An Extended FDTD Method for the Treatment of Partially Magnetized Ferrites

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Abstract - A formulation for the FDTD treatment of partially magnetized ferrites is presented. It is based on the derivation of a system of time-domain differential equations from an empirical expression of the permeability tensor, normally used for designing microwave devices. This approach is applied to the full-wave analysis of waveguides loaded with partially magnetized ferrites.

I. INTRODUCTION

Ferrites are basic materials in the realization of non-reciprocal and control devices such as isolators, circulators and phase shifters. The complexity of the electromagnetic fields in these devices suggests the use of numerical techniques for their analysis.

In recent years, various authors have proposed extensions of the finite-difference time-domain (FDTD) method for the treatment of saturated ferrites. Two different models have been used to describe the constitutive properties of ferrites in the time domain: one the frequency-domain constitutive equation relating the magnetic flux density to the magnetic field strength through the permeability tensor is transformed to the time domain by using the convolution theorem [1]; in the other, the equation of motion of the magnetization vector is used to describe the medium properties in the time domain. Several discretization schemes have been proposed to integrate the equation of motion simultaneously with Maxwell's time-dependent curl equations. An interesting discussion of the accuracy of these schemes can be found in [2].

On the other hand, in many practical applications, biased ferrites are not saturated and these formulations are not, therefore, valid. This paper presents an FDTD formulation capable of analyzing partially magnetized ferrite-loaded structures. The ferrite model consists of a permeability-tensor-based constitutive equation, which is transformed to the time domain by taking its inverse Fourier transform. As a result, a system of first-order differential equations is obtained. These equations are discretized by using the same scheme as in [3] for saturated ferrites. The validity of this new extension of the FDTD method is demonstrated by applying it to the analysis of the dispersion characteristics of partially magnetized ferrite-loaded waveguides. For this application, the numerical dispersion equation and a stability study of the resulting formulation are also carried out.

Manuscript received July 6, 1994.
This work was supported by the Spanish CICYT under the project TIC93-0671-C06-02.

II. FERRETE TREATMENT

Maxwell's time-dependent curl equations can be expressed as

\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E},
\]  
\[
\epsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H},
\]

where \(\vec{E}\) and \(\vec{H}\) are the electric and magnetic field strength, respectively, \(\vec{B}\) the magnetic flux density and \(\epsilon\) the permittivity. The electric constitutive equation \(\vec{D} = \epsilon \vec{E}\), where \(\vec{D}\) is the electric displacement, has been assumed in (1b).

For magnetized ferrites, in addition to (1), the following magnetic constitutive relation must be considered

\[
\vec{B} = [\mu] \vec{H},
\]

where \([\mu]\) is the permeability tensor. For a lossless ferrite partially magnetized in the z-direction to a value of magnetization \(M_s\), the non-zero elements of this tensor are given by the following empirical expressions [4]:

\[
\mu_{xx} = \mu_{yy} = \mu_0 \left[ \mu_0 + (1 - \mu_0) \left( M_s/M_s \right)^{3/2} \right],
\]

\[
\mu_{zz} = \mu_0 \mu_0 \left( 1 - M_s/M_s \right)^{1/2},
\]

\[
\mu_{xy} = -\mu_{yx} = j \omega_0 \omega / \omega,
\]

where \(M_s\) is the saturation magnetization and \(\omega = \gamma \mu_0 M_s\), where \(\gamma\) is the gyromagnetic ratio. In (3) \(\mu_s\) denotes the relative permeability of the ferrite for the completely demagnetized state. It is given by

\[
\mu_s = \frac{1}{3} \left( 1 + 2 \sqrt{1 - (\gamma \mu_0 M_s / \omega)^2} \right).
\]

The extended FDTD method for ferrite treatment leads to an iterative time-domain algorithm with three steps in each time iteration: first, \(\vec{B}\) is updated by using the difference form of (1a); second, \(\vec{H}\) is calculated by using an appropriate discretized time-domain model of (2); and, third, the difference form of (1b) is used to obtain \(\vec{E}\). The first and third steps are common to the standard (isotropic, non-dispersive) FDTD method while the second step must be introduced to
take into account the dispersive and anisotropic nature of partially magnetized ferrites. The next section introduces a formulation to carry out the second step.

III. TIME-DOMAIN MODEL AND DISCRETIZATION

Taking into account that \( F \left[ \frac{d}{dt} \right] = j \omega \), we can easily obtain the inverse Fourier transform of (2), resulting in the following system of first-order differential equations

\[
\frac{\partial B_x(t)}{\partial t} = \mu_0 \left( \mu_a B_x(t) - \omega_m H_y(t) \right), \quad (5a)
\]

\[
\frac{\partial B_y(t)}{\partial t} = \mu_0 \left( -\omega_m H_x(t) + \mu_a \frac{\partial H_z(t)}{\partial t} \right), \quad (5b)
\]

\[
H_z(t) = \mu_0 \mu_a (1 - M/M_z)^{5/2} H_x(t), \quad (5c)
\]

where \( \mu_a \) is an average value of \( \mu_a \) in the frequency band of interest.

As in the saturated ferrite case [3], (5c) is directly incorporated into (1a), and (5a) and (5b) are first discretized by using central finite differences and linear interpolation and then decoupled by solving for \( H_x^{\ast l+\delta l} \) and \( H_y^{\ast l+\delta l} \). This yields

\[
H_x^{\ast l+\delta l} = c_0 h_x^{\ast l} + c_1 (B_x^{\ast l+\delta l} - B_x^{\ast l}) + c_2 (B_y^{\ast l+\delta l} - B_y^{\ast l}) + c_3 h_y^{\ast l+\delta l}, \quad (6a)
\]

\[
H_y^{\ast l+\delta l} = c_0 h_y^{\ast l} + c_1 (B_y^{\ast l+\delta l} - B_y^{\ast l}) - c_2 (B_x^{\ast l+\delta l} - B_x^{\ast l}) - c_3 h_x^{\ast l+\delta l}, \quad (6b)
\]

where \( c_i = N_i / \text{Den} \) \((i = 0, \ldots, 3)\) with \( \text{Den} = \mu_0^2 (4 \mu_a^2 + \omega_m^2 \Delta t^2) \), \( N_0 = \mu_0^2 (4 \mu_a^2 - \omega_m^2 \Delta t^2) \), \( N_1 = 4 \mu_0^2 \mu_a \), \( N_2 = -2 \mu_0 \omega_m \Delta t \) and \( N_3 = -4 \mu_0^2 \mu_a \omega_m \Delta t \). In (6) the superscript \( n \) denotes the time \( t = n \Delta t \), and \( \Delta t \) is the time step. Equations (6) are now suitable for incorporation into the FDTD algorithm, allowing structures containing partially magnetized ferrites to be analyzed by the FDTD method.

IV. ANALYSIS OF FERRITE-LOADED WAVEGUIDES

The full-wave analysis of guiding structures can be formulated as a 2-D problem by noticing that for a uniform guide with an arbitrary cross-section, the electromagnetic fields can be expressed as \( F(x,y,z,t) = f(x,z,t) \exp(-j \beta y) \), where \( \beta \) is the propagation constant and \( y \) the direction of propagation. Substituting this expression into (1) and (6), and then discretizing the resulting set of 2D-equations by using central finite differences, we obtain the following difference equations (for simplicity, only TE_{00} modes are considered.)

\[
b_x^{\ast l+\delta l}(i) = j \Delta t \beta e_z^{\ast l}(i) + b_x^{\ast l}(i), \quad (7a)
\]

\[
b_y^{\ast l+\delta l}(i+\Delta y) = \Delta t \left( \frac{e_z^{\ast l+\delta l}(i+\Delta y) - e_z^{\ast l}(i)}{\Delta x} \right) + b_y^{\ast l}(i+\Delta y), \quad (7b)
\]

\[
h_x^{\ast l+\delta l}(i) = c_0 h_x^{\ast l+\delta l}(i) + c_1 \left[ b_y^{\ast l+\delta l}(i) - b_z^{\ast l+\delta l}(i) \right]
+ c_2 \left[ b_y^{\ast l+\delta l}(i) - b_y^{\ast l+\delta l}(i) \right] + c_3 h_y^{\ast l+\delta l}(i),
\]

\[
h_y^{\ast l+\delta l}(i+\Delta y) = c_0 h_y^{\ast l+\delta l}(i+\Delta y) + c_1 \left[ b_y^{\ast l+\delta l}(i+\Delta y) - b_y^{\ast l+\delta l}(i) \right]
- c_2 \left[ b_y^{\ast l+\delta l}(i+\Delta y) - b_y^{\ast l+\delta l}(i) \right] - c_3 h_x^{\ast l+\delta l}(i+\Delta y), \quad (7d)
\]

\[
e_z^{\ast l+1}(i) = \frac{\Delta t}{\epsilon_0} \left( j \beta h_x^{\ast l+\delta l}(i) + \delta y h_z^{\ast l+\delta l}(i+\Delta y) - \delta y h_z^{\ast l+\delta l}(i-\Delta y) \right) + \epsilon_e^{\ast l}(i), \quad (7e)
\]

where the index \( i \) denotes the spatial position \( x = i \Delta x \), and \( \Delta x \) the spatial discretization step. These equations allow the FDTD algorithm to be applied to the analysis of waveguides loaded with partially magnetized ferrites.

The determination of the dispersion characteristics involves selecting a desired value of the phase constant, \( \beta \), calculating the time-domain response and then obtaining the eigenfrequencies of the resonant modes of the cross-section of the waveguide from the spectral analysis of the time-domain response. Each resonant frequency, \( f_i \), corresponds to an excited propagating mode, which has the previously fixed value of \( \beta \) at the frequency \( f_i \). The complete dispersion diagram is obtained by changing the value of \( \beta \) and repeating this process.

A. Stability analysis

To investigate the stability of the difference scheme defined by (7) we adopt the Fourier series or von Neumann's method. This technique has also been used to study the stability of the standard FDTD method [1] and the FDTD method for frequency-dependent materials [5]. According to von Neumann's method, equations (7) have general solutions of the form \( f^\ast (i) = \xi^\ast \exp( j k_i i \Delta x) \), where \( k_i \) is the numerical wavenumber in the \( x \)-direction and \( \xi \) is a complex constant, often called the amplification factor. A necessary and sufficient condition for stability is that \( |\xi| < 1 \).

Substituting the above general solutions into (7) we obtain a homogeneous linear system whose determinant of the coefficient matrix must vanish. This results in the following second-order polynomial equation for \( \xi \):
\((A^2 + 1) \xi^2 + 2(A^2 + 2s^2 - 1) \xi + A^2 + 1 = 0\), \(A = \frac{\omega_m \Delta f}{2 \mu_o}\),

and where \(s\) is given by

\[ s = v \Delta f \left[ \frac{\beta^2 + 4 \sin^2(\theta_s)}{(\Delta x)^2} \right]^{1/2}, \]

with \(v = (\epsilon_0 \mu_0)^{-1}\) and \(\theta_s = k_s \Delta x / 2\).

From (8) we find that, independently of the value of the parameter \(A\), the present scheme is stable for

\[ 0 \leq s \leq 1. \]

In the whole stable range, the solutions of (8) have a modulus whose value is one. Consequently, this scheme is non-dissipative.

The stability condition (11) is the same as for the standard FDTD method for dispersion analysis [6]. Notice that for practical purposes the worst case for \(\sin^2(\theta_s)\), i.e. \(\sin^2(\theta_s) = 1\), is chosen in (10).

### B. Numerical dispersion analysis

As in the standard FDTD method, we substitute general solutions of the form \(\tilde{f}^* (t) = \tilde{f}_0 \exp \left( j \omega_n \Delta t - k_s i \Delta x \right)\) into (7) and again the determinant of the coefficient matrix of the resulting homogeneous system is made to vanish, which gives the following numerical dispersion equation

\[ \frac{\epsilon \mu_{\text{eff}}}{(\Delta t)^2} \sin^2(\theta_s) - \frac{\sin^2(\theta_s)}{(\Delta x)^2} + \frac{\beta^2}{4} = 0, \]

where \(\theta_s = \omega \Delta t / 2\) and \(\mu_{\text{eff}}\) is the numerical effective permeability given by

\[ \mu_{\text{eff}} = \frac{4 \mu^2 - \omega_m^2 \Delta t^2 \cotan^2(\theta_s)}{4 \mu_0}. \]

### V. RESULTS

To illustrate the stability behaviour of the formulation proposed in the preceding section, we show in figure 1 the modulus of the two roots of (8), \(|\xi_1|\) and \(|\xi_2|\), as a function of the stability factor \(s\), and for \(A = 1\). It can be observed that in the range \(0 \leq s \leq 1\), the scheme is stable and non-dissipative, i.e. \(|\xi_1| = |\xi_2| = 1\).

To show the numerical dispersion of the proposed formulation for the analysis of waveguides loaded with partially magnetized ferrites, we have considered the \(TE_{0m}\) mode of a rectangular ferrite-filled waveguide of width \(a\). Assuming a wavenumber \(k_s = \pi / a\) and selecting a value of \(\beta\), we can obtain the corresponding numerical eigen-frequency, \(f_{\text{FDTD}}\), by solving (12) analytically. The numerical dispersion error is then

\[ \epsilon = \frac{f_{\text{FDTD}}(\beta) - f(\beta)}{f(\beta)} \times 100, \]

where \(f\) denotes the exact eigen-frequency. Figure 2 shows the error, \(\epsilon\), as a function of the frequency, \(f\), for various values of the stability factor \(s\). Solid curves are results calculated by solving (12), while crosses are results obtained numerically by the FDTD method. An excellent agreement is observed between them. For this case, the ferrite has a dielectric constant \(\epsilon_r = 12\) and magnetization values of 4\(\pi M = 1000\)G and 4\(\pi M = 2000\)G. For both the numerical and the exact solutions we have taken \(\mu_0 = \mu_r = 0.75\) in the whole frequency range.

![Fig. 1. Modulus of the amplification factor as a function of the stability factor for \(A = 1\).](image)

Figure 3 shows the relative error obtained for the propagation constant of the \(TE_{0m}\) mode of a WR90 waveguide loaded with a centred ferrite slab. In this figure, exact results have been calculated for two cases: the solid curve has been obtained with a frequency-dependent demagnetized permeability; and the dashed curve has been obtained by taking \(\mu_0 = \mu_r = 0.865\) for the whole band, as in the FDTD results. Thus, the solid curve represents the numerical dispersion error plus the error associated with taking a constant value for \(\mu_0\) while the dashed curve corresponds to the numerical dispersion only.

Finally, we have considered a WR90 waveguide loaded with a ferrite slab located asymmetrically. The phase constant of the \(TE_{0m}\) mode for both the forward and backward waves
is shown in figure 4. Three curves have been plotted for each wave: the exact solution obtained with a frequency-dependent \( \mu_x \); the exact solution calculated with \( \mu_1 = \mu_2 = 0.865 \) for the whole frequency band; and the FDTD results obtained with \( \mu_1 = \mu_2 = 0.865 \). An excellent agreement is obtained between the exact and the FDTD solutions when both are calculated with \( \mu_1 = \mu_2 = 0.865 \). The discrepancies observed between the FDTD results and the exact solution calculated by using (4) can be reduced if we use a two-step procedure: for a value of \( \beta \), first an eigen-frequency is calculated by using \( \mu_x \); then this frequency is used to compute \( \mu_y(f) \), and the simulation is performed again with \( \mu_1 = \mu_2(f) \). Another way of improving the results is to include the dispersive nature of \( \mu_x \) in the FDTD algorithm by using the procedure described in [7].

![Figure 2: Relative dispersion error for the phase constant of the TE_{10} mode of a ferrite-filled waveguide with a width of 22.86 mm, for various values of the stability factor \( s = 1 \), \( s = 0.75 \), \( s = 0.5 \), and \( s = 0.25 \).](image)

![Figure 3: Relative error for the phase constant of the TE_{10} mode of a WR90 waveguide loaded with a centred ferrite slab. Ferrite data as in fig. 2. \( w = a/9 \) and \( \Delta x = a/27 \).](image)

**V. CONCLUSIONS**

The FDTD method has been extended for the treatment of partially magnetized ferrites. The approach used consists in expressing the medium equations of a partially magnetized ferrite by a system of first-order differential equations obtained from the commonly-used permeability-tensor-based constitutive relations. These differential equations have been discretized by using a scheme based on central differences and linear interpolation. The resulting formulation is very similar to that previously proposed for saturated ferrites. Thus, existing FDTD codes for the analysis of structures containing saturated ferrites can be easily adapted to partially magnetized ferrites, increasing the power of these CAD tools.

**REFERENCES**