Correction of errors to

An effective version of Pólya's theorem on positive definite forms

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The statement and proof of part 2 of Theorem 1.2 are both incorrect. In the original statement there is a typographical error. We meant to write $(lnd^2/\lambda) + dn$ instead of $(\ln d^2 + dn)/\lambda$. In the proof, there are two inequalities written in the wrong direction. We thank Victoria Powers and Dionne Bailey for pointing out these errors to us. The following is the corrected statement and a new proof. The proof provides the same bound for the Polya exponent that we included in the original article except for a factor of 2. Recently, using different techniques, Bruce Reznick has proved a new bound of $0.684\, lnd^2/(\lambda \ln(n))$ (personal communication).

Theorem 1.2 *Let $F(x_1, \dots, x_n)$ be a real homogeneous polynomial of degree d whose coefficients are bounded in absolute value by $l \geq 2$. Suppose that F is strictly positive in the non-negative orthant (minus the origin). Denote by λ the minimum of F in the unit simplex $\Delta = \{\sum_{i=1}^n x_i = 1, \quad x_i \geq 0 \quad \forall i\}$. Under these assumptions:*

1. *If F has integer coefficients, then $1/\lambda$ is bounded above by $l^{D^{O(n)}}$.*
2. *For any p greater than $(2lnd^2/\lambda) + dn$, the product $F(x_1, \dots, x_n) \cdot (x_1 + \cdots + x_n)^p$ has all its coefficients strictly positive.*

Proof: The proof of part one remains as before.

Now we present the proof of the second part: By Lemma 2.1, what we need to prove is that $2lnd^2/\lambda$ is an upper bound for the functional $\sum X$ on the semialgebraic region $G := \{X : F_+(X) - F_-(X + d) \leq 0, X \geq 0\}$. Here $X := (x_1, \dots, x_d)$, $\sum X := x_1 + \cdots + x_n$, $X + d := (x_1 + d, \cdots, x_n + d)$ and $F(X)$ is decomposed as $F_+(X) - F_-(X)$ where F_+ and F_- have only positive coefficients and no monomials in common.

We divide the region G into $G' := G \cap \{X : \sum X \geq nd(d-1)\}$ and $G'' := G \cap \{X : \sum X \leq nd(d-1)\}$. Obviously, $nd(d-1)$ is an upper bound for $\sum X$ in G''. We have $nd(d-1) < 2nd^2 \leq 2lnd^2/\lambda$ since $\lambda \leq F(1, 0, \dots, 0) \leq l$.

For G', we observe that the inequality $F_+(X) - F_-(X + d) \leq 0$ is equivalent to

$$1 \leq \frac{F_-(X + d) - F_-(X)}{F_+(X) - F_-(X)} = \frac{F_-(X + d) - F_-(X)}{F(X)}.$$

Thus, we can rewrite the region G' as

$$G' = \left\{ X : nd(d-1) \leq \sum X \leq \frac{(\sum X)(F_-(X+d) - F_-(X))}{F(X)}, X \geq 0 \right\}$$

We claim that $F_-(X+d) - F_-(X) \leq l(\sum(X+d))^d - l(\sum X)^d$ in the positive orthant, where $\sum(X+d)$ means the sum of coordinates of $(x_1+d, \ldots, x_n+d)$, which equals $\sum X + nd$. For this observe that $l(\sum X)^d - F_-(X)$ is a homogeneous polynomial of degree d with non-negative coefficients and, hence, it is monotonically increasing with respect to all the indeterminates. Using this inequality, at every point $X \in G'$ we have that

$$\sum X \leq \frac{l(\sum X)\left((\sum X + nd)^d - (\sum X)^d\right)}{F(X)}.$$

In the right hand side of this expression, the numerator is a polynomial of degree at most d with positive coefficients and the denominator is a homogeneous polynomial of degree d. Hence, the expression is monotonically decreasing with respect to every coordinate in the positive orthant and it achieves its maximum over G' in the simplex $(nd(d-1))\Delta = \{X : \sum X = nd(d-1), X \geq 0\}$. But in this simplex we have $\sum X = nd(d-1)$, $(\sum X + nd)^d - (\sum X)^d = (nd(d-1) + nd)^d - (nd(d-1))^d = (nd(d-1))^d((1+\frac{1}{d-1})^d - 1)$, and $F(X) \geq (nd(d-1))^d\lambda$. Hence, on G' we have

$$\sum X \leq \frac{nd(d-1)l}{\lambda}\left(\left(1+\frac{1}{d-1}\right)^d - 1\right) = \frac{nd^2 l}{\lambda}\left(\left(1+\frac{1}{d-1}\right)^{d-1} - \frac{d-1}{d}\right).$$

This gives the statement since $\left(1+\frac{1}{d-1}\right)^{d-1} - \frac{d-1}{d} \leq 2$ for any integer $d \geq 2$: it equals $3/2$ and $9/4 - 2/3$ for $d=2$ and $d=3$ and it is bounded by $e - 3/4 = 1.968$ for $d \geq 4$. $\square$