On the Muskat problem on a strip: Well-posedness and global properties*

Rafael Granero-Belinchón†
r.granero@icmat.es.

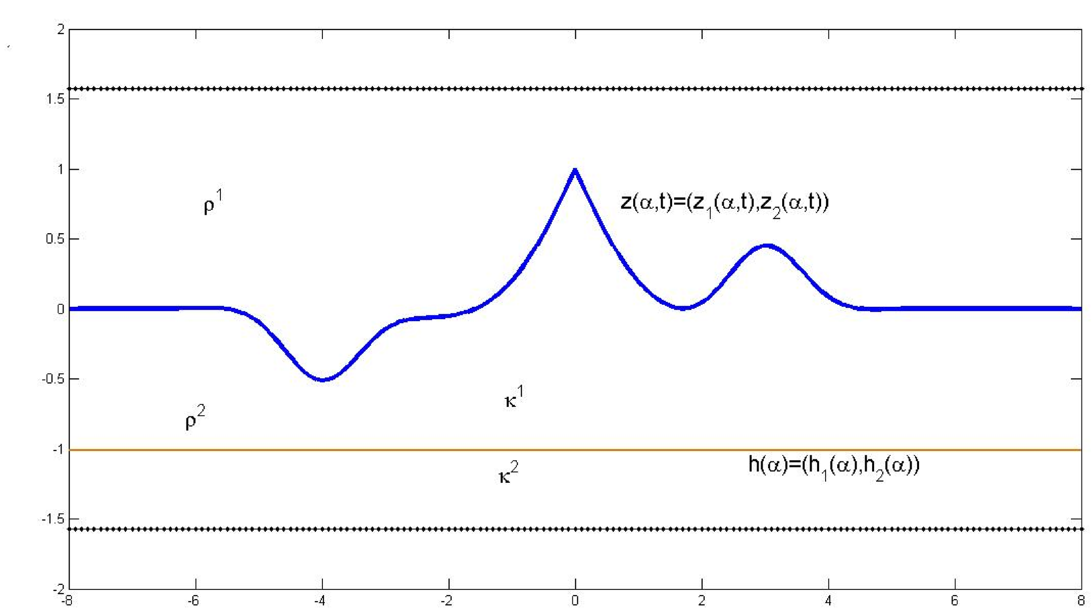
Abstract: We study the problem of the evolution of the interface given by two incompressible fluids with different densities in the porous strip $\mathbb{R}\times(-l,l)$ and where permeability can constant or a step function. This problem is known as the inhomogeneus Muskat problem. In this poster we state some of our main results.

1. The problem:

We consider the two-dimensional flat strip $S=\mathbb{R}\times(-l,l)\subset\mathbb{R}^2$ with $l>0$ as a porous medium. In this porous strip we have two immiscible and incompressible fluids with the same viscosity and different densities, ρ^1 in $S^1(t)$ and ρ^2 in $S^2(t)$, where $S^i(t)$ denotes the domain occupied by the i−th fluid *i.e.* $S^1(t)=\{(x,y)\in S:\ y>f(x,t)\}$ and $S^2(t)=S-S^1(t)$. The curve $z(x,t)=\{(x,f(x,t)):\ x\in\mathbb{R}\}$ is the interface between the fluids. We suppose that initially $|f_0(x)|<l$ for all x. We have the following system of equations

$$\begin{cases}\frac{v(x,y,t)}{\kappa(x,y)}=-\nabla p(x,y,t)-\rho(0,1)^t & \text{in } S,\ t>0,\ \text{(Darcy's Law)}\\ \nabla\cdot v(x,y,t)=0 & \text{in } S,\ t>0,\ \text{(Incompressibility)}\\ \partial_t\rho(x,y,t)+v\cdot\nabla\rho(x,y,t)=0 & \text{in } S,\ t>0,\ \text{(Mass conservation)}\\ f(x,0)=f_0(x) & \text{in } \mathbb{R},\end{cases}\tag{1}$$

with **impermeable boundary conditions for the velocity**.



In what follows $\kappa(x,y)\equiv 1$ or $\kappa(x,y)=\kappa^1\mathbf{1}_{-h_2<y<l}+\kappa^2\mathbf{1}_{-l<y<-h_2}$, $\kappa^i>0$. To simplify notation we take $\mathcal{K}=\frac{\kappa^1-\kappa^2}{\kappa^1+\kappa^2}$, $\kappa^1=1$, $l=\pi/2$ and $\rho^2-\rho^1=4\pi$. This problem can be formulated as an equation for the interface $f(x,t)$.

$$\partial_t f(x)=\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)-\cos(f(x)-f(\beta))}+\frac{(\partial_x f(x)+\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)+\cos(f(x)+f(\beta))}d\beta$$
$$+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_2(\beta)\left(\frac{(\sinh(x-\beta)+\partial_x f(x)\sin(f(x)+h_2))}{\cosh(x-\beta)-\cos(f(x)+h_2)}+\frac{(-\sinh(x-\beta)+\partial_x f(x)\sin(f(x)-h_2))}{\cosh(x-\beta)+\cos(f(x)-h_2)}\right)d\beta$$

with

$$\varpi_2(x)=2\mathcal{K}\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}+\frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta$$
$$+2\frac{\mathcal{K}^2}{\sqrt{2\pi}}G_{h_2,\mathcal{K}}*\left(\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}-\frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta\right).\tag{2}$$

where $G_{h_2,\mathcal{K}}$ is a (not explicit) Schwartz function.

The Rayleigh-Taylor condition is defined as

$$RT(x,t)=-(\nabla p^2(z(x,t))-\nabla p^1(z(x,t)))\cdot\partial_x^\perp z(x,t)$$

Due to the incompressiblity of the fluids the Rayleigh-Taylor condition reduces to the sign of the jump in the density. **The condition to be RT stable is** $RT=(\rho^2-\rho^1)>0$**.** This condition is satisfied if the denser fluid is below.

2. On existence and non-existence of solutions

We define the space $H^k_{l,h_2}=\{f\in H^k,\ \|f\|_{L^\infty}<l,\min_x f(x,0)>-h_2\}$.

Theorem (Well-posedness in the RT unstable regime): If the Rayleigh-Taylor condition is satisfied and the initial data $f_0(x)=f(x,0)\in H^k_{l,h_2}(\mathbb{R})$, $k\geq 3$, then, for any $\mathcal{K}$, there exists an unique solution of (2) with $f\in C([0,T],H^k_{l,h_2})$ and $T=T(\|f_0\|_{H^k},\|f_0\|_{L^\infty})$. Moreover, if $\mathcal{K}=0$, the solution can be continued analytically in the complex strip $\mathbb{B}$

$$\mathbb{B}=\{x+i\xi,\ |\xi|<k(f_0)t\}.$$

Theorem (Ill-posedness in the RT unstable regime): There exists a solution $\tilde f$ of (2) with $\rho^2<\rho^1$, $\mathcal{K}=0$ such that $\|\tilde f_0\|_{H^s(\mathbb{R})}<\epsilon$ and $\|\tilde f(\delta)\|_{H^s(\mathbb{R})}=\infty$, for any $s\geq 4$, $\epsilon>0$ and small enough $\delta>0$.

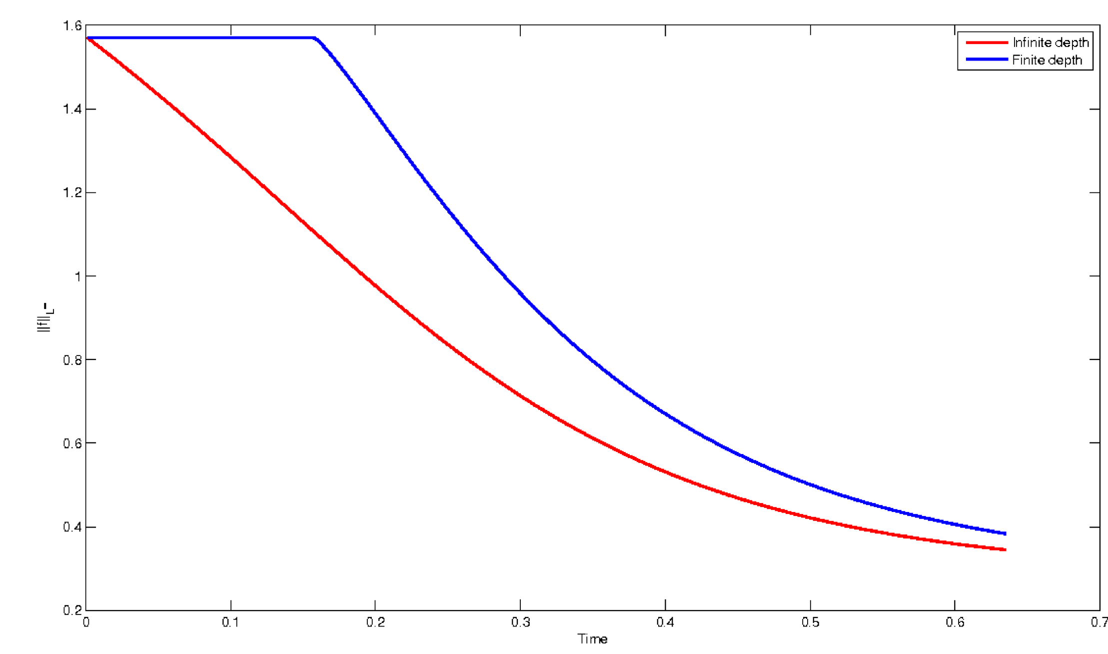
The key point in this result is that we do not require a family of global solutions for small initial data (compare with [1]).

3. Maximum principles

Theorem (Maximum principle for $\|f\|_{L^\infty}$): Let f be the unique classical solution of in the RT stable regime with $\mathcal{K}=0$. Then, f satisfies that $\|f(t)\|_{L^\infty}\leq\|f_0\|_{L^\infty}$. Moreove $f_0\in L^1\cap H^k_{l,0}(\mathbb{R})$, then the L^∞ norm satisfies the following inequality

$$\frac{d}{dt}\|f(t)\|_{L^\infty}\leq -c(\|f_0\|_{L^1},\|f_0\|_{L^\infty},\rho^2,\rho^1,l)\exp\left(-\frac{\pi\|f_0\|_{L^1}}{l\|f(t)\|_{L^\infty}}\right).$$

This result shows a big difference, when compared with the deep water case, in the decay of the amplit the solutions.



Numerical simulation showing the different decay of the solutions.

Theorem (Maximum Principle for $\|f\|_{L^2}$): Let f be the unique classical solution of in the RT stable regime with $\mathcal{K}=0$. Then, f satisfies the following energy balance

$$\|f(t)\|^2_{L^2(\mathbb{R})}+\frac{2}{\rho^2-\rho^1}\int_0^t\|v(s)\|^2_{L^2(S)}ds=\|f_0\|^2_{L^2(\mathbb{R})}.$$

Theorem (Maximum Principle for $\|\partial_x f\|_{L^\infty}$): Let f_0 be a smooth initial data in the stable regime with $\mathcal{K}=0$ such that the following conditions holds:

$$\|\partial_x f_0\|_{L^\infty}\leq 1,$$

$$\tan\left(\frac{\pi\|f_0\|_{L^\infty}}{2l}\right)\leq\|\partial_x f_0\|_{L^\infty}\tanh\left(\frac{\pi}{4l}\right),$$

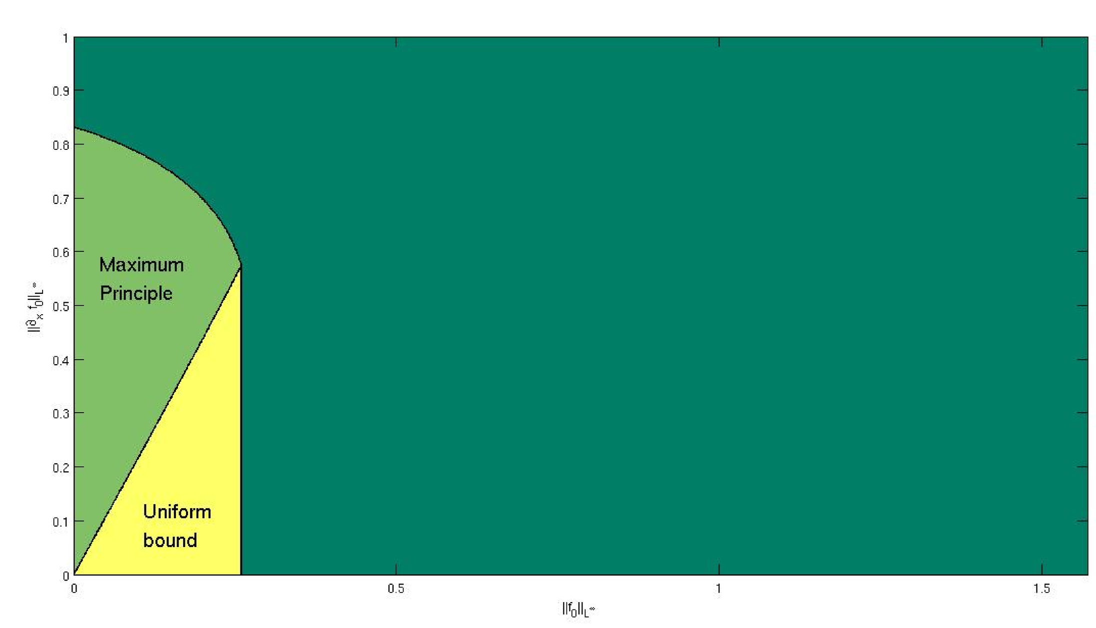
and

$$F(\|\partial_x f_0\|_{L^\infty},\|f_0\|_{L^\infty},l)\leq 0,$$

where F is explicit but intricate. Then $\|\partial_x f(t)\|_{L^\infty}\leq\|\partial_x f_0\|_{L^\infty}$.

This result is very intriguing because **in the deep water regime ($l=\infty$) the unique condition is (3) an the condition is not only on the size of the slope but also on the amplitude of the internal wave.**

The region in the $(\|f_0\|_{L^\infty},\|\partial_x f_0\|_{L^\infty})$ reference set:



References

[1] D. Córdoba and F. Gancedo. Contour dynamics of incompressible 3-D fluids in a porous mediu different densities. *Communications in Mathematical Physics*, 273(2):445–471, 2007.

[2] D. Córdoba, R. Granero-Belinchón and R. Orive, *The confined Muskat problem: differences w deep water regime.* arXiv:1209.1575 [math.AP]

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†Consejo Superior de Investigaciones Científicas, Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Madrid, Spain.