

On the number of facets of three-dimensional Dirichlet stereohedra I: Groups with reflections

Daciana Bochis*, Francisco Santos†

Departamento de Matemáticas, Estadística y Computación,
Universidad de Cantabria, Santander, Spain

March 14, 2000

Abstract

A Dirichlet stereohedron for a crystallographic group G is the Voronoi region of any point with trivial stabiliser in the Voronoi diagram of its orbit.

We prove that Dirichlet stereohedra for three-dimensional crystallographic groups containing reflections in three, two or one independent directions cannot have more than eight, eighteen and fifteen facets respectively. We show examples where the three bounds are attained.

As a tool for one of the cases, we study how many regions of a Dirichlet tessellation in the plane can be intersected by each region of another Dirichlet tessellation for the same group. We find out that the number is at most seven for all planar crystallographic groups except, perhaps, those of type pgg .

Introduction

This paper deals with the combinatorial types of Dirichlet stereohedra in three dimensions. More precisely, we find the exact maximum number of facets of three-dimensional Dirichlet stereohedra corresponding to groups which contain reflections.

A *discrete group of motions* in \mathbb{R}^d is a group of Euclidean isometries whose orbits are discrete. If it contains d -independent translations then it is a *crystallographic group*. Equivalently, a discrete group G is crystallographic if it admits a compact fundamental domain. A *fundamental domain* for G is any closed connected region with non empty interior and which covers \mathbb{R}^d by the action of G without gaps or overlaps.

A special manner to find fundamental domains for a crystallographic group is to compute the Voronoi diagram of the orbit of a point with trivial stabiliser. The (closed) regions in the Voronoi diagram are fundamental domains and they are called *Dirichlet domains*. In the theory of tessellations, a *stereohedron* is any bounded convex polyhedron which tiles the space by the action of a crystallographic group. Dirichlet domains are stereohedra and, hence, they are also called *Dirichlet stereohedra*.

The study of the combinatorial types of stereohedra is related to Hilbert's 18th problem, "*Building up the space with congruent polyhedra*" (see [11], [15]). Although Bieberbach (1910) and Reinhardt (1932) gave complete answers to the first two of

*dacib@matesco.unican.es

†santos@matesco.unican.es. Partially supported by Grant PB97-0358 of the Spanish Dirección General de Investigación Científica y Técnica.

Hilbert’s specific questions, other problems related to *monohedral tessellations* (i.e. tessellations whose tiles are congruent, such as those which appear in Hilbert’s 18th problem) remain open. An exhaustive account of them appeared in a survey article by Grünbaum and Shephard [9]. In particular, they ask to *determine the maximum number of $(d - 1)$ -dimensional faces (i.e. facets)—or, at least, a “good” upper bound—of convex polyhedral tiles of monohedral tessellations in \mathbb{R}^d* . This problem, for the case of stereohedral tessellations, is mentioned also in [18].

The only general result known concerning this is the *fundamental theorem of stereohedra* proved by Delone in 1961 [6]. It says that a stereohedron associated to a crystallographic group with a aspects in \mathbb{R}^d has at most $2^d(a + 1) - 2$ facets. Here, the number of *aspects* of a crystallographic group G is the index $[G : G_T]$ of its translational subgroup G_T . In other words, the number of translational lattices in which a generic orbit of G decomposes. As a corollary to our analysis of general properties of stereohedra in Section 1 we provide a proof of Delone’s theorem for the case of Dirichlet stereohedra (Theorem 1.8).

Starting with Fedorov, who studied some Dirichlet stereohedra with 14 facets in 1899, there have been several attempts to find three-dimensional stereohedra with “many” facets. Föppl, Novacki and Smith discovered Dirichlet stereohedra with 16, 18 and 20 facets, respectively, in 1916, 1935 and 1965 [7, 16, 19]. In 1972, Koch [12] found Dirichlet stereohedra with 23 facets and later together with Fischer [13] he produced some with 24 facets. With the data available at that time and some heuristic arguments, in 1978 Brunner and Laves conjectured 26 to be an upper bound for the number of facets of three-dimensional Dirichlet stereohedra and $3^d - 1$ for the d -dimensional case.

This conjecture was found to be false two years later when Engel, with computer-assisted techniques, found several families of Dirichlet stereohedra with facets from 17 to 38, for a cubic group with 24 aspects (see [8]). These are the stereohedra with the maximum number of facets known so far, while Delone’s upper bound in three dimensions is 390 since the maximum number of aspects of a three-dimensional crystallographic group is 48. There is an agreement among the experts that the maximum number of facets that a stereohedron can attain is closer to 38 than to 390 (see [8, page 214], [9, page 960], [18, page 50]).

Observe that in the constructions mentioned above only the particular case of Dirichlet stereohedra (also known as *plesiohedra*) is considered even if, as observed by Grünbaum and Shephard [9, page 965], “there seems to be no grounds to assume that all stereohedra are combinatorially equivalent to plesiohedra (though no example settling this question is known)”. The reason for this restriction is the absence of practical methods for constructing non-Dirichlet stereohedra. As the same authors state, “there seems to be no practical technique available which would be guaranteed to produce—if only carried out with sufficient perseverance—all combinatorial types of stereohedra”. Algorithms for the same task restricted to Dirichlet stereohedra are easy to design, since the existence of a particular combinatorial type for a particular affine equivalence class of crystallographic groups is a quantifier elimination problem over the reals. But the “perseverance” needed to accomplish this task seems beyond the capabilities of present computers.

In this paper and the subsequent ones [2] and [3] we intend to find a more realistic upper bound for the maximum number of facets of a three-dimensional Dirichlet stereohedron. The results we have obtained so far are: For groups with reflections, an optimal upper bound of 18 facets (this paper). For groups without reflections, upper bounds of 102 for non-cubic groups and 162 for the cubic ones (these can be found in [1]).

The method we use for studying Dirichlet stereohedra for a group G with reflections is based on the use of “reflection cells” of G , defined as the fundamental

domains of the reflection subgroup of G (or, equivalently, the regions in which the reflection planes of G divide the space). Every stereohedron for G is contained in a reflection cell (see Proposition 2.2). We first consider the number of facets that a Dirichlet stereohedron can have inside and on the boundary of the reflection cell, obtaining bounds of 16 and 6. Studying the dependence between the two types of facets, we get a global bound of 18.

Actually, we study separately the cases of groups having reflections in three, two or one independent directions (Sections 2.1 to 2.3). We get upper bounds of eight, eighteen and fifteen facets respectively (Theorems 2.4, 2.7 and 2.12) and examples where the three bounds are attained (Examples 2.5, 2.9 and 2.13). Summing up, our main result is:

Theorem *The exact maximum number of facets of a three-dimensional Dirichlet stereohedron corresponding to a group with reflections is 18.*

When studying the case of reflections in only one direction in Section 2.3 we will need to address the following problem: given two orbits GP and GQ of the same planar crystallographic group, how many regions of $Vor_{GP}(P)$ can intersect each region of $Vor_{GQ}(Q)$? Section 3 is devoted to study this problem. We prove that the number of regions is at most seven in all groups except perhaps those of type pgg , and that there is an explicit example where seven is attained, for the group $p2$. For pgg , we can only prove the bound of seven under the assumption that $GP \cup GQ$ is again an orbit of a crystallographic group, which is the case needed in Section 2.3.

All throughout the paper we use the International Crystallographic notation for planar and three-dimensional crystallographic groups; see e.g. [14].

1 Some general properties of Dirichlet stereohedra

For any discrete set of points $S \subset \mathbb{R}^d$, the (*open*) *Voronoi region* corresponding to a point $P \in S$, denoted $Vor_S(P)$, is the region of \mathbb{R}^d containing those points which are closer to P than to any other point of S , i.e.

$$Vor_S(P) = \{x \in \mathbb{R}^d : d(x, P) < d(x, Q), \forall Q \in S \setminus \{P\}\}.$$

The closure of $Vor_S(P)$ is a convex polyhedron and it can be described as

$$\overline{Vor_S(P)} = \{x \in \mathbb{R}^d : d(x, P) \leq d(x, Q), \forall Q \in S \setminus \{P\}\}.$$

Two points $P, Q \in S$ are called *neighbours* in the Voronoi diagram Vor_S if $\overline{Vor_S(P)}$ and $\overline{Vor_S(Q)}$ share a facet. Equivalently, if there exists a sphere \mathcal{C} which passes through P and Q and which does not contain any other point of S . Here are some straightforward properties of Voronoi diagrams that we will use:

Lemma 1.1 *Let S be a discrete set of points in \mathbb{R}^d .*

- (i) *Let S be partitioned in two parts S_1 and S_2 (i.e. $S_1 \cap S_2 = \emptyset$ and $S = S_1 \cup S_2$). If $P \in S_1$ and $Q \in S_2$ are neighbours in Vor_S , then $Vor_{S_1}(P) \cap Vor_{S_2}(Q) \neq \emptyset$.*
- (ii) *If $S' \subseteq S$ and $P \in S$, then $Vor_S(P) \subseteq Vor_{S'}(P)$.*
- (iii) *If $A, B, C, D \in S$ form a planar quadrilateral with vertices in that order, then:*

$$\angle A + \angle C \leq 180^\circ \Rightarrow A \text{ is not a neighbour of } C \text{ in } Vor_S.$$

□

The Voronoi diagram of an orbit of a crystallographic group is called a *Dirichlet tessellation*. Throughout the rest of this section G is a crystallographic group (of arbitrary dimension d) and GP is the orbit of a point P with trivial stabiliser. We are interested in the neighbours of P in Vor_{GP} . Each such neighbour corresponds to a facet of the Dirichlet stereohedron $Vor_{GP}(P)$. The following result implies that, for any $g \in G$, gP is a neighbour of P if and only if $g^{-1}P$ is also a neighbour.

Proposition 1.2 *Let $g \in G$. If f is the common facet of $Vor_{GP}(P)$ and $Vor_{GP}(gP)$, then $g^{-1}f$ is the common facet of $Vor_{GP}(P)$ and $Vor_{GP}(g^{-1}P)$. \square*

Our methods for bounding the number of neighbours of P in Vor_{GP} are based in decomposing GP via parallel subspaces (lines or planes) and bounding the number of neighbours in each of them separately.

Lemma 1.3 *Let α be an affine subspace and let $\beta \neq \alpha$ be a translate of α ; let $S_\alpha = GP \cap \alpha$ and $S_\beta = GP \cap \beta$. Let $\pi_\alpha : \mathbb{R}^d \rightarrow \alpha$ denote the orthogonal projection to α and let $S'_\alpha := \pi_\alpha(S_\beta)$. Let $P \in \alpha$ and $Q \in \beta$ be neighbours in Vor_{GP} . Then, $Vor_{S_\alpha}(P) \cap Vor_{S'_\alpha}(\pi_\alpha Q) \neq \emptyset$.*

Proof: Observe that S_β and S_α are disjoint. Hence, $Vor_{S_\beta}(Q) \cap Vor_{S_\alpha}(P) \neq \emptyset$ by parts (i) and (ii) of Lemma 1.1.

Let V denote the linear subspace orthogonal to α and β . Then $Vor_{S_\alpha}(P) = (Vor_{S_\alpha}(P) \cap \alpha) + V$ and $Vor_{S_\beta}(Q) = (Vor_{S_\beta}(Q) \cap \beta) + V = Vor_{S'_\alpha}(\pi_\alpha Q)$. This finishes the proof. \square

Lemma 1.4 *Let G_T be a crystallographic group generated by d linearly independent translations and $P \in \mathbb{R}^d$. Let $G_T^2 = \{t^2 : t \in G_T\}$ be the subgroup of G_T generated by translations of vectors double the vectors of G_T . Then the following equalities hold:*

$$\begin{aligned} & \{Q \in \mathbb{R}^d : Vor_{G_T P}(P) \cap Vor_{G_T Q}(Q) \neq \emptyset\} = \\ \stackrel{(1)}{=} & \{Q \in \mathbb{R}^d : \frac{P+Q}{2} \in Vor_{G_T P}(P) \cap Vor_{G_T Q}(Q)\} = \\ \stackrel{(2)}{=} & \{Q \in \mathbb{R}^d : \frac{P+Q}{2} \in Vor_{G_T P}(P)\} = \\ \stackrel{(3)}{=} & Vor_{G_T^2 P}(P). \end{aligned}$$

Proof: Let i_P be the inversion with center P (i_P^2 equal the identity and P is the unique fixed point of i_P). Then $i_P G_T P = G_T P$ and therefore $Vor_{G_T P}(P) = i_P Vor_{G_T P}(P) = i_{\frac{P+Q}{2}}(t_{PQ} Vor_{G_T P}(P))$, where t_{PQ} is the translation of vector PQ . Since $t_{PQ} Vor_{G_T P}(P) = Vor_{G_T Q}(Q)$, we have $Vor_{G_T P}(P) = i_{\frac{P+Q}{2}} Vor_{G_T Q}(Q)$. If the intersection $Vor_{G_T P}(P) \cap Vor_{G_T Q}(Q)$ is not empty, since it is convex, the point $\frac{P+Q}{2}$ has to belong to it and we are done with (1).

For the equality (2), it suffices to prove that $\frac{P+Q}{2}$ is in $Vor_{GP}(P)$ if and only if $\frac{P+Q}{2}$ is in $Vor_{GQ}(Q)$. This holds since $d(\frac{P+Q}{2}, P) \leq d(\frac{P+Q}{2}, tP)$, $\forall t \in G_T \Leftrightarrow d(\frac{P+Q}{2}, Q) \leq d(\frac{P+Q}{2}, -tQ)$, $\forall t \in G_T \Leftrightarrow d(\frac{P+Q}{2}, Q) \leq d(\frac{P+Q}{2}, tQ)$, $\forall t \in G_T$.

For (3), let Q be such that $\frac{P+Q}{2} \in Vor_{G_T P}(P) \Leftrightarrow d(\frac{P+Q}{2}, P) \leq d(\frac{P+Q}{2}, tP) \Leftrightarrow d(P, Q) \leq d(Q, 2tP)$, $\forall t \in G_T \Leftrightarrow Q \in Vor_{G_T^2 P}(P)$. \square

Note that the Voronoi region $Vor_{G_T^2}(P)$ is the dilation of factor 2 of the region $Vor_{G_T}(P)$ and therefore the open Voronoi region $Vor_{G_T^2}(P)$ contains at most 2^d points of any orbit GQ of G . With this and Lemma 1.4 we have:

Lemma 1.5 *Let G_T be a crystallographic group generated by d linearly independent translations and let $P, Q \in \mathbb{R}^d$. Then $Vor_{G_T P}(P)$ intersects at most 2^d regions of $Vor_{G_T Q}$.* \square

Corollary 1.6 *Let G be a crystallographic group with a aspects in \mathbb{R}^d . Let GP and GQ be two different orbits. Then, the Voronoi region $Vor_{GP}(P)$ intersects at most $2^d a$ Voronoi regions of Vor_{GQ} .* \square

Proposition 1.7 *Let G be a crystallographic group. Let α be an affine subspace of \mathbb{R}^d of a certain dimension k . Let G_0 be the subgroup of G which fixes all the affine subspaces that are translates of α , i.e. $G_0 = \{g \in G : g(t\alpha) = t\alpha, \forall \text{ translation } t\}$. Suppose moreover that G_0 is a crystallographic group, regarded as a group of motions in α (i.e., that G_0 contains k independent translations).*

Let $P \in \alpha$ be a point with trivial stabiliser under G and let $\beta \neq \alpha$ be a translate of α . Then, P has at most $2^k a_0$ neighbours on β in the Voronoi diagram Vor_{G_P} , where a_0 is the number of aspects of G_0 .

Proof: Let π be the orthogonal projection to the subspace α . By Lemma 1.3, if $Q \in \beta$ is a neighbour of $P \in \alpha$, then $Vor_{G_0 P}(P)$ intersects $Vor_{G_0 Q'}(Q')$, where $Q' = \pi_\alpha Q$. For each aspect of G_0 there are at most 2^k points $Q \in \beta$ such that $Vor_{G_0 Q'}(Q')$ intersects $Vor_{G_0 P}(P)$, by Lemma 1.5. \square

Although we will not need this result, the properties we have mentioned so far lead to a short proof of Delone's fundamental theorem of stereohedra for the particular case of Dirichlet stereohedra:

Theorem 1.8 (Delone) *Let G be a crystallographic group in \mathbb{R}^d with a aspects. Then, any Dirichlet stereohedron for the group G has at most $2^d(a+1) - 2$ facets.*

Proof: Let P be a point with trivial stabiliser, let GP be its orbit and $Vor_{GP}(P)$ the Dirichlet stereohedron we want to study. Let G_T be the translational subgroup of G . We decompose $GP = G_T P_1 \cup \dots \cup G_T P_a$, with $P_1 = P$. Then, $Vor_{GP}(P) \subset Vor_{G_T P}(P)$ and $Vor_{G_T P}(P)$ intersects at most 2^d regions in each $Vor_{G_T P_i}(P_i)$ for $i \neq 1$, which implies that P has at most $2^d(a-1)$ neighbours contained in $GP \setminus G_T P$.

Let G_T^2 be the subgroup of G_T consisting of translations of G_T doubled, as in Lemma 1.4. No neighbour of P can be in $G_T^2 P$ and each neighbour in $G_T P \setminus G_T^2 P$ will be in the relative interior of a facet of $Vor_{G_T^2}(P)$, by arguments similar to the ones in the proof of Lemma 1.4. Since G_T^2 has index 2^d in G_T , the "density" of points of $G_T P \setminus G_T^2 P$ in each region of $Vor_{G_T^2}$ is $2^d - 1$ points per region. We are interested in the points of $G_T P \setminus G_T^2 P$ which lie in the boundary of only two regions of $Vor_{G_T^2}$ and the number of those is at most $2(2^d - 1)$, as desired. \square

2 Dirichlet stereohedra for groups with reflections

Let G be a crystallographic group with reflections in dimension d , let G_r be the subgroup of G generated by its reflections. Note that G_r is not necessarily crystallographic, since it might not contain d linearly independent translations.

Definition 2.1 A reflection cell of G is the minimal region of \mathbb{R}^d with all facets lying in reflection hyperplanes. Equivalently, a reflection cell is a connected component of the complement of the union of all the reflection hyperplanes of G .

The reflection cells are all congruent to one another and tile the space, since they are Dirichlet domains of G_r . The reflection cells are bounded if and only if G has reflections in d independent directions. Equivalently, if G_r is crystallographic. No point in a reflection plane of the group G can lie in an open Voronoi region of Vor_{GP} . Hence, Vor_{GP} is a refinement of the tiling consisting of the reflection cells. Moreover:

Proposition 2.2 Let R be a reflection cell for G and $P \in R$. Let $S = GP \cap R$ be the set of points in the orbit GP of P inside R . Then,

$$Vor_{GP}(P) = R \cap Vor_S(P).$$

Proof: \subseteq Since $GP \supset S$ we have $Vor_{GP}(P) \subseteq Vor_S(P)$. In the same way $GP \supset G_r P$ and $R = Vor_{G_r P}(P)$ implies $Vor_{GP}(P) \subseteq R$. Therefore $Vor_{GP}(P) \subseteq Vor_S(P) \cap R$.

\supseteq Let $x \in R \cap Vor_S(P)$ and suppose $Q \in GP \setminus S$ (see Figure 1). Since there exists a reflection hyperplane l which separates x from Q , it follows that there is a point $Q' \in GP$ (obtained from Q by reflection in l) with $d(x, Q') < d(x, Q)$. If $d_0 = \min_{Q \in GP} \{d(x, Q)\}$ we have that $d_0 = \min_{Q \in S} \{d(x, Q)\} = d(x, P)$. Then, $d(x, P) < d(x, Q), \forall Q \in GP \Rightarrow x \in Vor_{GP}(P)$. \square

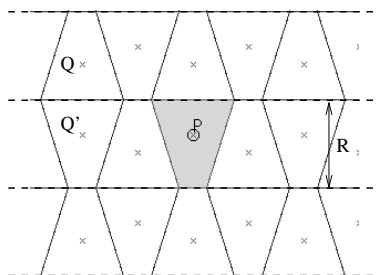


Figure 1: Proposition 2.2

Given a point P in a reflection cell R we call *external neighbours* (resp. *internal*) of P those neighbours of P which lie outside (resp. inside) R . By Proposition 2.2, the common facet between $Vor_{GP}(P)$ and the region of an external neighbour is part of a facet of R . Therefore P has at most as many external neighbours as the number of facets of R . Proposition 2.2 says also that the internal neighbours of P in $Vor_{GP}(P)$ are exactly the neighbours of P in $Vor_{GP \cap R}$ whose corresponding region intersects R .

Among the 219 affine equivalence classes of three-dimensional crystallographic groups, there are 100 with reflections. We will study separately the groups with reflections in three, two and one independent directions. By this we mean, respectively, that the normal vectors to the reflection planes span a 3, 2 or 1-dimensional space respectively.

2.1 Groups with reflections in three directions

There are 28 classes of groups with reflections in three directions: 4 in the orthorhombic system, 7 in the tetragonal system, 4 in the hexagonal system, and 13 in the cubic system. Let G be one of them.

Proposition 2.3 *Let P be a point with trivial stabiliser in G . Then, P has at most 6 external neighbours and at most 5 internal neighbours in Vor_{GP} .*

Proof: Every group generated by reflections is a direct product of groups whose fundamental domains are simplexes (see [5, page 190]). Therefore the reflection cells are products of simplices. The possibilities are a tetrahedron, a triangular prism and a rectangular prism, so that the maximal number of facets of the reflection cell is 6. The points of GP which are inside R are all at equal distance to the centroid of R , since every isometry $g \in G$ has $g(R) = R$. Hence, the points of $GP \cap R$ lie on a sphere. Their Voronoi diagram induces a monohedral tiling of this sphere and by Euler's formula each region in such a tiling has at most five edges. Hence, the number of internal neighbours of P is at most 5. \square

Theorem 2.4 *For any three-dimensional crystallographic group G with reflections in three independent directions and any point P with trivial stabiliser, the Dirichlet stereohedron $\text{Vor}_{GP}(P)$ has at most eight facets.*

Proof: Recall from the previous proof that the points of GP inside R lie on a sphere, centred at the centroid O of R . More precisely, they form an orbit of a finite discrete group of motions which we call G_O . Moreover, G_O is a subgroup of the symmetry group of the reflection cell R and it does not contain reflections. This implies that the only possibilities for G_O are (see [14, pages 124–130]):

1. A cyclic group consisting of rotations around a line. In this case P has only two internal neighbours.
2. A dihedral group generated by a rotation of order m ($m \in \{1, 2, 3, 4, 6\}$) around a line l and either an inversion (if m is odd) or a rotation of order two on a line perpendicular to l (for $m \geq 2$). Let us call 'vertical' the direction of l . If $m = 2$, P has at most three internal neighbours (since G_O has four elements). In any other case P has at most four internal neighbours: two in the horizontal plane containing P and two in another horizontal plane.

For the external neighbours, observe that R has to be a prism over either a square (and $m = 2$ or 4), a triangle (and $m = 3$) or a rectangle (and $m = 2$). $\text{Vor}_{GP}(P)$ is contained in an angular sector of $360/m$ degrees around l , which implies that it can be incident to at most two vertical facets of R in the cases where $m \neq 2$ and at most three in case $m = 2$. This gives at most four external neighbours for $m \neq 2$ and at most five for $m = 2$.

3. The group of direct symmetries of either a regular tetrahedron (12 elements) or a regular cube (24 elements). In both cases R has to be a regular cube, since a regular tetrahedron does not tile the space, and the group G_O contains the rotations of order three on the four diagonals l_1, l_2, l_3 and l_4 of the cube. Considering a particular diagonal l_i , $\text{Vor}_{GP}(P)$ has to be contained in an angular sector of 120 degrees at that diagonal. Such a sector intersects at most four facets of the cube (see Figure 2). More precisely, for each diagonal l_i of the cube there are two facets of the cube sharing an edge which does not meet l_i and which do not produce facets of $\text{Vor}_{GP}(P)$. This implies that at most three of the facets of the cube produce facets of $\text{Vor}_{GP}(P)$. Hence, P has at most three external neighbours. \square

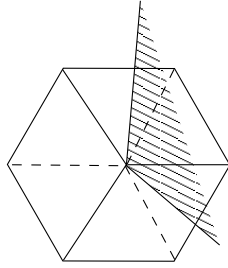


Figure 2: An angular sector of 120 degrees at a diagonal of a cube

Example 2.5 *A Dirichlet stereohedron with eight facets, for a group with three independent reflections.*

Consider the group $I_{\frac{4}{m}c\frac{2}{m}}$ generated by the reflections on the six facets of a square prism and the eight direct symmetries of the prism. Its reflection cell is the square prism and it has eight points of each generic orbit inside the reflection cell, forming the vertices of a square antiprism with bases parallel to those of the prism. In part (a) of Figure 3 the reflection cell is projected to a square and the vertices of the two square bases of the antiprism are drawn as black and white points respectively. The point P is labelled 1 and the others $2, \dots, 8$. Let us call vertical the direction of the prism and upper the plane containing P . Suppose first that the reflection cell is very long in height. Then P has seven neighbours: the four vertices adjacent to it in the skeleton of the antiprism (two in each square, labelled 2, 3, 7 and 8 in the figure) its reflection on the top facet of the reflection cell and its reflections on the two vertical facets of the reflection cell closer to P . See part (b) of Figure 3.

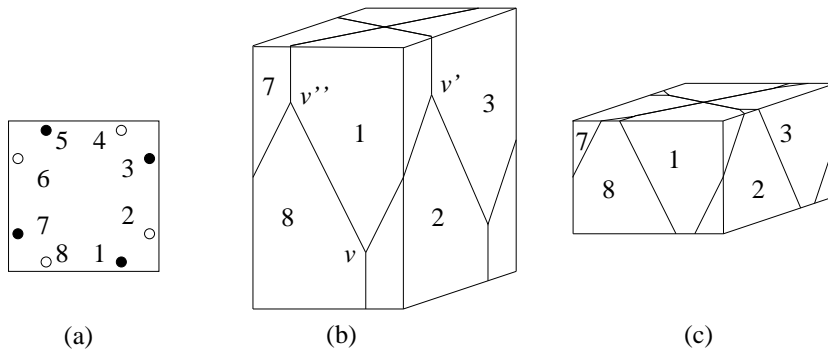


Figure 3: A Dirichlet stereohedron with 8 facets for a group with three independent reflections

Let O be the center of both the reflection cell and the antiprism with vertices in the orbit. Let v (resp. v' and v'') be the end-point of the edge 128 (resp. 123 and 178) in the boundary of the reflection cell. Clearly, the edges 123, 128 and 178 are all incident to O . This implies that v is the lowest point in the region of P . If we decrease the height of the reflection cell by moving its two square bases simultaneously towards the center, the first combinatorial change in the Voronoi region of P will occur when the bottom reflection plane cuts off the vertex v and the top reflection plane traverses the vertices v' and v'' . The change in v' and v'' does not affect the number of neighbours of P , but the change in v increases the number by one, since the bottom reflection plane creates now a facet in the Voronoi region. This is shown in part (c) of Figure 3, where the region 1 is still a neighbour

of the regions 2, 3, 7 and 8 and it has four facets on the boundary of the prism.

2.2 Groups with reflections in two directions

There are 40 groups with reflections in two directions: 10 in the orthorhombic system, 19 in the tetragonal system, 6 in the trigonal system and 5 in the hexagonal system. Let G be one of them and let R be one of its reflection cells. Let α be a plane perpendicular to all the reflection planes. The projection of R on the plane α is a bounded reflection cell of a two-dimensional crystallographic group, i.e. one of the five triangles or quadrilaterals shown in Figure 4. R is an infinite prism having as “base” one of the five polygons.

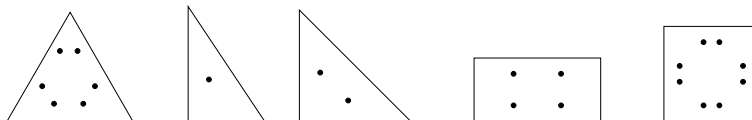


Figure 4: Bounded planar reflection cells

The points inside the prism R form a finite family of 1-dimensional lattices in the direction orthogonal to α , and the number of these lattices cannot exceed the number of symmetries of the planar projected reflection cell, since every element of G sending P to a point inside R fixes R . The maximum number of possible 1-dimensional lattices is shown by black dots in each case in Figure 4. The extremal case is that of a square, in which there are eight such dots.

Proposition 2.6 *Given a group G with reflections in two independent directions, any point P with trivial stabiliser has at most 16 internal neighbours and at most four external neighbours in Vor_{GP} .*

Proof: Since R has at most four facets, P has at most four external neighbours. By Proposition 1.7, P has at most two internal neighbours in each of the 1-dimensional lattices of GP inside R , hence it has at most 16 internal neighbours. \square

Theorem 2.7 *For any three-dimensional crystallographic group G with reflections in two independent directions and any point P with trivial stabiliser, the Dirichlet stereohedron $\text{Vor}_{GP}(P)$ has at most eighteen facets.*

Proof: We only need to deal with the case where the number of 1-dimensional lattices inside R is exactly eight, i.e. the case represented on the right-most picture of Figure 4. We will prove that one of the following two things occurs: either P has no neighbour in the 1-dimensional lattice containing it (in which case it has at most 14 internal neighbours) or P has at most two external neighbours.

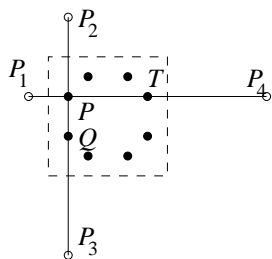


Figure 5: P has at most 16 neighbours

To prove this, suppose that P is a neighbour of one of the points in its 1-dimensional lattice. Hence, by Proposition 1.2 it is a neighbour of two points in that lattice. If we look at the Delaunay triangulation of GP , this implies that the line orthogonal to α and containing P consists of Delaunay edges. By the action of the symmetry group, the same occurs with all the other lines orthogonal to α containing points of GP . The end of the proof is illustrated in Figure 5. The points P_3 and P_4 , reflected images of P by the two facets of R further away from P , cannot be neighbours of P since this would produce Delaunay edges crossing two of the lines orthogonal to α . \square

For our next example we need the following result:

Lemma 2.8 *Let g be the screw rotation of order 8 around the vertical axis $x = y = 0$, with translation of length 1, i.e. $g(x, y, z) = (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}, z+1)$. Let $\langle g \rangle$ be the (infinite cyclic) group generated by g . Then, any point P not in the axis of g has sixteen neighbours in $\text{Vor}_{\langle g \rangle P}(P)$.*

Proof: Without loss of generality we can assume $P = (a, 0, 0)$, with $a > 0$.

Let us denote $P_i = g^i(P)$ for every $i \in \mathbb{Z}$. We will prove that P ($= P_0$) is a neighbour of the sixteen points P_i with $i \in \{-8, \dots, 8\}, i \neq 0$. It is clear that P is always a neighbour of P_1 since P and P_1 are equidistant to $(0, 0, 1/2)$ and closer to it than any other P_i . The same argument works for P_{-1} .

Consider now the following quadruples of points: $\{P_{-2}, P_{-1}, P_1, P_2\}$, $\{P_{-3}, P_{-2}, P_2, P_3\}$, $\{P_{-4}, P_{-3}, P_3, P_4\}$. By symmetry, the centres of spheres passing through them lie all in the X-axis. Their coordinates are:

$$O_1 = \left(\frac{-3}{\sqrt{2}a}, 0, 0 \right), \quad O_2 = \left(\frac{-5}{\sqrt{2}a}, 0, 0 \right) \quad \text{and} \quad O_3 = \left(\frac{-7}{(2-\sqrt{2})a}, 0, 0 \right).$$

For each quadruple, let us check that every P_i not in the quadruple is further from the sphere center than the ones in the quadruple. We only need to check this for the points $P_0 = (a, 0, 0)$, $P_1 = (a/\sqrt{2}, a/\sqrt{2}, 1)$, $P_2 = (0, a, 2)$, $P_3 = (-a/\sqrt{2}, a/\sqrt{2}, 3)$ and $P_4 = (-a, 0, 4)$. The ones with $i \in \{-4, -3, -2, -1\}$ are obtained by a rotation around the line $y = z = 0$ in which the three centres lie, and the rest of P_i 's are clearly further from that line. We have:

$$\begin{aligned} d(P_0, O_1)^2 - d(P_1, O_1)^2 &= 3\sqrt{2} - 4 & d(P_1, O_1)^2 &= d(P_2, O_1)^2 \\ d(P_3, O_1)^2 - d(P_1, O_1)^2 &= 2 & d(P_4, O_1)^2 - d(P_1, O_1)^2 &= 12 - 3\sqrt{2} \\ d(P_0, O_2)^2 - d(P_2, O_2)^2 &= 5\sqrt{2} - 4 & d(P_1, O_2)^2 - d(P_2, O_2)^2 &= 2 \\ d(P_2, O_2)^2 &= d(P_3, O_2)^2 & d(P_4, O_2)^2 - d(P_2, O_2)^2 &= 12 - 5\sqrt{2} \\ d(P_0, O_3)^2 - d(P_3, O_3)^2 &= \frac{16\sqrt{2}-4}{2-\sqrt{2}} & d(P_1, O_3)^2 - d(P_3, O_3)^2 &= \frac{22\sqrt{2}-16}{2-\sqrt{2}} \\ d(P_2, O_3)^2 - d(P_3, O_3)^2 &= \frac{12\sqrt{2}-10}{2-\sqrt{2}} & d(P_3, O_3)^2 &= d(P_4, O_3)^2 \end{aligned}$$

This implies that each pair of points in the same quadruple are neighbours in $\text{Vor}_{GP}(P)$. By symmetry, if P_i and P_j are neighbours, then P_{j-i} is a neighbour of P_0 . This finishes the proof. \square

Example 2.9 *A Dirichlet stereohedron with 18 facets, for a group with two independent reflections.*

Let G be a group of type $I \frac{4}{g} \frac{2}{m} \frac{2}{d}$ whose reflection cell is an infinite orthogonal prism over a square and generated by the following isometries:

1. Reflections on the four planes containing facets of the infinite prism. Assume without loss of generality that these planes are given by the equations $x = k$, $x = -k$, $y = k$ and $y = -k$ for a certain $k > 0$.

2. A screw rotation ρ of order four around the axis of the prism. We assume without loss of generality that ρ sends $(x, y, z) \mapsto (-y, x, z + 2)$.
3. A rotation σ of order two on a horizontal axis and which fixes the prism. Without loss of generality we take this axis to be $y = z = 0$, so that the rotation is $(x, y, z) \mapsto (x, -y, -z)$.

For any base point P in the reflection cell, the points in $GP \cap R$ are the point $Q = \sigma P$ obtained by the order-2 rotation and the points $\rho^i P$ and $\rho^i Q$ ($i \in \mathbb{Z}$) obtained by the screw rotation. We take as base point $P = (a \cos(\pi/8), a \sin(\pi/8), 1/2)$ for a certain $a > 0$. Then $\sigma(P) = (a \cos(-\pi/8), a \sin(-\pi/8), -1/2)$, which is the same point obtained by applying to P the inverse of the screw rotation g of Lemma 2.8. This implies that $GP \cap R$ is exactly the orbit $\langle g \rangle P$ which appears in Lemma 2.8 (we are using that $g^2 = \rho$). The Lemma implies that taking k sufficiently large, the Dirichlet region of P will have 16 facets in the interior of the prism. In order to obtain a concrete example, it is sufficient that k be such that the three sphere centres O_1, O_2 and O_3 in the proof of Lemma 2.8 are inside the reflection cell. I.e., it is sufficient that

$$k > \frac{7}{(2 - \sqrt{2})a} \simeq \frac{11.95}{a}.$$

Moreover, we can easily prove that the Dirichlet region has at least two external facets: Since all the Dirichlet regions inside the prism are equivalent, all of them have the same number of external facets. If all of them have only one external facet, then there have to be some vertical facets in the Voronoi diagram inside R . But this would imply that there are points in $GP \cap R$ with the same height, which is not the case: there is exactly one point in $GP \cap R \cap \{z = m + 1/2\}$ for each integer $m \in \mathbb{Z}$.

2.3 Groups with reflections in one direction

There are 32 classes of three-dimensional crystallographic groups with reflections in one direction: 5 in the monoclinic system, 16 in the orthorhombic system, 6 in the tetragonal system and 5 in the hexagonal system. Let G be any one of them, let R be a reflection cell of G and let P be a point in R and with trivial stabiliser. The reflection cell R is an infinite strip determined by 2 parallel reflection planes which we assume horizontal and at heights 1 and -1 (see Figure 6). Therefore the maximal number of external neighbours is at most 2.

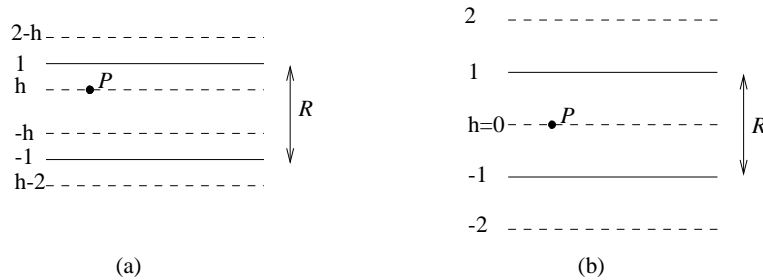


Figure 6: Reflection planes for groups with reflections in 1 direction (a) $h \neq 0$. (b) $h = 0$

Let h be the height of the horizontal plane which contains the point $P \in R$. Since every element of G brings horizontal planes to horizontal planes, all the points of the orbit GP lie on planes at distance $1 - h$ from some reflection plane. Then, inside R we have points of GP only on the planes h and $-h$. If $h = 0$, h and $-h$ is the same plane (see Figure 6).

Let $G_0 = \{g \in G : g(\alpha) = \alpha, \forall \text{ horizontal plane } \alpha\}$ (G_0 is the subgroup of G which contains the “horizontal motions”). We study the number of internal neighbours of P separately among points of GP on the planes h and $-h$ (if $h = 0$, the second is not necessary).

Lemma 2.10 $P \in h$ has at most 6 neighbours on the plane h .

Proof: Let $G_h = \{g \in G : g(h) = h\}$, then $GP \cap h = G_h(P)$.

If $Q \in h$ is a neighbour of $P \in h$ in the three-dimensional Delaunay triangulation there exists a sphere S passing through P and Q with no point from the orbit GP of P in its interior. The circle $S \cap h$ in the plane h passes through P and Q and has no interior point from $GP \cap h$, therefore Q is a neighbour of P also in the two-dimensional Dirichlet tessellation of G_0P in the plane h . The maximum number of neighbours of a point in a two-dimensional stereohedral tessellation is 6. \square

Lemma 2.11 $P \in h$ has at most seven neighbours on the plane $-h$.

Proof: Suppose $h \neq 0$; otherwise the statement follows from Lemma 2.10.

Observe that $GP \cap h = G_0P$ and, for any $Q \in GP \cap (-h)$, $GP \cap (-h) = G_0Q$. By Lemma 1.3, we can bound the number of neighbours of P in the plane $-h$ by the number of regions of $Vor_{G_0P}(Q)$ intersected by $Vor_{G_0P}(P)$, either considered as three-dimensional Voronoi diagrams or as two-dimensional Voronoi diagrams on the same plane. Moreover, $G_0P \cup G_0Q$, when projected to the horizontal plane at height 0, form an orbit of the subgroup of G which fixes this plane. By Theorem 3.1 of the next section, $Vor_{G_0P}(P)$ intersects at most seven regions of $Vor_{G_0P}(Q)$. \square

Adding up the bounds for internal and external neighbours we conclude that:

Theorem 2.12 For any three-dimensional crystallographic group G with reflections in only one independent direction and for any point P with trivial stabiliser, the Dirichlet stereohedron $Vor_{GP}(P)$ has at most fifteen facets. \square

Example 2.13 A Dirichlet stereohedron with 15 facets, for a group with only one independent reflection.

Let G_0 be a two dimensional crystallographic group of type $p2$ generated by rotations of order two on the four vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ of a square. The dashed region in Figure 7 is the Dirichlet region of the point $(1/2, \epsilon)$ for a certain $\epsilon \in (\frac{1}{2}, 1)$, whose orbit is represented by black dots. The Voronoi diagram of the orbit of $(\epsilon, 1/2)$ (white dots) is also shown. The small ovals in the Figure represent rotation centres of order two. The Dirichlet region of one tessellation intersects seven regions of the other, and the two orbits together form an orbit of a group cm .

Let now G be a three-dimensional crystallographic group of type $P \frac{2}{m} \frac{2}{c} \frac{2}{c}$ having G_0 as its subgroup of horizontal translations and generated by G_0 together with

- the reflections in parallel horizontal planes $z = k$ and $z = -k$ and
- the rotation of order two on the axis $x = y = 0$, which fixes one of the reflection cells exchanging its two boundary planes.

Let $P = (\frac{1}{2}, \epsilon, h)$ and $Q = (\epsilon, \frac{1}{2}, -h)$. Figure 7 can be regarded as the projection of the two orbits G_0P and G_0Q considered in the proof of Lemma 2.11. If the two reflection planes bounding R are sufficiently far, then P has exactly six neighbours on the plane h , seven on the plane $-h$ and one facet in the top reflection plane. If we move closer the two reflection planes, the first combinatorial change occurs when the bottom reflection plane crosses the lowest vertex of $Vor_{GP}(P)$, which increases by one the number of neighbours, in a way similar to Example 2.5.

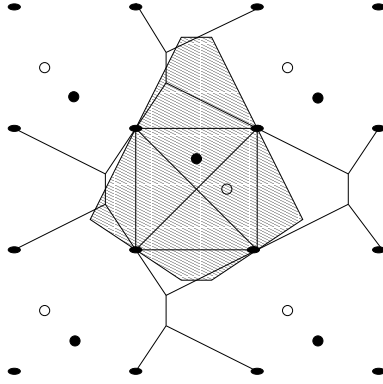


Figure 7: The Dirichlet intersection number of a group $p2$ is at least seven

3 How two two-dimensional Dirichlet tessellations of the same group intersect

The proof of Lemma 2.11 relied on the answer to the following question: *Given two different orbits GP and GQ of the same two-dimensional crystallographic group G , how many regions of the Dirichlet tessellation Vor_{GQ} can each region of Vor_{GP} intersect?* We will call this number the *Dirichlet intersection number* of the group G .

In this section we prove that the Dirichlet intersection number of every planar crystallographic group is bounded above by seven, except perhaps for the groups of type pgg . For pgg we have to make the extra assumption that the union $GP \cup GQ$ is again an orbit of a crystallographic group, which is the case that we needed in Lemma 2.11. The following statement summarises our results:

Theorem 3.1 *Let G be a planar crystallographic group and let GP and GQ be two orbits of points with trivial stabiliser. Then, the number of Dirichlet regions of one of the orbits intersected by each Dirichlet region of the other orbit is bounded above by:*

- one, for groups of types pmm , $p3m1$, $p4m$ and $p6m$.
- two, for groups of types cm , cmm , $p31m$ and $p4g$.
- four, for groups of types $p1$, $p3$, $p4$, $p6$, pmg and pm .
- seven, for the groups $p2$ and pg .
- seven, for the group pgg , if $GP \cup GQ$ is again an orbit of a two-dimensional crystallographic group.

The bounds are tight in all cases except perhaps for pg and pgg .

Proof: The groups of types $p2$, $p3$, $p4$, $p6$, pg and pgg require an elaborate proof, and will be dealt with in Sections 3.1 to 3.4. For the other eleven classes of planar crystallographic groups, we have:

- Lemma 1.5 directly implies that the number is at most four for any group $p1$, with one aspect. Independently of the metric parameters of the group, if the base points P and Q for the two orbits are chosen sufficiently generic, then the Voronoi region $Vor_{GP}(P)$ will intersect exactly four regions of Vor_{GQ} .

- Proposition 2.2 implies that the Dirichlet tessellation for a group generated by reflections does not depend on the chosen orbit, so the Dirichlet intersection number for the groups $p6m$, $p4m$, $p3m1$ and pmm equals one.
- For the other groups which contain reflections in two independent directions ($p31m$, $p4g$ and cm), all the points of the orbit inside a reflection cell R must have the same distance to the centroid of R (every motion sending R to itself fixes the centroid). Hence, $R \cap GP$ and $R \cap GQ$ are orbits of the same finite discrete group and, clearly, a region in one Voronoi diagram intersects two regions in the other, except in degenerate cases where the two Dirichlet tessellations coincide.
- For the three groups which contain reflections in only one direction (cm , pm and pmg), the part of GP (or of GQ) inside R consists of one (for cm) or two (for pm and pmg) 1-dimensional translational lattices, placed in lines parallel to the reflection lines. Using Lemmas 1.3 and 1.5 we conclude that each region of Vor_{GP} can intersect only the regions of at most two points in each 1-dimensional lattice. Hence the Dirichlet intersection number is at most two for cm and at most four for pm and pmg . It is easy to check that both bounds are tight. \square

Observe that the groups pg and $p2$ have only two aspects, so Corollary 1.6 already implies that their Dirichlet intersection number is at most eight.

Remark 3.2 We conjecture that the Dirichlet intersection number of pgg is at most seven, too. In Section 3.3 we prove it to be at most eleven. Applying the methods used there in a more sophisticated way we can show that it is at most nine, but we do not include a proof of this.

The number seven equals the maximum number of facets of a two dimensional Dirichlet stereohedron plus one. It is an interesting possibility that the Dirichlet intersection number of any crystallographic group in any dimension might be bounded above by the maximum number of facets of a Dirichlet stereohedron for that group plus one.

3.1 Groups of type $p3$, $p4$ and $p6$

Proposition 3.3 *If $G \in \{p3, p4, p6\}$ and GP , GQ are orbits of G , then $Vor_{GP}(P)$ intersects at most 4 regions of Vor_{GQ} .*

Proof: For any of these three groups, the action of G tiles \mathbb{R}^2 into two classes of triangles with vertices in rotation centres of the group, as shown in Figure 8. We call them “black” and “white”, respectively. Any black triangle is adjacent to three white ones and vice-versa. The union of any pair of adjacent black-white triangles is a fundamental domain for the group.

Without loss of generality we suppose P in a black triangle $C_1C_2C_3$. Considering the bisectors between P and the points of GP obtained from P by the rotations at C_1 , C_2 and C_3 we conclude that the Voronoi region of P is contained in the union of $C_1C_2C_3$ and its three adjacent white triangles. If Q is in a white triangle, this argument shows that $Vor_{GP}(P)$ intersects only the three regions of Vor_{GQ} corresponding to points in the triangles adjacent to $C_1C_2C_3$. Hence, in the sequel we will assume without loss of generality that P and Q lie both in the same black triangle $C_1C_2C_3$.

In this case, our previous argument implies that $Vor_{GP}(P)$ can intersect only the following regions in Vor_{GQ} : the region of Q and the five (in p_4 and p_6) or six

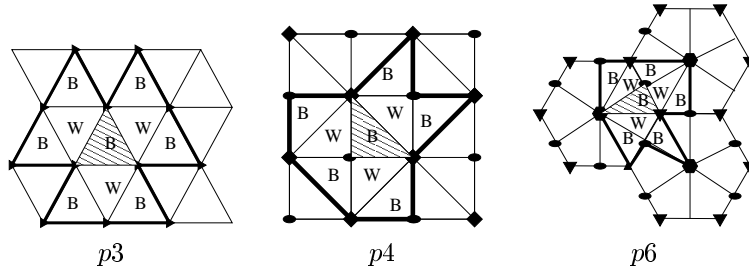


Figure 8: The groups p_3 , p_4 and p_6 tile the plane into “black” and “white” triangles

(in p_3) regions corresponding to points of GQ in the black triangles at distance two from $C_1C_2C_3$. These are the triangles contained in the thick polygons of Figure 8. We will prove now that only three from the six or five black triangles other than $C_1C_2C_3$ produce regions which intersect $Vor_{GP}(P)$, for each fixed P and Q . For this observe that the structure of the Dirichlet tessellation Vor_{GP} for any point P in the interior of $C_1C_2C_3$ is as follows: each region consists of a full black triangle and part of its three adjacent white triangles, in such a way that a unique new vertex of the tessellation, apart from the rotation centres, appears in the interior of any white triangle. See Figure 3.1, where this is shown for the group p_4 .

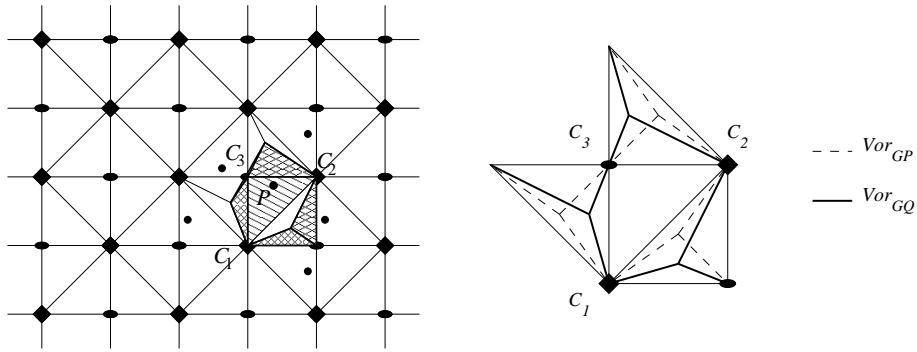


Figure 9: The Dirichlet intersection number of p_4 is four

Of course, the decompositions of the white triangles into three sub-triangles are all equivalent by the action of the group. In a generic situation (and we only need to consider this case since degeneracies can only decrease the number of regions intersected) exactly one of the new vertices of Vor_{GP} will be contained in $Vor_{GQ}(Q)$. Say that it is the vertex in the white triangle incident to the edge C_1C_2 , as in Figure 9. Then $Vor_{GP}(P)$ does not intersect any Voronoi region of Vor_{GQ} other than $Vor_{GQ}(Q)$ in that white triangle. With similar arguments, in the other two white triangles $Vor_{GP}(P)$ intersects respectively one and two regions of Vor_{GQ} other than $Vor_{GQ}(Q)$. \square

Actually, using the conventions of “black” and “white” triangles the proof of the previous lemma says:

Corollary 3.4 *If $G \in \{p_3, p_4, p_6\}$ and GP, GQ are orbits of G , such that Q and P do not belong to a triangle of the same colour, then $Vor_{GP}(P)$ intersects three regions of Vor_{GQ} . If they belong to triangles of the same colour and they are sufficiently generic, then $Vor_{GP}(P)$ intersects four regions of Vor_{GQ} . \square*

3.2 Groups of type $p2$

Let G be a group of type $p2$, i.e. generated by two translations of vectors which we denote a and b and a rotation of order two. The order-two rotation centres of G form an affine two-dimensional lattice and the translation vectors are exactly twice the differences between rotation centres. The plane can be naturally tiled with congruent triangles with vertices in the rotation centres and without obtuse angles. There is a unique way of doing this, unless the translation vectors a and b are orthogonal, in which case we choose one of the two possibilities. Any of these triangles, together with its three adjacent ones, forms a fundamental domain for G . One is shown in Figure 10.

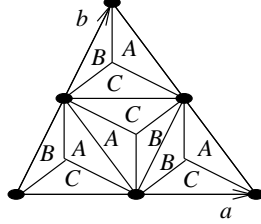


Figure 10: A fundamental domain for $G = p2$

We divide each of the triangles into three “ortho-regions” labelled A , B and C by drawing the altitudes from the vertices to the orthocentre (see again Figure 10). For symmetry reasons we will rather speak of *rhombi* of types A , B or C , where a rhombus is the union of two adjacent ortho-regions labelled with the same letter.

For any choice of the point P in a certain rhombus labelled A , the region $Vor_{GP}(P)$ is contained in the shaded region of Figure 11(a). We call this the *extended Dirichlet region* associated to the rhombus A . To prove this assertion, the rest of the plane has been divided into pieces associated to certain points in the orbit GP , displayed in the Figure. It is easy to check that the piece containing a certain point $P' \in GP$ consists of points which are closer to P than to P' for any choice of P in the rhombus. The way to verify this is to consider the bisector of P and P' , for varying P (taking into account in each case the isometry sending P to P'). Part (b) of the same Figure shows the extended Dirichlet region of a rhombus labelled B .

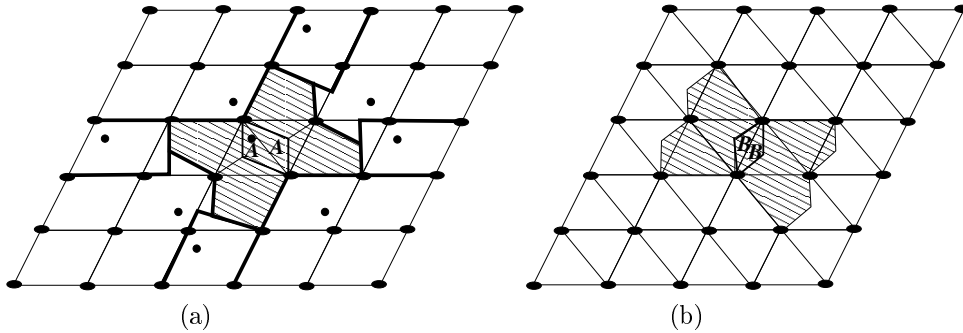


Figure 11: Extended Dirichlet regions for the rhombi A and B

Of course, we can consider the extended Dirichlet region for any other rhombus of type A , B or C . In order to bound the number of regions of Vor_{GQ} intersected by $Vor_{GP}(P)$ we will use the obvious fact that if $Vor_{GP}(P) \cap Vor_{GQ}(Q') \neq \emptyset$ then the extended Dirichlet regions of the rhombi containing P and Q' overlap. Without loss

of generality we suppose that P lies in a rhombus labelled A and we will consider the cases of Q lying in a rhombus A or B separately. The case of a rhombus C would be analogous to B .

$[A, A]$

There are thirteen rhombi of type A whose extended Dirichlet region intersects a given one also of type A . The easiest way to see that this is true is observe that the initial extended Dirichlet region is contained in the union of a parallelogram with its vertices in rotation centres and the four parallelograms adjacent to it. Hence, the rhombi we are interested in are contained in the “Aztec diamond” formed by these five parallelograms and the eight ones adjacent to them. This is shown in Figure 12.

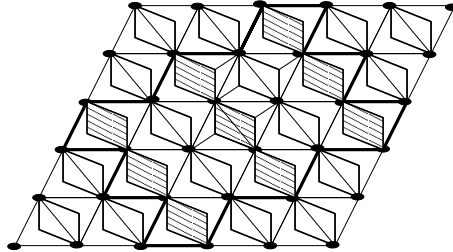


Figure 12: Possible intersecting extended Dirichlet regions in the case $[A, A]$

Now, these thirteen rhombi belong to two orbits by the action of G , one with nine rhombi (shaded in Figure 12) and the other with only four. Hence, we only need to deal with the case where GQ lies in the first orbit of rhombi. In this case, without loss of generality we suppose that P and Q lie in the same rhombus, which we denote A_P .

The other eight shadowed rhombi are the ones obtained by the four rotations on the vertices of the parallelogram containing A_P and the four translates $\tau_a(A_P)$, $\tau_{-a}(A_P)$, $\tau_b(A_P)$ and $\tau_{-b}(A_P)$. We claim that only one of $\tau_a(A_P)$ and $\tau_{-a}(A_P)$ can contain a point of GQ whose region intersects $Vor_{GP}(P)$ (and the same for b). This reduces the number of possible intersecting regions from nine to seven, as desired.

To prove the claim, observe that the bisectors of P and $\tau_a P$ and of P and $\tau_{-a} P$ form a strip of width $|a|$ which contains $Vor_{GP}(P)$. In the same way, the bisectors of Q and $\tau_a Q$ and of Q and $\tau_{-a} Q$ forms a strip parallel to and of the same width as the previous one, separating $Vor_{GQ}(\tau_a Q)$ and $Vor_{GQ}(\tau_{-a} Q)$. Hence, only one of these two regions can intersect the strip containing $Vor_{GP}(P)$.

$[A, B]$

There are fourteen rhombi of type B whose extended Dirichlet region intersects a given one of type A , as shown in Figure 13. Again, they divide into two orbits by the action of G , each having seven elements. Hence, only seven rhombi can contain points of the same orbit GQ .

3.3 Groups of type pgg

Let G be a group of type pgg , i.e. generated by two orthogonal glide reflections γ_a and γ_b , so that $\gamma_a^2 := \tau_a$ and $\gamma_b^2 := \tau_b$ are two orthogonal translations of vectors a and b . G contains rotations of order two whose centres form a rectangular grid, and any of the primitive rectangles in this grid (whose sides equal half the translation vectors a and b) is a fundamental domain. See Figure 14. The two medial axes of the rectangle are the axes of two glide-reflections generating G . We will call

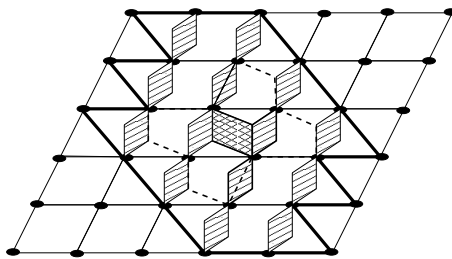


Figure 13: Possible intersecting extended Dirichlet regions in the case [A,B]

fundamental subdomains of G the four sub-rectangles in which the two medial axes divide a fundamental domain.

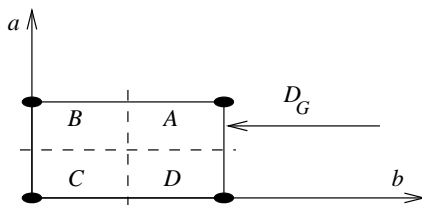


Figure 14: A fundamental domain for $G = pgg$

Let GP and GQ be two orbits of G . Without loss of generality we assume that P and Q lie in the same fundamental domain D_G and call A the fundamental subdomain containing P . Let us call B , C and D the other three fundamental subdomains in D_G , in a circular order (see again Figure 14).

In the same way as we did for $p2$, we compute an extended Voronoi region for the fundamental subdomain A , i.e. a region containing the Voronoi region $Vor_{GP}(P)$ for every point $P \in A$. The region obtained is the shaded one in Figure 15. As in Figure 11(a), the rest of the plane is divided into pieces containing each a distinguished element of GP so that for any choice of the point $P \in A$ the piece of a point P' contains only points closer to P' than to P .

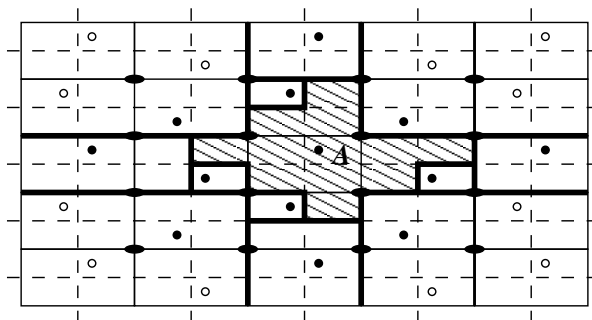


Figure 15: Extended Dirichlet region for A

We can consider the extended Dirichlet region for any other fundamental subdomain in the plane. We call *influence region* of the subdomain A the union of all the subdomains whose extended Dirichlet region intersects the extended Dirichlet region of A . Clearly, any point $Q' \in GQ$ such that $Vor_{GQ}(Q') \cap Vor_{GP}(P) \neq \emptyset$ will lie in the influence region.

To compute the influence region, first observe that it must be contained in the “Aztec diamond” consisting of D_G , the four fundamental domains adjacent to D_G , and the eight fundamental domains other than D_G adjacent to these four. Then, compute the extended Dirichlet region of the thirteen shaded subdomains of Figure 16 (i.e., apply to the extended Dirichlet region of A the appropriate transformation of G) and check that they do not intersect the extended Dirichlet region of A .

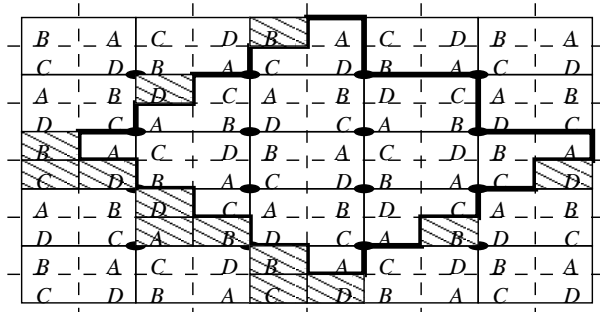


Figure 16: Influence region of A

The influence region of A contains twelve fundamental subdomains of type A , eleven of type C and eight of types B and D . Actually, using the same argument as in the proof of the case $[A, A]$ in Section 3.2 the number of subdomains of type A can be reduced to eleven (only two of the four subdomains obtained from A by translation can produce intersecting regions). This immediately shows that the Dirichlet intersection number of G is at most eleven. With more sophisticated arguments we can decrease this to nine, but we will not do it here.

From now on we will concentrate in the case when $GP \cup GQ$ is an orbit of a crystallographic group, as needed in Section 2.3.

Lemma 3.5 *Let G be planar crystallographic group of type pgg and P a point with trivial stabiliser, in a certain rectangular fundamental domain D_G . Let $Q \in D_G$ be such that the union $GP \cup GQ$ is an orbit of another crystallographic group H . Then, the element of H sending P to Q is an involutive symmetry of D_G .*

Proof: Let g be the element of H with $gP = Q$. If $P = Q$ then g is the identity and there is nothing to prove. Suppose $P \neq Q$.

Once we prove that g is a symmetry of the rectangle D_G , the fact that g is an involution is easy: let $g' \in H$ such that $g'Q = P$. Applying the lemma with P and Q exchanged, we conclude that g' is a symmetry of D_G too. Now, if $g \neq g'$, then H contains two non-trivial elements which fix D_G and the index of G in H would be at least three, which is false. Hence, $g = g'$ and $g^2P = P$. Since P has trivial stabiliser in G , g^2 is the identity.

Let us now prove that g is a symmetry of D_G . We have the following two possibilities: either H has exactly the same rotations of order two as G , in which case it is clear that g is a symmetry of D_G since it preserves the rectangular grid, or g sends the rectangular grid of rotation centres of G to another grid. We assume this is the case. Observe that, then, H is the group generated by G and any of the new rotations.

Any of the two grids of rotation centres is invariant under the rotations of the other one. There are only three ways in which this can occur: (a) the second grid has rotation centres in the center of the rectangles of the first, (b) the second grid has rotation centres in the midpoints of vertical edges of the rectangles of the first grid and (c) the same with horizontal edges.

In the three cases H contains a non-trivial symmetry of the rectangle D_G . In case (a), the rotation of order two at the centre of the rectangle D_G . In case (b), the reflection at the vertical axis of D_G and in case (c) the reflection at the horizontal axis. In the last two cases the reflection in question is obtained as a composition of a rotation of order two in H and a glide reflection of G .

This symmetry of D_G sends P to an element of GQ in D_G , i.e. to Q . \square

There are six possible involutions of D_G , displayed in Figure 17: the identity, the rotation of order two at the centre of the rectangle, the reflections on the horizontal and vertical axes of the rectangle and, if the rectangle is a square, (i.e. if $|a| = |b|$) the reflections on the two diagonals of the square. We name these cases $[A, A]$, $[A, C]$, $[A, B]$, $[A, D]$, $[A, A']$ and $[A, C']$ respectively, and study them separately.

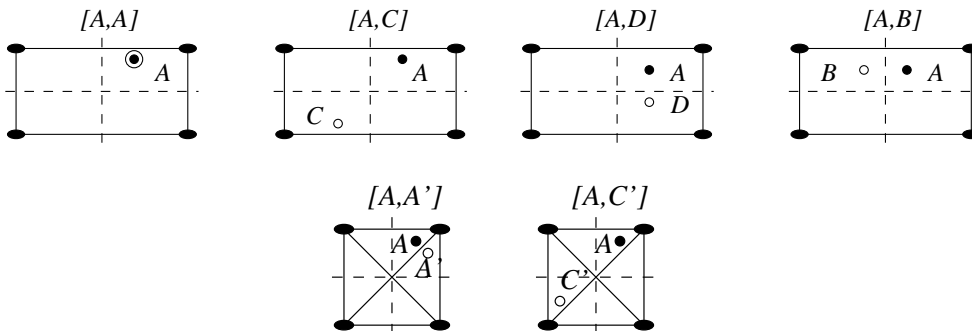


Figure 17: The six possible involutions of D_G

$[A, A]$

$Vor_{GP}(P)$ intersects only one region of the Voronoi diagram Vor_{GQ} , since $P = Q$.

$[A, B]$ and $[A, D]$

We just need to discard one of the eight fundamental subdomains of types B and D in the influence region of Figure 16. We do this with type B , since type D is analogous.

Let P be our base point in the subdomain A of our fundamental domain D_G and let Q be the reflected on the vertical axes, lying in the subdomain of type B . Let P' and Q' denote the points of GP and GQ in the fundamental domain adjacent to the subdomains B and C of D_G (the one next to the central one on the left, in Figure 16). We have $P' - P = Q' - Q$ so that the bisectors of P and P' and of Q and Q' are parallel. Moreover, the first one is closer to P than the second one, and hence these two bisectors separate the regions $Vor_{GP}(P)$ and $Vor_{GQ}(Q')$.

$[A, C]$

In the case $[A, C]$, we want to prove that among the 11 fundamental subdomains of type C inside the influence region there are at most seven whose Voronoi regions intersect $Vor_{GP}(P)$. In Figure 18 we show the parts of the orbits GP and GQ which lie inside the influence region, with black and white points, respectively. We will prove that the regions corresponding to the points labelled $2'$, $4'$, $5'$ and $6'$ in Figure 18 (where $P = 0$ and $Q = 0'$) do not intersect $Vor_{GP}(P)$.

Observe first that the bisector $bi(0, 1)$ is parallel to $bi(1', 2')$ and with $bi(0, 1)$ closer to 0 and $bi(1', 2')$ closer to $2'$. Thus, the Voronoi region of $2'$ can not intersect the one of 0 . In the same way we eliminate the point $4'$ using the bisectors $bi(0, 3)$ and $bi(3', 4')$.

In order to show that the Voronoi region of $5'$ in Vor_{GQ} does not intersect the one of 0 in Vor_{GP} , we will concentrate on the eight points $0, 0', 1, 1', 3, 3', 5$ and

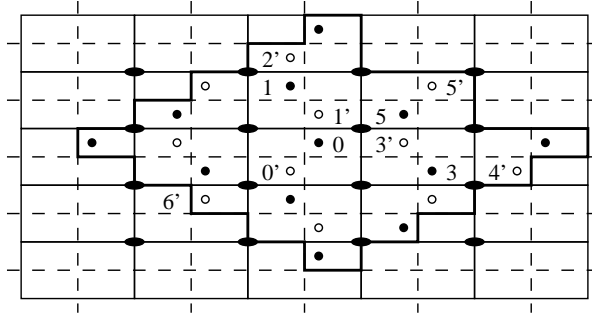


Figure 18: The case $[A, C]$

$5'$, displayed in Figure 19. The Voronoi diagram of the four points $0, 1, 3$ and 5 is shown with solid lines. The Voronoi diagram has to be combinatorially as shown in the figure since 0 and 5 are equidistant to the centre of the figure and closer than 3 and 1 . The Voronoi diagram of $0', 1', 3'$ and $5'$ is shown in dashed lines, and obtained from the other one by a reflection in either a vertical or a horizontal line.

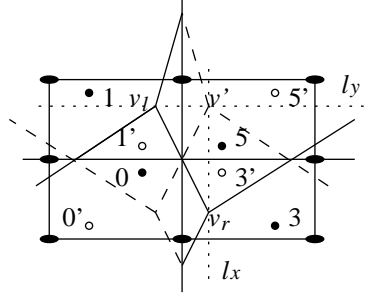


Figure 19: The case $[A, C]. \text{Vor}_{GQ}(5')$ does not intersect $\text{Vor}_{GP}(0)$

Let v' be the unique vertex of the region of $5'$ (incident to the regions of $1'$ and $3'$). By symmetry, v' is on the same vertical line r as the vertex v_r between the regions of $0, 3$ and 5 and on the same horizontal line l as the vertex v_l between the regions of $0, 1$ and 5 . Then, the region $\text{Vor}_{\{0,1,3,5\}}(0)$ is contained in one of the quadrants determined by l and r while $\text{Vor}_{\{0',1',3',5'\}}(5')$ is contained in the union of the other three quadrants. Hence, the two regions do not intersect, and, by part (ii) of Lemma 1.1, the corresponding regions in Vor_{GP} and Vor_{GQ} do not intersect either.

The same reasoning as for $5'$, applied taking $6'$ as base point, implies that the regions of 0 and $6'$ do not intersect.

$[A, A']$ and $[A, C']$

The remaining cases $[A, A']$ and $[A, C']$ only make sense if the fundamental domain is a square, i.e. if the two orthogonal translation vectors a and b of our group have the same length. In this case we compute an extended Dirichlet region again, except that we now divide the fundamental domain into eight congruent triangles instead of four rectangles, as shown in Figure 20. The extended Dirichlet region for the triangle A is shown in the left, and the associated influence region on the right. There are only six triangles of each of the types A' and C' inside the influence region of A . Hence, the number of regions of Vor_{GQ} which overlap $\text{Vor}_{GP}(P)$ is at most six. \square

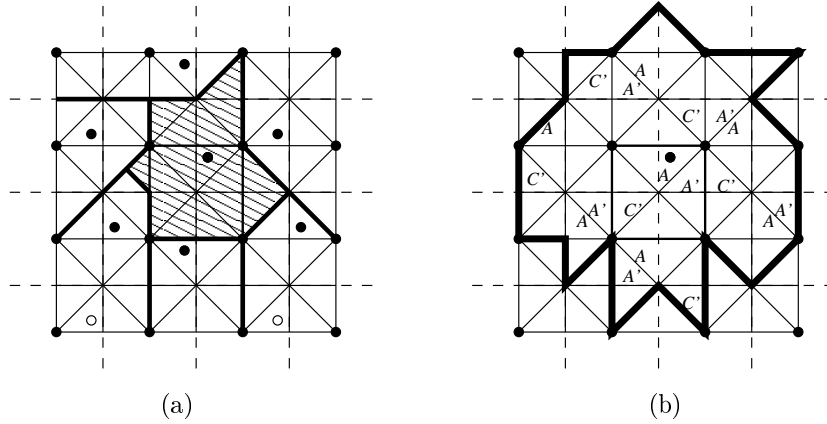


Figure 20: For $|a| = |b|$, (a): extended Dirichlet region for A (b): Influence region for A

3.4 Groups of type pg

Let G be a planar crystallographic group of type pg , i.e generated by two orthogonal translations of vectors a and b and a family of glide reflections with translation of vector $\frac{a}{2}$ on lines parallel to a and distant $|\frac{b}{2}|$ to one another. Any translation of the rectangle $a \times \frac{b}{2}$ with two sides on glide-reflection axes is a fundamental domain for G .

Let P be any point and let s denote the segment parallel to b containing P and joining the closest glide-reflection axes (we assume that P is not on a reflection axis). Using the methods of the previous sections we can easily construct an extended Dirichlet region for the segment s , which is shown in part (a) of Figure 21.

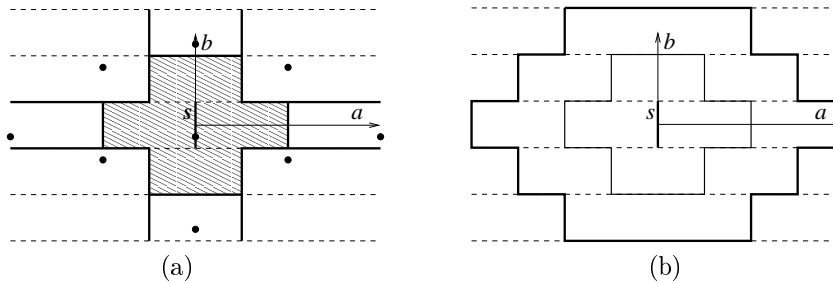


Figure 21: Extended Dirichlet region and influence region for the group pg

Similarly, we construct the influence region of s , defined as the union of all the segments parallel to b and joining two consecutive glide-reflection axes whose extended Dirichlet region intersects the one of s . This is shown in part (b) of Figure 21. The influence region is contained in the union of eight Dirichlet domains of pg : two in between the same glide-reflection axes as P and s , two in each of the two adjacent strips between reflection axes and one more in each of the next two strips. In principle this produces a Dirichlet intersection number of eight, but we can reduce it to seven using the same argument as in the case $[A, A]$ of the group $p2$: for any point Q , the Voronoi region $Vor_{GQ}(Q)$ is contained between the bisectors $bi(Q, \tau_b Q)$ and $bi(Q, \tau_b^{-1} Q)$, which form a band of width $|\frac{b}{2}|$ parallel to a . In particular, only one of the two regions $Vor_{GQ}(\tau_b Q)$ and $Vor_{GQ}(\tau_b^{-1} Q)$ can intersect $Vor_{GP}(P)$. This reduces to one the two possible intersecting regions produced by the two fundamental domains in the strips farther from s .

References

- [1] D. BOCHIŞ, *Estereoedros de Dirichlet en 2 y 3 dimensiones*, Ph. D. Thesis, Universidad de Cantabria, 1999.
- [2] D. BOCHIŞ AND F. SANTOS, On the number of facets of 3-dimensional Dirichlet stereohedra II: non-cubic groups. In preparation.
- [3] D. BOCHIŞ AND F. SANTOS, On the number of facets of 3-dimensional Dirichlet stereohedra III: cubic groups. In preparation.
- [4] G. O. BRUNNER AND F. LAVES, How many faces has the largest space-filling polyhedron?, *Zeitschr. Kristallographie* **147** (1978), 39–43.
- [5] H.S.M. COXETER, *Regular Polytopes*, Dover Publications, INC. New York, 1973.
- [6] B. N. DELONE (OR DELAUNAY), A proof of the fundamental theorem of the theory of stereohedra, *Dokl. Akad. Nauk. SSSR* **138** (1961), 1270–1272 (in Russian).
- [7] L. FÖPPL, Der Fundamentalbereich des Diamantgitters, *Phys. Zeitschr.* **15** (1914), 191–193.
- [8] P. ENGEL, *Geometric Crystallography: An Axiomatic Introduction to Crystallography*, D. Reidel Pub. Co. Dordrecht, 1986.
- [9] B. GRÜNBAUM AND G. C. SHEPHARD, Tilings with congruent tiles, *Bull. Amer. Math. Soc.* **3** (1980), 951–973.
- [10] B. GRÜNBAUM AND G. C. SHEPHARD, *Tilings and Patterns*, W.H. Freeman and company, New York, 1987.
- [11] D. HILBERT, Mathematische Probleme, in: D. Hilbert, *Gesammelte Abhandlungen* Vol 3, Springer-Verlag, 1935, pp. 290–329.
- [12] E. KOCH, *Wirkungsbereichpolyeder und Wirkungsbereichsteilungen zu kubischen Gitterkomplexen mit weniger als drei Freiheitsgraden*, Ph. D. Thesis, Philipps-Universität Marburg/Lahn, 1972.
- [13] E. KOCH AND W. FISCHER, Wirkungsbereichstypen einer verzerrten Diamantkonfiguration mit Kugelpackungscharakter, *Zeitschr. Kristallographie* **135** (1972), 73–92.
- [14] E. H. LOCKWOOD AND R. H. MACMILLAN, *Geometric symmetry*, Cambridge University Press, 1978.
- [15] J. MILNOR, Hilbert’s 18 problem: on crystallographic groups, fundamental domains, and on sphere packing, in: *Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society*, Northern Illinois University, Dekalb, Illinois, 1974, pp. 491–507.
- [16] W. NOVACKI, *Homogene Raumteilung und Kristallstruktur*, Ph. D. Thesis, E. T. H. Zürich, 1935.
- [17] D. SCHATTSCHNEIDER, The Plane Symmetry Groups: their Recognition and Notation, *American Mathematical Monthly* **85** (1978), 439–450.
- [18] D. SCHATTSCHNEIDER AND M. SENECHAL, Tilings, in: *Handbook of Discrete and Computational Geometry* (J. E. Goodman, J. O’Rourke, eds), CRC Press, 1997, pp. 43–63.

- [19] F. W. SMITH, The structure of aggregates—a class of 20-faced space-filling polyhedra, *Canad. J. Physics* **43** (1965), 2052–2055.