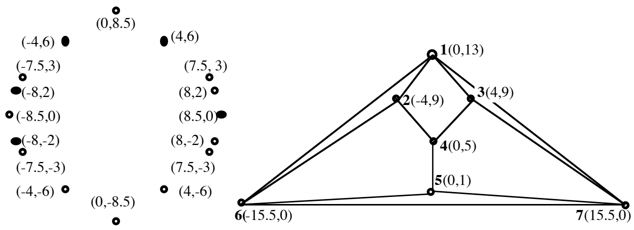

Figure 4

We do not know either if every diagram with convex contour and strictly convex regions has the shape of a Delaunay diagram for some strictly convex distance, nor if for every strictly convex distance there exists a seven-point Delaunay diagram with non Euclidean shape. Nevertheless, in the proof of Theorem 3 we will see that any strictly convex distance, provided it is not affinely equivalent to the Euclidean distance, produces some nine-point Delaunay diagram with one of the shapes shown in Figure 5. The proof that no Euclidean Delaunay diagram has these shapes is similar to the proofs made for the diagrams in Figure 3, and left to the reader.

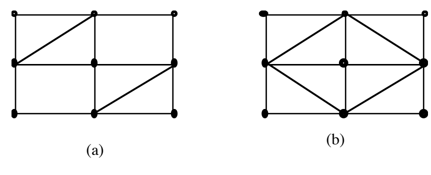

Figure 5

3. Proofs of Theorems

Proof of Theorem 1.(i). To prove that f is affine it suffices to see that it sends collinear points to collinear points. By the Fundamental Theorem of Affine Geometry any bijection from the plane onto itself preserving collinearity is affine.

Let P, Q and R be three collinear points and, without loss of generality, suppose that R is in the segment

$[P,Q]$. Then, by the strong triangle inequality on d, $d(P,Q) = d(P,R) + d(R,Q)$. Thus, $\delta(f(P), f(Q)) = \delta(f(P), f(R)) + \delta(f(R), f(Q))$ and by the strong triangle inequality on δ, $f(P)$, $f(Q)$ and $f(R)$ are collinear.

It remains only to study how f trasforms Voronoi diagrams. We recall the definition of Voronoi regions:

$$R_{S,d}(P) = \{X \in R^2 \colon d(X,P) < d(X,Q), \forall Q \in S - \{P\}\}$$

Now:

$$d(X,P) < d(X,Q) \iff \delta(f(X), f(P)) < \delta(f(X), f(Q))$$

and then $X \in R_{S,d}(P) \iff f(X) \in R_{f(S),\delta}(f(P))$. We conclude that $f(R_{S,d}(P)) = R_{f(S),\delta}(f(P))$, i.e. that f preserves the Voronoi regions. As f is bijective, the same happens for the union of the regions and so also for the Voronoi diagrams. ∎

The proof of Theorem 1.(ii) is based in the following sequence of lemmas:

Lemma 1. In the hypothesis of (ii), for every d-circle C_d with center at X, its image $f(C_d)$ is a δ-circle with center at $f(X)$.
Proof. Let C_d be a d-circle with center at a point X and let us prove that $f(C_d)$ is contained in a certain δ-circle C_δ with center $f(X)$. By a similar argument we would prove that $f^{-1}(C_\delta)$ is contained in a certain d-circle C'_d with center at $f^{-1}(f(X)) = X$ and necessarily $C'_d = C_d$ and so $f(C_d) = C_\delta$, and the lemma holds.

Let P, Q and R be three points in C_d. Then the distance from X to any of them is the same and X is the only point having this property, because any point having this property would be the center of another d-circle intersecting C_d in at least the three points P, Q and R. Then,

$$Bi_d(P,Q) \cap Bi_d(Q,R) = \{X\}.$$

Now, f being bijective and preserving bisectors implies that

$$f(Bi_d(P,Q)) \cap f(Bi_d(Q,R)) = \{f(X)\}, \qquad \text{and}$$

$$Bi_\delta(f(P), f(Q)) \cap Bi_\delta(f(Q), f(R)) = \{f(X)\},$$

so $f(P)$, $f(Q)$ and $f(R)$ lie in a certain δ-circle C_δ with center at $f(X)$.

To prove that for any other $T \in C_d$, $f(T)$ also lies in C_δ, it suffices to make the same considerations for the points P, Q and T and conclude that they lie in a certain δ-circle with center at $f(X)$. This circle must be the same C_δ because it has the same center $f(X)$ and the same radius $\delta(f(X), f(P))$. ∎

Lemma 2. In the hypothesis of (ii), if R is the midpoint of P and Q, then $f(R)$ is the midpoint of $f(P)$ and $f(Q)$.
Proof. The midpoint R of a segment $[P,Q]$ is the only point of intersection of $Bi_d(P,Q)$ with the d-circle with center P and radius $r = d(P,Q)/2$. As we already know that f transforms d-circles to δ-circles and d-bisectors to δ-bisectors, it follows that $f(R)$ is the only point of intersection of $Bi_\delta(f(P), f(Q))$ with a certain δ-circle C_δ centered at $f(P)$. In these conditions, the radius of C_δ must be $\delta(f(P), f(Q))/2$ and thus $f(R)$ is the midpoint of $f(P)$ and $f(Q)$. ∎

Lemma 3. In the hypothesis of (ii), f is an homeomorphism.
Proof. We know by Lemma 1 that f sends each d-circle C_d to a δ-circle $C_\delta = f(C_d)$. Let us see now that it also sends the region B_d bounded by C_d to the region B_δ bounded by C_δ. Let O be the center of C_d and r its radius. Consider a point P with $d(O,P) = 2r$ and let R be the midpoint of O and P, which is on C_d. As f preserves midpoints, the point $f(R)$, which lies on C_δ, is the midpoint of $f(O)$ and $f(P)$.

Any point X in B_d belongs to a d-circle centered in O and with radius smaller than r, and thus not intersecting the bisector $Bi_d(O,P)$. Its image $f(X)$ must be then in a certain δ-circle centered in $f(O)$ and not intersecting the bisector $Bi_\delta(f(O), f(P))$. So $\delta(f(O), f(X)) < \delta(f(O), f(P))/2 = \delta(f(O), f(R))$, concluding that $f(X)$ belongs to B_δ.

As the families of B_d's and B_δ's are both basis for the usual topology, f is an homeomorphism. ∎

Lemma 4. In the hypothesis of (ii), f sends collinear points to collinear points.
Proof. Let P, Q and R be three collinear points and, without loss of generality, suppose that Q is between P and R. We can construct a sequence of "midpoints" in segment $[P,R]$ having Q as limit: let X_1 be the midpoint of P and R, X_2 the midpoint of the halfsegment in which Q is contained and so on. As f preserves midpoints, the image sequence $f(X_n)$ is contained in segment $[f(P), f(R)]$ and, being f continuous, $f(X_n)$ has $f(Q)$ as limit and $f(Q)$ belongs to the segment $[f(P), f(R)]$. ∎

Proof of Theorem 1.(ii). Firstly, f is affine because any bijection in the plane sending collinear points to collinear points is affine, by the Fundamental Theorem of Affine Geometry.

It remains only to prove that $d = k\,\delta \circ f$, for some constant $k > 0$.

Let P and R be any two points and let us compute $\delta(f(P), f(R))$. For that, let C_d be the d-circle centered at P with radius 1 and let Q be the unique point of intersection of the half line from P through R with C_d. By definition of d we have $d(P,R) = ||P-R||/||P-Q||$.

By Lemma 1, $f(C_d)$ is some δ-circle centered at $f(P)$ and radius $r = \delta(f(P), f(Q))$.

To measure $\delta(f(P), f(R))$, note that f being affine implies:

$$\frac{||P-R||}{||P-Q||} = \frac{||f(P)-f(R)||}{||f(P)-f(Q)||},$$

and, by definition of δ we have:

$$\frac{||f(P)-f(R)||}{||f(P)-f(Q)||} = \frac{\delta(f(P), f(R))}{\delta(f(P), f(Q))}$$

So $\delta(f(P), f(R)) = \delta(f(P), f(Q))\frac{||P-R||}{||P-Q||} = r \cdot d(P, R)$, where we have used that:

$$r = \delta(f(P), f(Q)) \text{ and } d(P, R) = ||P-R||/||P-Q||.$$

Thus, $\delta \circ f = r \cdot d$ and $d = k \cdot \delta \circ f$, with $k = 1/r$.

Finally, $k \cdot \delta \circ f = \delta \circ (k \cdot f)$ (any normed distance commutes with homotecies) and we are in the conditions of (i). ∎

Proof of Theorem 2. First we are going to see that f sends collinear points to collinear points, and thus f is affine. Let P, Q and R be three collinear points. If $f(P)$, $f(Q)$ and $f(R)$ were not collinear, they would be δ-cocircular (because of δ being smooth, recall property (ii) of strictly convex distances) and the center O' of a δ-circle passing through them would belong to the closures of the three regions in $Vor_\delta(\{f(P), f(Q), f(R)\}$ (by property (iii) of Voronoi diagrams).

Now, $Vor_\delta(\{f(P), f(Q), f(R)\})$ has the same topological shape than $Vor_d(\{P, Q, R\})$ and thus there exists some common point O in the closures of the three Voronoi regions in $Vor_d(\{P, Q, R\})$. This implies that O has the same distance to P, Q and R, and thus there is a d-circle passing through P, Q and R, which is impossible because they are collinear.

Now let us see that f sends any d-circle with center at O to a δ-circle with center at $f(O)$.

Let C_d be any d-circle and O its center. For every finite collection S of three or four points in C_d, O belongs to the closure of each of the Voronoi regions in $Vor_d(S)$ (again by property (iii) of Voronoi diagrams). As f preserves the topological shape of Voronoi diagrams of at most four points, there must exist a point O' in the closure of each of the Voronoi regions of $Vor_\delta(f(S))$. Then, all the points in $f(S)$ are in a certain δ-circle C_S with center at O'. This C_S is uniquely determined by S, because two δ-circles cannot have three common points.

Now, if $S \subset S'$, (i.e. if $S = \{P, Q, R\}$ and $S' = \{P, Q, R, T\}$), then clearly $C_S = C_{S'}$, for they intersect in $f(S)$. For arbitrary S and S', say $S = \{P, Q, R\}$ and $S' = \{P', Q', R'\}$, we can consider the intermediate sets:

$$S_1 = \{P, P', Q, R\}, S_2 = \{P', Q, R\}, S_3 = \{P', Q', Q, R\},$$

$$S_4 = \{P', Q', R\} \quad \text{and} \quad S_5 = \{P', Q', R', R\},$$

and conclude that:

$$C_S = C_{S_1} = C_{S_2} = C_{S_3} = C_{S_4} = C_{S_5} = C_{S'}.$$

So, C_S is the same for every chosen S and we can call it simply C_δ. We conclude that $f(C_d) \subset C_\delta$, where C_δ is some δ-circle and a similar argument (applied to f^{-1}) proves the converse; thus $f(C_d) = C_\delta$. Moreover, f being affine, the center O' of C_δ must coincide with $f(O)$, because affine maps preserve centers of symmetry.

Once we know that f is affine and that it sends d-circles to δ-circles we can finish the proof of $d = k \cdot \delta \circ f$ as in Theorem 1.(ii). ∎

Proof of Theorem 3. The reasoning will be made with the Delaunay diagrams, rather than with the Voronoi ones; due to the considerations made in Section 2, a Voronoi diagram has the same topological shape of an Euclidean Voronoi diagram if and only if its dual Delaunay diagram has the shape of an Euclidean Delaunay diagram.

Let C be the unit d-circle of the given distance d. C is a closed curve, with strictly convex interior and symmetrical respect to the origin. We give the following lemma on C, whose proof we do not reproduce for it is quite long and uses analytical methods having nothing to do with Voronoi diagrams.

Lemma 5. Let C be any closed curve, symmetrical with respect to the origin and with convex interior. Then there exists a certain ellipse E centered at the origin containing C in its inside and intersecting C in at least two pairs of opposite points (see Figure 6). ∎

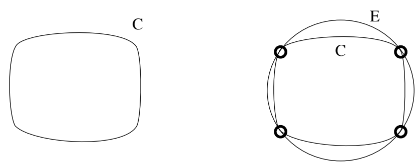

Figure 6

The lemma says, roughly speaking, that there exists an ellipse *circumscribed* to C by four points. For the sake of simplicity we can suppose, moreover, that the ellipse E is actually a circle; this produces no loss of generality because, by theorems 1 and 2, an affine bijective transformation to distance d makes no changes in the topological shapes obtained for Voronoi diagrams.

Now, the two pairs of opposite points in which E and C coincide, E being a circle, are the vertices of a rectangle in the plane and we can form a nine-point figure with four copies of this rectangle, as indicated in Figure 7(a). Clearly the Delaunay diagram of these nine points, both for the Euclidean distance and for the distance d, consists on the four rectangles **1245**, **2365**, **5698** and **4587** because the vertices of each rectangle lie both in a circle and in a scaled translation of C (which is a d-circle).

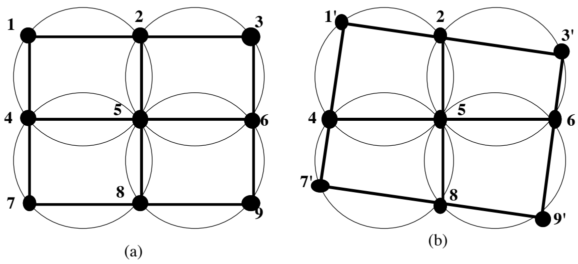

Figure 7

We are now going to move a little the four vertices of the bigger rectangle (i.e. the points **1**, **3**, **7** and **9** in the figure) in order to change the shape of the Delaunay diagram: note that if we move one of these vertices, say **1**, along the circle passing by it, the other three vertices can be moved accordingly in their respective circles, in such a way that the contour (**1'23'69'87'4**) is still a rectangle with the four points points **2**, **4**, **6** and **8** in its sides (as in Figure 7(b)).

Using this property we can move the four vertices **1**, **3**, **7** and **9** to a position in which at least one of them (say **1'**) does not lie on the corresponding d-circle (this is possible because, by hypothesis, the d-circles are not circles). In these conditions, point **1'** must be exterior to the d-circle passing through vertices **2**, **4** and **5** (for the d-circle is "inscribed in the circle"), and this makes the segment joining points **2** and **4** to appear as a new edge in the Delaunay diagram of the nine points for the distance d.

By symmetry, the same thing occurs at the point **9'** opposite to **1'**. For the other two vertices **3'** and **7'**, two possibilities may happen: either they lie on their corresponding d-circles or are exterior to them. In the first case no more edges appear and the Delaunay diagram is the one in Figure 5(a), and in the second case the edges **26** and **48** are also in the Delaunay diagram, and it has the shape of Figure 5(b). Neither 5(a) nor 5(b) have the shape of any Euclidean Delaunay diagram, so the proof is complete. ∎

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