On the Muskat problem on a strip: Well-posedness and global properties*

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Abstract: We study the fluid problem of the evolution of the interface given by two incompressible fluids with different densities in the porous strip $\mathbb{R}\times[-l,l]$. This problem is known as the Muskat problem and is analogous to the two phase Hele-Shaw cell. In this poster we state our main results.

1. The problem:

We consider the two-dimensional flat strip $S=\mathbb{R}\times(-l,l)\subset\mathbb{R}^2$ with $l>0$ as a porous medium. In this porous strip we have two immiscible and incompressible fluids with the same viscosity and different densities, ρ^1 in $S^1(t)$ and ρ^2 in $S^2(t)$, where $S^i(t)$ denotes the domain occupied by the $i-$th fluid *i.e.* $S^1(t)=\{(x,y)\in S:\ y>f(x,t)\}$ and $S^2(t)=S-S^1(t)$. The curve

$$z(x,t)=\{(x,f(x,t)):\ x\in\mathbb{R}\}$$

is the interface between the fluids. We suppose that initially $|f_0(x)|<l$ for all x. We have the following system of equations

$$\begin{cases}\frac{\mu}{\kappa}v(x,y,t)=-\nabla p(x,y,t)-g\rho e_2 \quad \text{in } S,\ t>0,\ \text{(Darcy's Law)}\\ \nabla\cdot v(x,y,t)=0 \quad \text{in } S,\ t>0,\ \text{(Incompressibility)}\\ \partial_t\rho(x,y,t)+v\cdot\nabla\rho(x,y,t)=0 \quad \text{in } S,\ t>0,\ \text{(Mass conservation)}\\ f(x,0)=f_0(x) \quad \text{in } \mathbb{R},\end{cases} \tag{1}$$

with impermeable boundary conditions for the velocity.
This problem can be formulated as an equation for the interface $f(x,t)$. Indeed, we obtain the following equation for the evolution of the interface in our bounded domain S

$$\partial_t f(x,t)=\frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\left[\frac{(\partial_x f(x)-\partial_x f(x-\eta))\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)-\cos(\frac{\pi}{2l}(f(x)-f(x-\eta)))}+\frac{(\partial_x f(x)+\partial_x f(x-\eta))\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)+\cos(\frac{\pi}{2l}(f(x)+f(x-\eta)))}\right]d\eta. \tag{2}$$

The second term becames singular when f reaches the boundaries.

2. The Rayleigh-Taylor condition

The Rayleigh-Taylor condition is defined as

$$RT(x,t)=-(\nabla p^2(z(x,t))-\nabla p^1(z(x,t)))\cdot\partial_x^{\perp}z(x,t)$$

Due to the incompressiblity of the fluids the Rayleigh-Taylor condition reduces to the sign of the jump in the density. If $RT=g(\rho^2-\rho^1)>0$ we say that the Rayleigh-Taylor condition is satisfied. This condition is satisfied if the densest fluid is below.

3. Well-Posedness

In order to ensure the local existence of classical solution we need that the Rayleigh-Taylor condition is satisfied. Indeed, we have the following result:

Theorem: If the Rayleigh-Taylor condition is satisfied and the initial data $f_0(x)=f(x,0)\in H^k_l(\mathbb{R})$, $k\geq 3$, then there exists an unique solution of (2) with $f\in C^1([0,T],H^k_l)$ and $T=T(||f_0||_{H^k},||f_0||_{L^\infty})$.

where the space is defined as $H^3_l=\{f\in H^3,\ ||f||_{L^\infty}<l\}$.
Sketch of the proof: In order to proof the well-posedness we use the energy method. We assume $l=\pi/2$ and we obtain some *a priori* estimates for the following energy

$$E[f](t)=||f||^2_{H^3}(t)+||d[f]||_{L^\infty}(t),$$

where $d[f]:\mathbb{R}^2\times\mathbb{R}^+\mapsto\mathbb{R}^+$ is defined as

$$d[f](x,\eta,t)=\frac{1}{\cosh(\eta)+\cos(f(x)+f(x-\eta))}.$$

We observe that $||d[f]||_{L^\infty}<\infty$ implies that $||f||_{L^\infty}<\frac{\pi}{2}$. So this is the natural *'energy'* associated to the space $H^3_l(\mathbb{R})$. This definition of $d[f]$ is *'ad hoc'* to the equation (2).

4. Maximum Principle for $||f||_{L^\infty}$

We have

Theorem: Let f be the unique classical solution of (2) under the assumptions of the well-posedness Theorem. Then, f satisfies that

$$||f||_{L^\infty}(t)\leq||f_0||_{L^\infty}.$$

Sketch of the proof: The proof relies in the smoothness of the solution $f(x,t)$. This regularity gives us the Lipschitz condition for the maximum (or minimum) value, $\max_x f(x)$ or $\min_x f(x)$. Then using Rademacher Theorem we can obtain the (ordinary differential) equation for $f(x_t)=||f||_{L^\infty}(t)$, and thus we conclude the proof.

5. Decay Rate for $||f||_{L^\infty}$

Theorem: Let $f_0\geq 0$, $f_0\in L^1\cap H^k_l(\mathbb{R})$. Then the L^∞ norm satisfies the following inequality

$$\frac{d}{dt}||f||_{L^\infty}(t)\leq -c(||f_0||_{L^1},||f_0||_{L^\infty},\rho^2,\rho^1,l)e^{-\frac{\pi||f_0||_{L^1}}{l||f||_{L^\infty}(t)}},$$

Sketch of the proof: We have (see the Maximum Principle Theorem)

$$\frac{d}{dt}f(x_t)=-\int_{\mathbb{R}}\frac{1}{\cosh^2(\eta/2)}\Pi(x,\eta,t)d\eta,$$

with

$$\Pi\geq\frac{\tan(\theta)}{(\tan^2(||f_0||_{L^\infty})+1)^2+\tan^2(||f_0||_{L^\infty})}\frac{1}{1+\tan^2(||f_0||_{L^\infty})},$$

where $2\theta=f(x_t)-f(x_t-\eta)$. Fix the interval $[-r,r]$. We consider the sets

$$\mathcal{U}_1=\left\{\eta:\eta\in[-r,r],\theta\geq\frac{f(x_t)}{4}\right\}\ \text{ and }\ \mathcal{U}_2=\left\{\eta:\eta\in[-r,r],\theta<\frac{f(x_t)}{4}\right\}.$$

The Conservation of the total mass gives us a control for the measure of these sets. now we use the set $\mathcal{U}_1$ to bound the integral and the set $\mathcal{U}_2$ to control the measure of the previous set. Then we choose $r=2\frac{||f_0||_{L^1}}{f(x_t)}$.

6. Maximum Principle for $||\partial_x f||_{L^\infty}$

Theorem: Let f_0 be a smooth initial data such that the following conditions holds:

$$\tan\left(\frac{\pi||f_0||_{L^\infty}}{2l}\right)\leq||\partial_x f_0||_{L^\infty}\tanh\left(\frac{\pi}{4l}\right),$$
$$||\partial_x f_0||_{L^\infty}\leq 1,$$

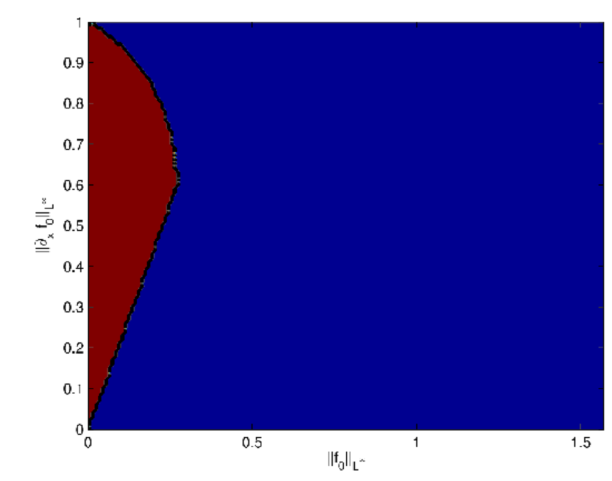
and

$$4\tan\left(\frac{\pi||f_0||_{L^\infty}}{2l}\right)+4||\partial_x f_0||_{L^\infty}+\left(\frac{||\partial_x f_0||_{L^\infty}}{3}+||\partial_x f_0||^3_{L^\infty}+\frac{2}{3}||\partial_x f_0||^5_{L^\infty}\right)\frac{\pi}{l}-8||\partial_x f_0||_{L^\infty}\cos\left(\frac{\pi||f_0||_{L^\infty}}{2l}\right)\leq 0,$$

then

$$||\partial_x f||_{L^\infty}(t)\leq||\partial_x f_0||_{L^\infty}.$$

Sketch of the proof: The proof is similar to the Maximum Principle for $||f||_{L^\infty}$.
The region in the $(||f_0||_{L^\infty},||\partial_x f_0||_{L^\infty})$ reference set:



7. Other Properties

Some other properties for this problem can be shown. For instance we can show that if we take the limit $l\to\infty$ we recover the equation in the full space (see [1]):

$$\partial_t f=\frac{\rho^2-\rho^1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(x-\eta))\eta}{\eta^2+(f(x)-f(x-\eta))^2}d\eta. \tag{3}$$

We also prove that ρ defined as before is a weak solution of the conservation of mass equation present in (1) if and only if the interface verifies the equation (2).

References

[1] D. Córdoba, F. Gancedo, *Contour dynamics of incompressible 3-D fluids in a porous medium with different densities.* Comm. Math. Phys. 273 (2007), no. 2, 445–471.

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