The inhomogeneous Muskat problem

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Resumen

En esta tesis estudiamos la dinámica de una interfase separando dos fluidos incompresibles con diferentes densidades en un medio poroso heterogéneo.

Primero consideramos una banda porosa con anchura igual a $2l$, *i.e.* el dominio es $S = \mathbb{R} \times (-l, l)$. La permeabilidad del medio poroso se anula fuera de estas paredes y es idénticamente uno dentro, es decir, la región S es homogénea y sus fronteras son impermeables. Este caso se conoce como el problema de Muskat confinado y homogéneo. En el Capítulo 3 obtenemos la ecuación explícita para la interfase en este modelo y otras propiedades, por ejemplo, estudiamos la convergencia del modelo confinado al modelo no acotado cuando la profundidad tiende a infinito. Además, en la primera parte de la tesis, para este problema, probamos:

- Existencia local en espacios de Sobolev cuando la condición de Rayleigh-Taylor se satisface, es decir, cuando el fluido más denso está encima del menos denso (ver Capítulo 4).
- Existencia de solución en cierto espacio de funciones analíticas usando un Teorema de Cauchy-Kowalevski. Como consecuencia obtenemos que la ecuación para la interfase tiene solución local única incluso si no se satisface la condición de Rayleigh-Taylor (ver Capítulo 5).

Uno de los objetivos principales de esta investigación es estudiar las diferencias entre el caso con profundidad infinita y el caso de un medio poroso acotado. Para hacer esto, en la segunda parte de esta tesis, estudiamos varias propiedades cualitativas de las soluciones (ver Capítulo 7). En particular probamos

- Si la condición de Rayleigh-Taylor se satisface entonces la solución se vuelve analítica para todo tiempo positivo (ver Capítulo 6).
- En el caso inestable, el problema está mal propuesto en espacios de Sobolev (ver Capítulo 6). Un aspecto importante de este Teorema es que no requiere de una familia de soluciones con existencia global para ser probado (comparar con [23] y [54]).
- Una ley de balance de energía para $\|f(t)\|_{L^2(\mathbb{R})}$, donde $(x, f(x,t))$ es la interfase.
- Un principio del máximo para la amplitud, $\|f(t)\|_{L^\infty(\mathbb{R})}$.
- Una estimación del decaimiento para la amplitud $\|f(t)\|_{L^\infty(\mathbb{R})}$ en un medio acotado. Este decaimiento es mucho más lento que en el caso en el que la profundidad es infinita. Esto es así porque la condición de frontera natural para la velocidad, $v \cdot n = 0$, implica que, si la interfase está cerca de la pared, la evolución del máximo es muy lenta. Como consecuencia obtenemos que la única solución estacionaria es la idénticamente nula. Este decaimiento se estudia numericamente en la Sección 13.2.
- Un principio del máximo para para la pendiente, $\|\partial_x f\|_{L^\infty(\mathbb{R})}$, si el dato inicial, f_0, pertenece a una determinada región del plano $(\|f_0\|_{L^\infty(\mathbb{R})}, \|\partial_x f_0\|_{L^\infty(\mathbb{R})})$. Aquí el efecto de las paredes es muy importante y las hipótesis que debemos hacer sobre el dato inicial son mucho mas restrictivas que en el caso de *aguas profundas*, es decir, el caso con profundidad infinita. Así

nuestro resultado nos da condiciones sobre el tamaño de $\|f_0\|_{L^\infty(\mathbb{R})}$ y $\|\partial_x f_0\|_{L^\infty(\mathbb{R})}$ comparados con la profundidad. Es decir, si nuestro dato inicial está en lo que se llama *régimen de onda larga* (amplitud pequeña y longitud de onda larga), entonces no hay *giro*, *i.e.* la pendiente no crece indefinidamente. Observamos que si tomamos el límite $l \to \infty$ en estas condiciones recuperamos el resultado para aguas profundas contenido en [24].

- Una cota uniforme para la pendiente, $\|\partial_x f\|_{L^\infty(\mathbb{R})}$, para datos iniciales pertenecientes a una segunda región del plano $(\|f_0\|_{L^\infty(\mathbb{R})}, \|\partial_x f_0\|_{L^\infty(\mathbb{R})})$.
- En el Capítulo 8 mostramos, sin ninguna hipótesis soble la condición de Rayleigh-Taylor, la existencia de solución para datos iniciales que son curvas analíticas arbitrarias.
- En el Capítulo 8 probamos existencia de singularidades. Dichas singularidades son explosiones de la pendiente máxima, $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$. Físicamente estas singularidades significan que algunas curvas *'giran'*. Es más, nosotros comparamos este resultado con su análogo para el caso de aguas profundas (ver [10]). Para ello primero obtenemos evidencia numérica (ver Sección 13.2 en el Capítulo 13) y despues demostramos con una prueba asistida por ordenador (ver Sección 13.4 del mismo capítulo) que existen datos iniciales tales que, en tiempo finito, la solución de (2.9) pasa al régimen inestable sólo cuando la profundidad es finita. Si la profundidad es infinita las mismas curvas se vuelven grafos (ver Evidencia Numérica 13.1 y Teorema 13.1). Este resultado también se aplica al problema de las *water waves* (ver Corolario 13.1).
- Para datos iniciales Lipschitz satisfaciendo ciertas hipótesis que relacionan la pendiente, la amplitud y la profundidad, existe una solución Lipschitz global en tiempo (ver Capítulo 9).

Tras este estudio, en la tercera parte de la tesis, presentamos el problema de Muskat inhomogéneo en diferentes dominios: $S = \mathbb{R}^2, \mathbb{T}\times\mathbb{R}$ (profundidad infinita) y $\mathbb{R}\times(-\pi/2, \pi/2)$ (profundidad finita). Dentro de estos dominios tenemos una línea recta (que es conocida y fija) $h(\alpha) = \{(\alpha, -h_2) : \alpha \in \mathbb{R}\}$, que separa dos regiones con distintos valores de la permeabilidad (ver Figura 2.4). Para estos problemas, obtenemos explícitamente las ecuaciones para la interfase (ver Capítulo 10) y mostramos:

- La existencia local de solución en espacios de Sobolev (ver Capítulo 11).
- Una ley de balance de energía para la norma L^2 dependiendo de las permeabilidades. Esta ley generaliza el balance de energía obtenido anteriormente para el caso homogéneo y confinado (ver Capítulo 11).
- En el Capítulo 12 mostramos la existencia singularidades en tiempo finito para las interfases cuando los parámetros físicos están en una cierta región.

En la última parte de la tesis, el Capítulo 13, mostramos unas simulaciones y obtenemos evidencia numérica que muestra que realmente las ecuaciones de la interfase presentan singularidades para cualquier valor de los parámetros. Finalmente, dicha evidencia numérica se demuestra rigurosamente con una prueba asistida por ordenador en la Sección 13.4 (ver Teorema 13.2).

Abstract

In this thesis the dynamics of an interface between two incompressible fluids with different densities in a heterogeneous porous medium is addressed.

First, we consider a porous strip with width $2l$, *i.e.* the domain is $S = \mathbb{R} \times (-l, l)$. The permeability vanishes outside and it is identically one inside, *i.e.* the region S is homogeneous and its boundaries are impervious. This problem is known as the confined and homogeneous Muskat problem. In Chapter 3 we obtain explicitly the equation for the interface in this model and some other properties, for instance, we study the convergence of the confined model towards the unconfined one when the depth tends to infinity. In the first part of this thesis, for this problem, we prove:

- Local existence in Sobolev spaces when the Rayleigh-Taylor condition is satisfied *i.e.* when the denser fluid is above the lighter one (see Chapter 4).
- Local existence of solutions in some space of analytic functions using a Cauchy-Kowalevski Theorem. As a consequence we obtain that the evolution equation is locally well-posed even in the Rayleigh-Taylor unstable case (see Chapter 5).

One of the main goals of this research is to study the differences between the case with infinite depth and the confined case. Thus, in the second part of this thesis, we study some qualitative properties of the solutions (see Chapter 7). In particular:

- We prove that in the Rayleigh-Taylor stable case the solution becomes analytic for every positive time (see Chapter 6).
- On the other hand, in the unstable case, the problem is ill-posed in Sobolev spaces (see Chapter 6). The key point of this result is that we do not need global existence for some class of solutions to prove the result (compare with [23] and [54]).
- An energy balance for $\|f(t)\|_{L^2(\mathbb{R})}$, where $(x, f(x,t))$ is the interface.
- A maximum principle for the amplitude, $\|f(t)\|_{L^\infty(\mathbb{R})}$.
- A decay estimate for the amplitude, $\|f(t)\|_{L^\infty(\mathbb{R})}$, in a confined porous medium. Let us observe that the natural boundary condition for the velocity, $v \cdot n = 0$, imposes that if our initial interface is close enough to the boundary the evolution of the maximum is very slow. Due to this fact, we obtain the a decay slower than the decay in the infinitely deep case. As a corollary we conclude that the unique stationary solution is the rest state. This decay rate is studied numerically in Section 13.2.
- A maximum principle for the slope, $\|\partial_x f\|_{L^\infty(\mathbb{R})}$, if the initial datum belongs to a known region in the plane $(\|f_0\|_{L^\infty(\mathbb{R})}, \|\partial_x f_0\|_{L^\infty(\mathbb{R})})$. Here the effect of the boundaries is very important in such a way the conditions that we obtain are much more restrictives than in the deep water regime (the case with infinite depth). Our result gives us conditions on the smallness of $\|f_0\|_{L^\infty(\mathbb{R})}$, $\|\partial_x f_0\|_{L^\infty(\mathbb{R})}$ compared with the depth. So, roughly speaking, the Theorem says that if we are in the long wave regime (small amplitude and large wavelength)

then there is no *turning effect*, *i.e.* the slope does not blow up. We remark that if we take the limit $l \to \infty$ we recover the result for the deep water regime contained in [24].

- An uniform bound for the slope, $\|\partial_x f\|_{L^\infty(\mathbb{R})}$, for initial data in a second region in the plane $(\|f_0\|_{L^\infty(\mathbb{R})}, \|\partial_x f_0\|_{L^\infty(\mathbb{R})})$.
- In Chapter 8 we prove the existence of solutions for initial data, not necessarily in the stable case, which are arbitrary analytic curves.
- In Chapter 8 we also prove the existence of singularities. These singularities are blow-ups of the slope, $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$. Physically, these singularities mean that some curves *'turn over'*. Moreover, we can compare this result with its analogous one in the deep water case (see [10]). To do this, we first obtain numerical evidence (see Section 13.2 in Chapter 13) and then we prove with a computer assisted proof (see Section 13.4) the existence of initial data such that, in finite time, the solution of (2.9) reach the unstable regime only if the depth is finite. If the depth is infinite the same curves become graphs (see Numerical Evidence 13.1 and Theorem 13.1). This result also applies for the *water waves* problem (see Corollary 13.1).
- We prove that, for Lipschitz continuous initial data satisfying some conditions related to the amplitude, slope and depth, there are global in time Lipschitz continuous solutions (see Chapter 9).

In the third part of the thesis, we present the inhomogeneous Muskat problem posed in different domains: $S = \mathbb{R}^2, \mathbb{T} \times \mathbb{R}$ (infinitely deep) y $\mathbb{R} \times (-\pi/2, \pi/2)$ (finite depth). In this domains we have a straight line (known and fixed) $h(\alpha) = \{(\alpha, -h_2) : \alpha \in \mathbb{R}\}$, separating two regions with different permeabilities (see Figure 2.4). For these problems, we obtain explicitly the evolution equations (see Chapter 10) and we prove:

- Local existence in Sobolev spaces (see Chapter 11).
- An energy balance for the L^2 norm depending on the permeabilities. This energy balance is more general than the energy balance for the homogeneous and confined Muskat problem (see Chapter 11).
- In Chapter 12 we prove the existence of finite time singularities for the interfaces when the physical parameters are in some region.

In the last part of the thesis, Chapter 13, we perform some numerical simulations and we obtain numerical evidence showing that the evolution equation for the inhomogeneous Muskat problem present singularities for any set of parameters. Finally, this numerical evidence is rigorously proved with a computer assisted proof in Section 13.4 (see Theorem 13.2).

"It is a well-known experience that the only truly enjoyable and profitable way of studying mathematics is the method of "filling in details" by one's own efforts."
'Applied Analysis',
Cornelius Lanczos.

Chapter 1

Introducción, resultados y conclusiones

1.1 Motivación

Uno de los *problemas del Milenio* que quedan por resolver es el de la existencia y unicidad global de solución clásica para la ecuación de Navier-Stokes tridimensional. Resolver esta cuestión tiene un premio de un millón de dólares, además del reconocimiento que seguiría al anuncio. Relacionado con este problema está el de la existencia y unicidad de soluciones clásicas globales en tiempo para las ecuaciones de Euler tridimensional.

Las ecuaciones de Euler (Leonhard Euler, 1707-1783) para un fluido con densidad ρ en d dimensiones, $d \geq 2$, son

$$\begin{aligned}
\overbrace{\rho}^{\text{Masa}} \overbrace{(\partial_t u + (u \cdot \nabla) u)}^{\text{Aceleración}} &= \overbrace{-\nabla p}^{\text{Fuerzas internas}} \quad \text{(Conservación del momento)},\\
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \quad \text{(Conservación de la masa)}
\end{aligned}$$

donde $\nabla = (\partial_{x_1}, ..., \partial_{x_d})$ es el gradiente, $u = (u_1, u_2, ...u_d)$ es el campo de velocidades del fluido y el escalar p es la presión. Esta función refleja *fuerzas internas* entre las partículas, que intentan empujar y apartar a sus vecinas. Es la segunda ley de Newton en el caso de un continuo de partículas. Nos sobra una incógnita para poder cerrar el sistema, así que añadimos la hipótesis de incompresibilidad del fluido

$$\nabla \cdot u = 0, \quad \text{(Incompresibilidad)}.$$

Esta última condición de tener divergencia cero significa que se preserva el volumen del fluido, y matemáticamente convierte las ecuaciones de Navier-Stokes y de Euler en ecuaciones integro-diferenciales. Esto es así porque la presión es un operador integral cuadrático en u. Para verlo basta con tomar formalmente la divergencia de la ecuación de conservación del momento de donde resulta

$$-\Delta p = \nabla \cdot (u \cdot \nabla) u.$$

Una cantidad clave en el estudio de este tipo de fluidos es la *'vorticidad'*

$$\omega = \text{curl } u.$$

Esta cantidad nos da una idea de cuánto giran las partículas del fluido. En dos dimensiones la ecuación de Euler escrita en la formulación de la vorticidad (que ahora es un escalar) es

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & x \in \mathbb{R}^2, t \in \mathbb{R}^+ \\ u = \nabla^{\perp}(\Delta)^{-1}\omega. \end{cases}$$

Un problema que aparece al considerar esta formulación es el del *'vortex patch'*. En él se considera el problema de la evolución de un *'parche'* de vorticidad, es decir, una región del plano donde la vorticidad tiene un valor constante ω_0, mientras que en el resto del plano la vorticidad es idénticamente nula:

$$\omega(0, x) = \omega_0 \mathbf{1}_{\Omega_0}.$$

Este problema atrajo mucha atención hace años porque se propuso como una explicación de los flujos turbulentos, es decir, cuando las propiedades de dicho flujo cambian de manera estocástica y caótica (ver [43]). Del problema de entender la turbulencia, Richard Feynman dijo que era *el problema sin resolver más importante de la física clásica.*

Para entender mejor la ecuación de la vorticidad en tres dimensiones se investigan los *'escalares activos'*. Estos modelos son el transporte de un escalar de manera que podemos recuperar el fluido del propio escalar. Como ejemplo tenemos la ecuación quasi-geostrófica bidimensional (SQG) (ver [14] y [15])

$$[\text{SQG}] \begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, & x \in \mathbb{R}^2, t \in \mathbb{R}^+ \\ u = (-R_2, R_1)\theta, \end{cases}$$

donde θ es un escalar y R_i es la transformada de Riesz $i-$ésima (ver [55]).

Matemáticamente la ecuación SQG es importante porque tiene muchas propiedades parecidas a la ecuación de Euler 3D escrita en su formulación para la vorticidad (ver [45]). Por ejemplo ambas tienen criterios de existencia de solución clásica parecidos. En efecto, para Euler (o Navier-Stokes) tenemos el famoso *criterio de Beale-Kato-Majda* (ver [3]) que afirma que si

$$\int_0^T \|\omega\|_{L^\infty} < \infty,$$

entonces existe solución clásica hasta tiempo T. También se sabe que si se forma una singularidad la integral anterior se hace infinita. Para SQG la cantidad análoga es

$$\int_0^T \|\nabla^{\perp}\theta\|_{L^\infty},$$

y se tiene el mismo resultado.
Ahora, si tomamos el operador $\nabla^{\perp}$ formalmente en la ecuación SQG, obtenemos

$$\partial_t \nabla^{\perp}\theta + (u \cdot \nabla)\nabla^{\perp}\theta = (\nabla^{\perp}\theta \cdot \nabla)u, \quad x \in \mathbb{R}^2, t \in \mathbb{R}^+,$$

que guarda gran semejanza con la ecuación para la vorticidad en Euler 3D,

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \quad x \in \mathbb{R}^3, t \in \mathbb{R}^+.$$

Además del interés puramente matemático que ya hemos mencionado, la ecuación SQG modela la evolución de la temperatura en la atmósfera, y es muy estudiada como modelo de la *'frontogénesis'* (la formación de frentes de aire a distinta temperatura). Por lo tanto el modelo análogo al vortex patch para el caso de SQG es el estudio de la evolución de θ donde el dato inicial es

$$\theta(0, x) = \theta_1 \mathbf{1}_{\Omega_1} + \theta_2 \mathbf{1}_{\mathbb{R}^2 - \Omega_1}.$$

El lector interesado puede consultar [31].

Otro problema de gran importancia es el estudio de las interfases entre fluidos. Por ejemplo podemos pensar en las olas en el mar o el caso, antes mencionado, de un frente donde se encuentran masas de aire a distintas temperaturas. Así, el problema de las *'water waves'* (ver [6], [8], [9], [10], [21], [41] y [42],) u olas en el mar consiste en estudiar la interfase entre el vacío, un fluido cuya densidad se anula, y el agua bajo las hipótesis de que el agua es no viscosa e irrotacional fuera de la ola, de modo que la vorticidad se concentra en la interfase. El modelo matemático de las *'water waves'* es entonces

$$\text{[Water waves]}\begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0, \ \text{(Conservación de la masa)} \\ \nabla\cdot u = 0, \ \text{(Incompresibilidad)} \\ \partial_t u + (u\cdot\nabla)u = -\nabla p - \rho(0,1)^t, \ \text{(Conservación del momento)} \end{cases}$$

donde

$$\rho = \mathbf{1}_{\{y<f(x,t)\}}$$

y $f(x,t)$ es la interfase entre el fluido y el vacío.

Motivados por estos modelos y problemas, en este texto estudiamos la evolución de las interfases entre fluidos incompresibles en *medios porosos*, entendiendo como tal aquellos medios que tienen un esqueleto sólido (también llamado *matriz*) y unos poros de tamaño uniforme. Este problema es el análogo al de las olas en el mar bajo una ley de conservación del momento distinta (ésta ecuación se conoce como ley de Darcy) para poder reflejar el efecto del roce con el medio sólido donde se mueven los fluidos

$$\text{[IPM]}\begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0, \ \text{(Conservación de la masa)} \\ \nabla\cdot u = 0, \ \text{(Incompresibilidad)} \\ \frac{\mu}{\kappa}u = -\nabla p - \rho(0,1)^t, \ \text{(Conservación del momento)} \end{cases}$$

donde

$$\rho = \rho^2\mathbf{1}_{\{y<f(x,t)\}} + \rho^1\mathbf{1}_{\{y>f(x,t)\}}$$

y $f(x,t)$ es la ola interna que separa ambos fluidos. Así escrito salta a la vista la similitud entre una interfase para [IPM] y [Water waves]. Sin embargo, operando, podemos escribir de manera equivalente el sistema anterior obteniendo

$$\text{[IPM2]}\begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0, \ \text{(Conservación de la masa)} \\ \frac{\mu}{\kappa}u = (R_1R_2, -R_1^2)\rho, \ \text{(Conservación del momento)} \end{cases}$$

Esta segunda manera [IPM2] de entender el sistema [IPM] muestra similitudes con [SQG].

1.2 Dinámica de los fluidos en medios porosos

La evolución de un fluido en un medio poroso es un problema importante en las Ciencias aplicadas e Ingeniería (ver [4] y [48]), pero también lo es en Matemáticas (ver, por ejemplo, [22]). El efecto del medio tiene importantes consecuencias, siendo la más importante que las ecuaciones de la conservación de momento usuales de la dinámica de fluidos, *i.e.* las ecuaciones de Euler o Navier-Stokes, deben cambiarse por una ley empírica: la ley de Darcy. El nombre lo recibe en honor de H. Darcy (1803-1858), un ingeniero francés que fue pionero del estudio de los fluidos en medios porosos. Darcy construyó un aparato (ver Figura 1.2) y midiendo la decarga Q (volumen total entre tiempo) y la presión en los extremos de dicho aparato, P_a y P_b, derivó la fórmula siguiente:

$$\mu\frac{Q}{A} = -\kappa\frac{P_b - P_a}{L},$$

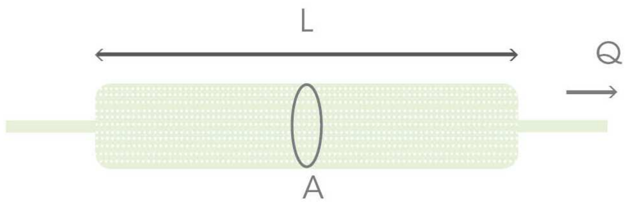

Figure 1.1: Aparato de Darcy.

donde μ es la viscosidad del fluido, κ es la permeabilidad del medio poroso y, A es el área de una sección de la tubería por donde pasa el fluido y L es la longitud del medio (ver Figura 1.2).

Si escribimos esta fórmula en el caso continuo y además consideramos los efectos de la gravedad obtenemos

$$\frac{\mu}{\kappa}v = -\nabla p - g(0,\rho).$$

Observamos que esta ecuación es aristotélica. En efecto, la fórmula anterior iguala velocidades y fuerzas.

Hay varias maneras de derivar *'teóricamente'* la ley de Darcy. Por ejemplo, uno puede considerar las ecuaciones de Euler para un fluido incompresible en dos dimensiones (ver [40] para más detalles)

$$\begin{cases} \rho\left(\partial_t v + (v\cdot\nabla)v\right) = -\nabla p - g\rho(0,1) \\ \nabla\cdot v = 0. \end{cases}$$

Estas ecuaciones son la traducción a la mecánica de los continuos de la Segunda Ley de Newton en el caso de que las partículas que forman el fluido no presenten fricción entre ellas. Debido a la resistencia del medio a ser atravesado podemos añadir un término de rozamiento dinámico. Si además asumimos que las velocidades son regulares y pequeñas, podemos escribir

$$\rho\left(\cancelto{0}{\partial_t v + (v\cdot\nabla)v}\right) = -\nabla p - g\rho(0,1) - \alpha v,$$

donde reconocemos la ley de Darcy. Necesitamos una constante de proporcionalidad α (que tiene las unidades apropiadas) para que la fórmula tenga sentido.

Para obtener una derivación matemáticamente rigurosa uno debe usar técnicas de homogenización (ver [36, 53]). Ahora consideramos un medio poroso, $\mathfrak{B}^\epsilon$, como el de la Figura 1.2 con ϵ la escala del poro. Este medio está lleno de un fluido viscoso moviéndose lentamente. para simplicar la exposición despreciaremos los efectos de la gravedad. Un fluido con las características anteriores sigue las ecuaciones de Stokes para la velocidad reescalada en función del tamaño del poro

$$\begin{cases} \epsilon^2\Delta u = \nabla p,\ x\in\mathfrak{B}^\epsilon, \\ \nabla\cdot u = 0,\ x\in\mathfrak{B}^\epsilon. \end{cases}$$

Debido a la viscosidad, en la frontera del poro tenemos condiciones de borde Dirichlet homogénas $u=0$ en Γ^ϵ. Ahora asumimos el siguiente *ansatz*

$$u^\epsilon(x) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + ...,$$

$$p^\epsilon(x) = p_0(x,y) + \epsilon p_1(x,y) + \epsilon^2 p_2(x,y) + ...,$$

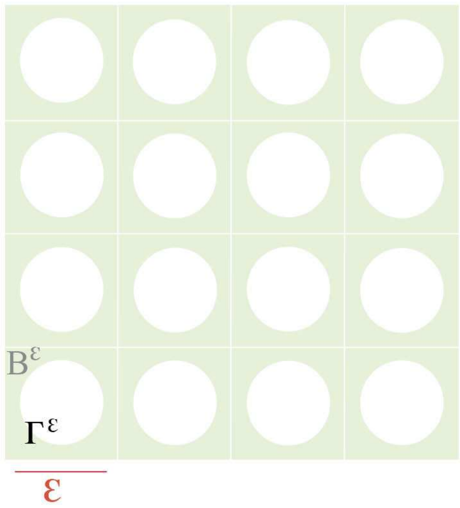

Figure 1.2: El medio poroso.

para $y = x/\epsilon$ y donde los coeficientes u_i, p_i son periódicas en la variable y variable. Insertando este *ansatz* en las ecuaciones e igualando las potencias en ϵ, obtenemos

$$\nabla_y p_0 = 0, \nabla_y \cdot u_0 = 0,$$

$$\Delta_y u_0(x, y) = \nabla_y p_1(x, y) + \nabla_x p_0(x).$$

Entonces, usando *separación de variables*, podemos considerar diferentes *problemas en las celdas* (solo involucran derivadas en la variable y):

$$\begin{cases} \Delta_y w_j = \nabla_y \pi_j - e_j, \\ \nabla_y \cdot w_j = 0. \end{cases}$$

donde e_j es el $j-$ésimo vector coordenado en $\mathbb{R}^d$, $d = 2, 3$, y $w_j = 0$ en la frontera del poro. Estos problemas de Stokes tienen una única solución clásica. Usando la linealidad del problema, podemos escribir

$$u_0(x, y) = -\sum_{k=1}^{d} w_j(y) \partial_{x_j} p_0(x).$$

Definimos la velocidad promedio como

$$\bar{u}(x) = \int_{\mathfrak{B}} u_0(x, y) dy = -K \nabla p_0,$$

para una matriz K con coeficientes constantes. Esta ley para la velocidad promedio es justamente la ley de Darcy. Quedaría probar la incompresibilidad, pero esta se obtiene integrando por partes (ver [36]).

Una parte importante de la teoría de flujos en medios porosos es la relativa a la coexistencia de dos fluidos inmiscibles con diferentes propiedades en el mismo volumen. Este problema se conoce como problema de Muskat. Este problema (ver [47]) recibe su nombre en honor de Morris Muskat (1906-1998), un ingeniero estadounidense.

1.3 Dinámica en una celda de Hele-Shaw

El problema que consideramos ahora es la evolución de un fujo de Stokes en tres dimensiones entre dos paredes planas que están separadas por una distancia infinitesimal (cuando se compara con las otras dimensiones del problema).

Para obtener las ecuaciones del problema de Hele-Shaw empezamos considerando las ecuaciones de Stokes. Escribiendo las coordenadas adecuadamente obtenemos

$$\nabla_{x,y,z}p + g\rho(0,1,0) = \mu\Delta_{x,y,z}u; \quad \nabla_{x,y,z}\cdot u = 0.$$

Como las paredes están muy cerca una de la otra (escribimos b para la distancia) podemos pensar que el fluido sólo se mueve en las otras dos dimensiones, *i.e.* si las paredes están en el eje z, $u_3 = 0$. Insetando este *ansatz* en las ecuaciones obtenemos

$$p(x,y,z) = p(x,y).$$

También se tiene que las derivadas en la dirección z de u_1, u_2 son mayores que en las otras direcciones. Así obtenemos el sistema

$$\partial_x p = \mu\partial_z^2 u_1; \;\; \partial_y p + g\rho = \mu\partial_z^2 u_2.$$

la solución de este sistema se puede escribir explícitamente como

$$u = (\nabla p + g\rho(0,1))\frac{z^2 - zb}{2\mu}.$$

Ahora definimos $\bar{u}$ como la velocidad media en la dirección z. Obtenemos la ecuación

$$\bar{u} = \frac{(\nabla p + g\rho(0,1))}{2b\mu}\int_0^b \left(z^2 - zb\right) dz = \frac{-b^2}{12\mu}(\nabla p + g\rho(0,1)),$$

y además esta nueva velocidad es incompresible. El problema de la celda de Hele-Shaw recibe su nombre en honor de Henry Selby Hele-Shaw (1854-1941), un ingeniero inglés que lo estudió en [35]. Desde entonces este problema ha sido profundamente investigado en [12, 16, 29].

1.4 El problema de Muskat confinado

La evolución de la interfase entre dos fluidos incompresibles con la misma viscosidad en una banda plana bidimensional es uno de los problemas principales estudiados en esta tesis. Este problema tiene interés, entre otras cosas, porque es un modelo de un acuífero o un pozo petrolífero (ver [47]). En estos fenómenos la velocidad del fluido en el medio poroso satisface la ley de Darcy

$$\frac{\mu}{\kappa}v = -\nabla p - g\rho e_2, \tag{1.1}$$

donde μ es la viscosidad dinámica, κ es la permeabilidad del medio, g es la aceleración debida a la gravedad, ρ es la densidad del fluido, p es la presión del fluido y v es el campo de velocidades (ver [4, 48]).

La ecuación (1.1) se ha considerado un modelo válido de la velocidad de las células en el crecimiento tumoral (ver [30, 51] y las referencias allí indicadas).

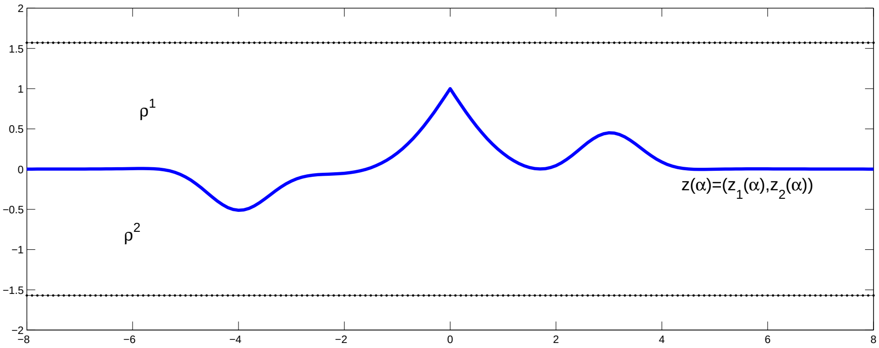

Figure 1.3: La interfase $z(\alpha, t)$ en la banda $\mathbb{R} \times (-l, l)$.

El movimiento de un fluido en un medio poroso bidimensional es análogo al movimiento de un fluido en una celda de Hele-Shaw. En este caso el fluido se encuentra atrapado entre dos pareces cercanas. La velocidad media del fluido satisface

$$\frac{12\mu}{b^2} v = -\nabla p - g\rho e_2,$$

donde b es la (pequeña) distancia entre las paredes.

Nosotros consideramos la banda bidimensional $S = \mathbb{R} \times (-l, l) \subset \mathbb{R}^2$ con $l > 0$. En esa banda tenemos dos fluidos incompresibles con la misma viscosidad y distintas densidades, ρ^1 en $S^1(t)$ y ρ^2 en $S^2(t)$, donde $S^i(t)$ es el dominio ocupado por el $i-$ésimo fluido. La curva

$$z(\alpha, t) = \{(z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}$$

es la interfase entre los fluidos y suponemos que la interfase inicial, $f_0(x)$, es un grafo tal que $|f_0(x)| < l$ para todo punto x (ver Figura 1.3). El ser un grafo se preserva al menos por un tiempo corto (ver Sección 4.1 en el Capítulo 4). La condición de Rayleigh-Taylor se define como

$$RT(\alpha, t) = -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t) > 0.$$

Si la curva se puede parametrizar como un grafo, debido a la incompresibilidad de los fluidos, se tiene que la condición de Rayleigh-Taylor es

$$RT = g(\rho^2 - \rho^1) > 0.$$

Esta condición se satisface si el fluido mas denso está debajo.

Consideramos el campo de velocidades v, la presión p y la densidad ρ

$$\rho(t) = \rho^1 \mathbf{1}_{S^1(t)} + \rho^2 \mathbf{1}_{S^2(t)}, \tag{1.2}$$

en todo el dominio S. La densidad debe seguir la ecuación de continuidad, por lo tanto, buscamos soluciones del sistema

$$\begin{cases} \frac{\mu}{\kappa} v(x, y, t) = -\nabla p(x, y, t) - g\rho(0, 1) \quad \text{en } S,\ t > 0, \\ \nabla \cdot v(x, y, t) = 0 \quad \text{en } S,\ t > 0, \\ \partial_t \rho(x, y, t) + v \cdot \nabla \rho(x, y, t) = 0 \quad \text{en } S,\ t > 0, \\ v_2(x, l, t), v_2(x, -l, t) = 0 \quad \text{en } \mathbb{R},\ t > 0, \\ f(x, 0) = f_0(x) \quad \text{en } \mathbb{R}, \end{cases} \tag{1.3}$$

donde las condiciones de borde implican la impermeabilidad de las paredes (ver Sección 3.2 en el Capítulo 3). También asumimos que los efectos de la tensión superficial son despreciables en la evolución.

La ecuación para la evolución de la interfase en nuestro medio poroso acotado, la cual se deduce en la Sección 3.2, es

$$\partial_t f(x,t) = \frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\Big[(\partial_x f(x) - \partial_x f(x-\eta))\,\Xi_1(x,\eta,f) + (\partial_x f(x) + \partial_x f(x-\eta)\,\Xi_2(x,\eta,f)\Big]d\eta = \frac{\rho^2-\rho^1}{4l}A[f](x), \quad (1.4)$$

donde los núcleos Ξ_1 y Ξ_2 se definen como

$$\Xi_1(x,\eta) = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) - \cos(\frac{\pi}{2l}(f(x) - f(x-\eta)))},$$

correspondiendo al carácter singular del problema, y

$$\Xi_2(x,\eta) = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) + \cos(\frac{\pi}{2l}(f(x) + f(x-\eta)))},$$

que se vuelve singular si f toca las fronteras. El símbolo P.V. denota que la integral es en valor principal. Como en el caso con profundidad infinita (ver [13, 25]) el operador espacial $A[f](x)$ se puede escribir como una derivada en x. En efecto,

$$A[f](x) = \frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x\left(\arctan\left(\frac{\tan\left(\frac{\pi}{2l}\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)}\right)\right)d\eta + \frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x\left(\arctan\left(\tan\left(\frac{\pi}{2l}\frac{f(x)+f(x-\eta)}{2}\right)\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)\right)\right)d\eta.$$

Si no parametrizamos la curva como un grafo, *i.e.*, consideramos la curva dada por la parametrización $z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t))$, obtenemos la ecuación

$$\partial_t z = \frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\left[\frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta)) - \cos(z_2(\alpha)-z_2(\eta))} + \frac{(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\eta), \partial_\alpha z_2(\alpha) + \partial_\alpha z_2(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta)) + \cos(z_2(\alpha)+z_2(\eta))}\right]d\eta.$$

Definimos el siguiente parametro adimensional (ver [6] y las referencias allí presentes)

$$\mathcal{A} = \frac{\|f_0\|_{L^\infty}}{l}.$$

Este parámetro se llama *parámetro de nolinealidad* (o *parámetro de amplitud*) y tenemos $0 \le \mathcal{A} \le 1$. En esta tesis consideramos el caso $0 < \mathcal{A} < 1$.

El caso $\mathcal{A} = 1$ es el caso donde f alcanza la frontera y recibe el nombre de *régimen de gran amplitud*. Este caso se estudia por ejemplo en [39]. En dicho trabajo consideran una gota bidimensional con tensión superficial en el vacío sobre una pared.

El caso $\mathcal{A} = 0$ se conoce como *régimen de aguas profundas* en el cual la ecuación para la interfase es

$$\partial_t f = \frac{\rho^2 - \rho^1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x f(x) - \partial_x f(x-\eta))\eta}{\eta^2 + (f(x) - f(x-\eta))^2} d\eta. \tag{1.5}$$

Este caso ha sido estudiado ampliamente (ver [1, 7, 10, 20, 22, 23, 24, 27, 28, 38, 54] y las referencias allí presentes)

En el caso $0 < \mathcal{A} < 1$, mostramos que la densidad ρ definida en (1.2) es una solución débil de la ecuación de conservación de la masa en (1.3) si, y sólo si, la interfase verifica la ecuación (3.1) (ver Proposición 3.3 en la Sección 3.4). También mostramos (ver Proposición 3.2) que si tomamos el límite $\mathcal{A} \to 0$ recuperamos la ecuación (2.10) (ver [23]).

Tras esos resultados, en el Capítulo 4 probamos que en el caso Rayleigh-Taylor estable y dato inicial en

$$H^s_l = H^s(\mathbb{R}) \cap \{f : \|f\|_{L^\infty} < l\},$$

con $s \geq 3$ existe solución local en tiempo en dicho espacio de Sobolev (ver Teorema 4.1). Este resultado sigue las mismas ideas que su análogo en el caso de profundidad finita en [23] para la ecuación (2.10). Sin embargo, debido a las paredes, la prueba en el caso confinado es más técnica.

En el Capítulo 5 demostramos que si el dato inicial es analítico existe una solución analítica local en tiempo y este resultado es independiente de la condición de Rayleigh-Taylor (ver Teorema 5.1). Este resultado también es interesante porque no asumimos que ninguna norma integral del dato sea finita.

En la segunda parte de esta tesis estudiamos algunas propiedades cualitativas de las soluciones y la posible formación de singularidades, o, por otro lado, su existencia global. En el Capítulo 6 demostramos que las soluciones anteriores en el caso Rayleigh-Taylor estable (ver Teorema 4.1) se vuelven analíticas (ver Teorema 6.1). Como esta ecuación presenta un *efecto de suavizado*, esperamos que el problema esté mal propuesto hacia atrás, o, análogamente, que esté mal propuesto en el caso inestable (ver Teorema 6.2). Ambos teoremas tienen su análogo con profundidad finita (ver [23, 10]). El punto principal del Teorema 6.2 es que no requiere de una familia de soluciones globales para su demostración ni tampoco de ningún reescalado particular (comparar con el resultado en [23]). En el Capítulo 7 obtenemos un principio del máximo para la amplitud de la interfase (ver Teorema 7.1),

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})},$$

y una tasa de decaimiento para datos iniciales con un signo e integrables

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1(\mathbb{R})}, \|f_0\|_{L^\infty(\mathbb{R})}, \rho^2, \rho^1, l) \exp\left(-\frac{\pi\|f_0\|_{L^1(\mathbb{R})}}{l\|f(t)\|_{L^\infty(\mathbb{R})}}\right). \tag{1.6}$$

Esta tasa de decaimiento es muy distinta de la tasa en [24]. En efecto, en el caso con profundidad infinita la tasa de decaimiento es

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1}, \|f_0\|_{L^\infty}, \rho^1, \rho^2)\|f(t)\|^2_{L^\infty(\mathbb{R})},$$

que es sensiblemente más rápido. También probamos un principio del máximo para $\|f(t)\|_{L^2(\mathbb{R})}$ (ver Teorema 7.2)

$$\|f(t)\|^2_{L^2(\mathbb{R})} + \frac{2}{\rho^2 - \rho^1} \int_0^t \|v(s)\|^2_{L^2(S)} ds = \|f_0\|^2_{L^2(\mathbb{R})}.$$

Usando este resultado podemos probar que, para cualquier dato inicial en H^3_l, la tasa de decaimiento es

$$\frac{d}{dt}\|f(t)\|_{L^\infty} \leq -c(\|f_0\|_{L^2}, \|f_0\|_{L^\infty}, \rho^2, \rho^1, l) \exp\left(-\frac{2\pi}{l}\frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\left(1 + \frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\right)\right).$$

Como corolario de este resultado obtenemos que no hay soluciones estacionarias.

En el Capítulo 7, obtenemos un principio del máximo para $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ para datos iniciales que cumplan unas hipótesis que relacionan la amplitud, la pendiente y la profundidad (ver Teorema 7.4). Este resultado es interesante porque en el caso con profundidad infinita la condición en el dato inicial concierne sólo a su pendiente (ver [24]). Como corolario obtenemos que si el dato inicial es tal que $\|f_0\|_{L^\infty(\mathbb{R})} > l/2$ entonces las condiciones anteriores no se pueden cumplir incluso si la pendiente es muy pequeña. También obtenemos una región donde la derivada de la interfase puede crecer pero estará acotada uniformemente. Esa región no aparece en el caso con profundidad infinita.

En el Capítulo 8 demostramos que para dato inicial grande en C^1 existen lo que se llaman *turning waves*, *i.e* interfases que presentan una explosión para $\|\partial_x f\|_{L^\infty}$ (ver Teorema 8.1). Para probar este resultado definimos una familia de curvas suaves a trozos tales que la velocidad tenga las propiedades correctas. Estas curvas las aproximamos por curva analíticas y aplicamos un teorema de Cauchy-Kowalevsky (ver Teorema 8.3). Ahora podemos cerrar el argumento usando que el problema está bien propuesto hacia delante en tiempo y también hacia atrás. Observamos que el tamaño de $\|f(t)\|_{L^\infty(\mathbb{R})}$ no juega ningún papel. Es más, podemos construir curvas con una amplitud tan pequeña como se quiera tales que giran (ver Teorema 8.2).

En el Capítulo 9, obtenemos existencia global de soluciones Lipschitz para la ecuación (1.4) y datos iniciales satisfaciendo las hipótesis previas que relacionan la amplitud, la pendiente y la profundidad. Para estos datos iniciales probamos que la pendiente y la amplitud están uniformemente acotadas y por lo tanto la unica singularidad posible sería una explosión para la curvatura con primera derivada finita, *i.e.* la singularidad sería *una esquina*. Este resultado indica que no pueden formarse *picos* (explosiones de las dos primeras derivadas) ni *turning waves*, y deja abierta la existencia (o la no existencia) de esquinas. Observamos que en el límite $l \to \infty$ recuperamos el resultado presente en [13].

Todos estos resultados están disponibles en [26, 34]. Para otros resultados pueden consultarse las referencias [1, 7, 10, 22, 38, 54]. Observamos que en todos estos resultados la permeabilidad es constante $\kappa = \kappa^1 \mathbf{1}_{|y|<l}$. Por lo tanto una pregunta natural es: ¿qué pasa si la permeabilidad viene dada por $\kappa = \kappa^1 \mathbf{1}_{\Omega^1} + \kappa^2 \mathbf{1}_{\Omega^2}$?

1.5 El problema de Muskat inhomogéneo

En la tercera parte de la tesis estudiamos la evolución de una interfase entre dos fluidos incompresibles con la misma viscosidad y diferentes densidades en un medio poroso cuando la permeabilidad del medio toma dos valores distintos. Este problema es de interés práctico porque se usa como modelo de una reserva geotermal (ver [11] y las referencias allí presentes). La velocidad de un fluido en un medio poroso satisface la ley de Darcy antes mencionada (ver [4, 47, 48])

$$\frac{\mu}{\kappa(\vec{x})} v = -\nabla p - g\rho(\vec{x})(0,1),$$

donde μ es la viscosidad dinámica, $\kappa(\vec{x})$ es la permeabilidad del medio, g es la aceleración de la gravedad, $\rho(\vec{x})$ es la densidad del fluido, $p(\vec{x})$ es la presión del fluido y $v(\vec{x})$ es el campo de velocidades. En nuestras unidades favoritas podemos asumir que $g = \mu = 1$.

Los dominios espaciales que consideramos en esta parte de la tesis son $S = \mathbb{R}^2, \mathbb{T} \times \mathbb{R}$ (profundidad infinita) y $\mathbb{R} \times (-\pi/2, \pi/2)$ (profundidad finita). Los fluidos tienen densidades distintas; ρ^1 es la densidad del fluido que ocupa el dominio superior $S^1(t)$ y ρ^2 es la densidad del fluido que ocupa

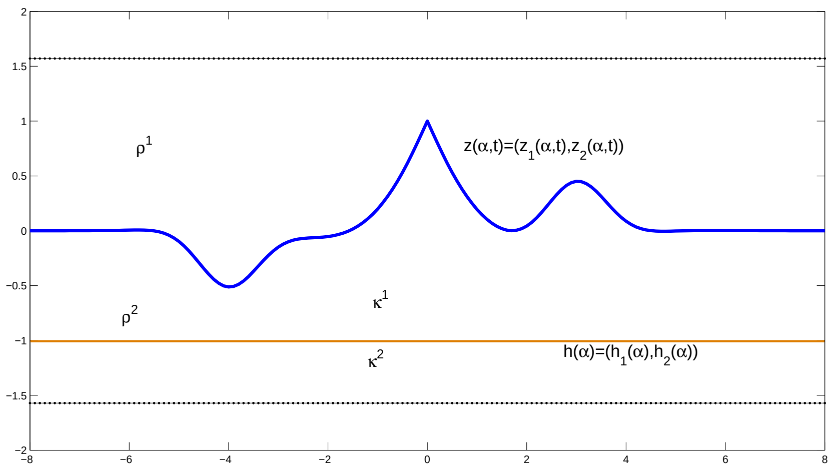

Figure 1.4: La situación.

el dominio inferior $S^2(t)$. La curva

$$z(\alpha,t) = \{(z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\}$$

es la interfase entre los fluidos. En partícular se tiene que S^1 y S^2 son una partición de S están separados por la curva z (ver Figura 1.4).

Como en el caso homogéneo, el sistema está en el caso Rayleigh-Taylor estable si el fluido más denso está debajo del más ligero, *i.e.* $\rho^2 > \rho^1$ (ver Sección 2.3).

Ahora suponemos que la permeabilidad $\kappa(\vec{x})$ es una función escalón, o, de manera más precisa, tenemos una recta

$$h(\alpha) = \{(\alpha, -h_2) : \alpha \in \mathbb{R}\}$$

que separa las dos regiones con diferentes valores de la permeabilidad (ver Figura 2.4). Para este problema estudiamos el caso con profundidad infinita y dato inicial periódico o plano en el infinito pero también el caso con profundidad finita e igual a $\frac{\pi}{2}$. En la región por encima de la curva $h(\alpha)$ la permeabilidad es $\kappa(\vec{x}) \equiv \kappa^1$, mientras que en la región por debajo de la curva $h(\alpha)$ la permeabilidad es $\kappa(\vec{x}) \equiv \kappa^2 \neq \kappa^1$. Nótese que la curva $h(\alpha)$ es conocida y fija. Definimos el parámetro

$$\mathcal{K} = \frac{\kappa^1 - \kappa^2}{\kappa^1 + \kappa^2}.$$

Este parámetro es un número adimensional similar al número de Atwood.

Se sigue de la ley de Darcy que la vorticidad es

$$\omega(\vec{x}) = \varpi_1(\alpha,t)\delta(\vec{x} - z(\alpha,t)) + \varpi_2(\alpha,t)\delta(\vec{x} - h(\alpha)),$$

donde ϖ_1 se corresponde a la diferencia de densidades, ϖ_2 se corresponde a la diferencia de permeabilidades y δ es la distribución de Dirac usual. Las amplitudes de la vorticidad son bastante diferentes, mientras que ϖ_1 es una derivada, la amplitud ϖ_2 tiene carácter no-local (ver (1.8), (1.10) y la Sección 10.2). La ecuación para la interfase cuando los fluidos llenan todo el plano es

$$\partial_t f(x) = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_x f(x)-\partial_x f(\beta))(x-\beta)}{(x-\beta)^2+(f(x)-f(\beta))^2}d\beta + \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(\beta)(x-\beta+\partial_x f(x)(f(x)+h_2))}{(x-\beta)^2+(f(x)+h_2)^2}d\beta, \quad (1.7)$$

con

$$\varpi_2(x) = \frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)(h_2+f(\beta))}{(x-\beta)^2+(-h_2-f(\beta))^2}d\beta. \quad (1.8)$$

Si los fluidos llenan todo el plano pero la curva inicial es periódica la ecuación anterior se reduce a

$$\partial_t f(x) = \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{T}} \frac{\sin(x-\beta)(\partial_x f(x)-\partial_x f(\beta))d\beta}{\cosh(f(x)-f(\beta))-\cos(x-\beta)} + \frac{1}{4\pi}\text{P.V.}\int_{\mathbb{T}} \frac{(\partial_x f(x)\sinh(f(x)+h_2)+\sin(x-\beta))\varpi_2(\beta)d\beta}{\cosh(f(x)+h_2)-\cos(x-\beta)}, \quad (1.9)$$

donde la segunda amplitud de la vorticidad se puede escribir como

$$\varpi_2(x) = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\frac{\kappa^1-\kappa^2}{\kappa^1+\kappa^2}\text{P.V.}\int_{\mathbb{T}} \frac{\sinh(h_2+f(\beta))\partial_x f(\beta)d\beta}{\cosh(h_2+f(\beta))-\cos(x-\beta)}. \quad (1.10)$$

En el caso con profundidad finita e igual a $\pi/2$, la ecuación para la interfase es

$$\begin{aligned}\partial_t f(x) &= \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_x f(x)-\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)-\cos(f(x)-f(\beta))}d\beta \\ &+\frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_x f(x)+\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)+\cos(f(x)+f(\beta))}d\beta \\ &+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(\beta)(\sinh(x-\beta)+\partial_x f(x)\sin(f(x)+h_2))}{\cosh(x-\beta)-\cos(f(x)+h_2)}d\beta \\ &+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(\beta)(-\sinh(x-\beta)+\partial_x f(x)\sin(f(x)-h_2))}{\cosh(x-\beta)+\cos(f(x)-h_2)}d\beta,\end{aligned}$$

donde

$$\begin{aligned}\varpi_2(x) &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}} \partial_x f(\beta)\frac{\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \\ &-\mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}} \partial_x f(\beta)\frac{\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta \\ &+\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}} * \text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \\ &-\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}} * \text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta,\end{aligned}$$

con

$$G_{h_2,\mathcal{K}}(x) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}{1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}\right)$$

una función del espacio de Schwartz.

Para estas ecuaciones, en el Capítulo 11, obtenemos un principio del máximo para $\|f(t)\|^2_{L^2(\mathbb{R})}$ (ver Teorema 11.1)

$$\|f(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \frac{\|v\|^2_{L^2(\mathbb{R}\times(-h_2,\pi/2))}}{\kappa^1(\rho^2-\rho^1)} + \frac{\|v\|^2_{L^2(\mathbb{R}\times(-\pi/2,-h_2))}}{\kappa^2(\rho^2-\rho^1)} ds = \|f_0\|^2_{L^2(\mathbb{R})}.$$

También mostramos la existencia local de solución en espacios de Sobolev (ver Teoremas 11.2 y 11.3). Estas pruebas siguen las mismas ideas que las expuestas en el Capítulo 4. Un hecho interesante es que no necesitamos ninguna condición en el signo de $\mathcal{K}$ para asegurar la existencia de dichas soluciones.

En el Capítulo 12, obtenemos una familia de datos iniciales que dependen de los parámetros físicos h_2 y $\mathcal{K}$, tales que la pendiente explota en tiempo finito (ver Teoremas 12.1 y 12.2). Estas singularidades son análogas a las singularidades del Capítulo 8.

Estos resultados se han publicado en [5].

1.6 Una comparación entre los modelos

Para entender mejor la dinámica de las ecuaciones (1.4), (1.7) y (1.9), hemos hecho unas simulaciones siguiendo las ideas en [25] (ver Capítulo 13). Consideramos el mismo dato inicial evolucionando siguiendo (1.5) y (1.4) y hemos comparado la evolución de la amplitud. Así hemos observado que el caso confinado es mas singular que el caso con profundidad finita, *i.e.* la tasa de decaimiento de la amplitud es más lenta. Estas simulaciones concuerdan con los resultados en este sentido contenidos en el Capítulo 7 y la estimación (1.6).

También comparamos el caso homogéneo (1.5) y el caso con diferentes permeabilidades (1.7). En estas simulaciones observamos que las componentes de $\|f\|_{C^1}$ decaen pero de una manera diferente dependiendo de $\mathcal{K}$. Si $\mathcal{K}<0$ el decaimiento $\|f\|_{L^\infty(\mathbb{R})}$ es más rápido que en el caso $\mathcal{K}=0$. Cuando se estudia la evolución de $\|\partial_x f\|_{L^\infty}$ la situación se revierte. Ahora las simulaciones correspondiendo al caso $\mathcal{K}>0$ tienen el decaimiento más rápido. Con estos resultados no podemos definir un régimen *estable* para $\mathcal{K}$ en el que la evolución fuese *más suave*. Observamos que no hay ninguna condición qen el signo de $\mathcal{K}$ necesaria a la hora de probar existencia de soluciones (ver Teoremas 11.2 y 11.3).

En la Sección 13.4 demostramos con una prueba asistida por ordenador que existen curvas tales que giran si la profundidad es finita, pero si la profundidad es infinita las mismas curvas se vuelven grafos (ver Teorema 13.1). La idea es dar signo rigurosamente, usando aritmética de intervalos, a determinadas integrales. Estos resultados han surgido de una colaboración con Javier Gómez, y todos los detalles relacionados con la aritmética de intervalos, los códigos y otros asuntos técnicos están contenidos en [32]

Como corolario (ver Corolario 13.1) obtenemos que existen soluciones al problema de las *water wave* tales que giran sólo si la profundidad es finita mientras que en el caso con profundidad infinita las mismas curvas se vuelven grafos.

Como dijimos antes, el caso $\mathcal{K}>0$ parece ser más estable (desde el punto de vista de la formación de singularidades). Sin embargo, esto no es cierto (ver Evidencia numérica 13.2 y el Teorema 13.2). Este último resultado indica que la diferencia de permeabilidades no puede prevenir la explosión de la pendiente para todas las curvas. Estos resultados aparecerán en [33].

1.7 Conclusiones

En esta tesis hemos continuado el estudio de olas en medios porosos contenido en los trabajos [1, 7, 10, 13, 22, 23, 24, 25, 38, 54] y las referencias allí presentes. En concreto hemos investigado el efecto de las fronteras y las regiones con distinta permeabilidad en la evolución de una ola interna en un medio poroso. Este análisis ha dado lugar a los trabajos [5, 26, 33, 34].

Cuando los resultados para el problema confinado (ver Capítulos 3-9 y Capítulo 13) se comparan con los resultados conocidos para el caso con profundidad finita (ver [10, 13, 23, 24]), aparecen tres diferencias principales:

1. El decaimiento de la amplitud es más lento en el caso confinado.
2. Hay curvas suaves con energía infinita que giran en el caso confinado y se vuelven grafos en el caso con profundidad infinita.
3. Para evitar las explosiones de la pendiente en el caso confinado necesitamos imponer condiciones en la amplitud, $\|f_0\|_{L^\infty(\mathbb{R})}$, y la pendiente, $\|\partial_x f_0\|_{L^\infty(\mathbb{R})}$, relacionadas con la profundidad. En el caso con profundidad finita sólo se requiere una condición sobre la pendiente para obtener el mismo resultado. Es más, en el caso confinado obtenemos una region de datos iniciales cuya pendiente permanece acotada pero podría crecer. En ambos casos (la región con principio del máximo y la región con cota uniforme), el Teorema 9.1 nos garantiza la existencia global de soluciones Lipschitz.

Tras estos resultados, podemos afirmar que las paredes hacen el problema más singular.

En el caso homogéneo y confinado quedan abiertas algunas preguntas. Por ejemplo,

1. la existencia de interfases tales que la pendiente crezca pero que permanezca acotada,
2. la existencia de una interfase con pendiente pequeña y que, debido a la distancia a las tapas, crezca,
3. la existencia (o no existencia) de singularidades de tipo *esquina* cuando el dato inicial es pequeño en $W^{1,\infty}(\mathbb{R})$.

En el caso inhomogéneo (ver Capítulos 10-12 y Capítulo 13) sólo se conocen resultados preliminares, siendo la falta de una condición de estabilidad para el signo de la permeabilidad uno de los más interesantes en nuestra opinión. La ausencia de principios del máximo u otras cantidades conservadas y el papel del signo de $\mathcal{K}$ son preguntas muy importanets que nos gustaría contestar en futuros trabajos. Otro problema abierto es la existencia global para alguna familia de datos iniciales y la existencia (sea esta local o global en tiempo) de solución cuando el dato inicial toca la curva donde la permeabilidad cambia.

Chapter 2

Overture, results and conclusion

2.1 Dynamics of fluids in porous media

The evolution of a fluid in a porous medium is important in the Applied Sciences and Engineering (see [4] and [48]) but also in Mathematics (see, for instance, [22]). The effect of the medium has important consequences and the usual equations for the conservation of momentum, *i.e.* the Euler or Navier-Stokes equations, must be replaced with an empirical law: Darcy's Law. The name came in honor of H. Darcy (1803-1858), a french engineer who studied this phenomenon. He built a device (see Figure 2.1) that allowed him to measure the total flow discharge Q (the total volume per time) and the pressure in two spatially separated parts of the flow P_a and P_b. Experimentally, he derived the following formula:

$$\mu\frac{Q}{A} = -\kappa\frac{P_b - P_a}{L},$$

where μ is the dynamic viscosity of the fluid, κ is the permeability of the porous medium, A is the area of a section of the device and L is the length of the porous medium (see Figure 2.1).
If we write this formula in the 2D case in a continuous manner, and add in the effect of the gravity, we obtain

$$\frac{\mu}{\kappa}v = -\nabla p - g(0,\rho).$$

We observe that this equation is Aristotelian. Indeed, the formula equals velocities and forces.
There are many derivations for Darcy's Law. For instance, to motivate *'theoretically'* Darcy's Law, one can consider the Euler equations for an incompressible fluid in two dimensions (see [40] for

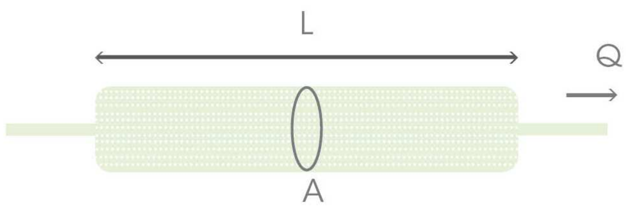

Figure 2.1: Darcy's device.

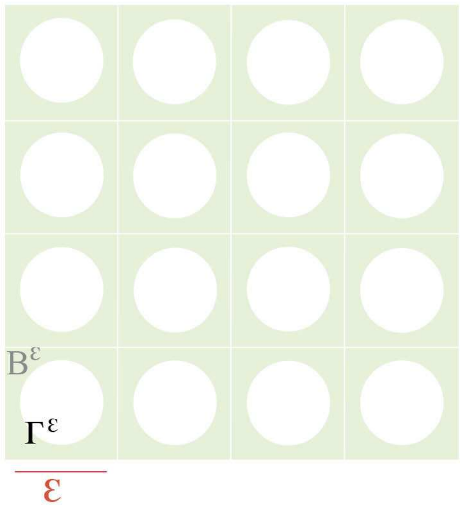

Figure 2.2: The considered porous medium.

more details)

$$\begin{cases} \rho\left(\partial_t v + (v\cdot\nabla)v\right) = -\nabla p - g\rho(0,1) \\ \nabla\cdot v = 0. \end{cases}$$

These equations are the translation of the Newton's second law to continuum mechanics in the case without internal friction, *i.e* the friction between fluid particles. We can add a dynamical friction, linear in v, as a model of the friction between the fluid and the porous medium, thus an *'external'* friction. If we additionally suppose that the velocities are smooth and small we can write

$$\rho\,\underbrace{\left(\partial_t v + (v\cdot\nabla)v\right)}_{\to 0} = -\nabla p - g\rho(0,1) - \alpha v$$

where we recognize Darcy's Law. We need a constant of proportionality α with units in order to be able to balance the units in the whole formula. We need a constant α, with the appropriate units, to be dimensionally correct.

A mathematical derivation is possible using homogenization techniques (see [36, 53]). We consider the porous medium, $\mathfrak{B}^\epsilon$, as periodic spatial domain as in Figure 2.1 with ϵ as pore scale. This medium is filled with a viscous, incompressible fluid flowing slowly. For the sake of simplicity we do not consider the effects of gravity. For a fluid in this regime, the Stokes equation (with the velocity rescaled to the pore typical size) is

$$\begin{cases} \epsilon^2\Delta u = \nabla p, \ x\in\mathfrak{B}^\epsilon, \\ \nabla\cdot u = 0, \ x\in\mathfrak{B}^\epsilon. \end{cases}$$

Due to the viscosity in the boudaries of the pores we have homogeneus Dirichlet boundary conditions $u = 0$ on Γ^ϵ. Then we assume the *ansatz*

$$u^\epsilon(x) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y)...,$$

and

$$p^\epsilon(x) = p_0(x,y) + \epsilon p_1(x,y) + \epsilon^2 p_2(x,y)...,$$

for $y = x/\epsilon$ and where the coefficients u_i, p_i are periodic in the y variable. Inserting this *ansatz* into the equations and comparing the different powers of ϵ we obtain

$$\nabla_y p_0 = 0, \nabla_y \cdot u_0 = 0,$$

and

$$\Delta_y u_0(x,y) = \nabla_y p_1(x,y) + \nabla_x p_0(x).$$

Then, using separation of variables, we consider different *cell problems* (only involving $y-$derivatives):

$$\begin{cases} \Delta_y w_j = \nabla \pi_j - e_j, \\ \nabla \cdot w_j = 0. \end{cases}$$

where e_j is the $j-$esime coordinate vector in $\mathbb{R}^d$, $d = 2,3$, and $w_j = 0$ in the pore boundaries. These Stokes problems have a unique classical solution. Using the linearity of the problem, we recover

$$u_0(x,y) = -\sum_{k=1}^{d} w_j(y)\partial_{x_j} p_0(x).$$

Defining the average velocity

$$\bar{u}(x) = \int_{\mathfrak{B}} u_0(x,y)dy = -K\nabla p_0,$$

for some matrix K with constant entries, we arrive to Darcy's Law. The incompressibility can be proved using integration by parts (see [36]).

A very important part of the theory of flow in porous media studies the coexistence of two immiscible fluids with different qualities in the same volume. The case of two immiscible and incompressible fluids is known as the Muskat o Muskat-Leverett problem.

The Muskat problem (see [47]) receives its name in honour of Morris Muskat (1906-1998), an american engineer.

2.2 Dynamics in a Hele-Shaw cell

The problem that we want to consider is the evolution of a Stokes 3D flow between two parallel flat plates separated by an infinitesimal (when compared with the other dimensions in the problem) gap.

To obtain the equation for the Hele-Shaw cell problem we start with the Stokes system. Looking how we write the coordinates (see the pictures before) we have

$$\nabla_{x,y,z} p + g\rho(0,1,0) = \mu\Delta_{x,y,z} u; \quad \nabla_{x,y,z} \cdot u = 0.$$

As the walls are very close (write b for the distance) the flow only moves in other the directions, *i.e.* if the walls are in the $z-$axis, $u_3 = 0$. Inserting this *ansatz* in the equations we obtain

$$p(x,y,z) = p(x,y).$$

Now we have that the derivatives in the z direction of u_1, u_2 are bigger that the derivatives in the other directions, so we arrive to the following system

$$\partial_x p = \mu\partial_z^2 u_1; \quad \partial_y p + g\rho = \mu\partial_z^2 u_2.$$

The solution of thus system can be explicitly obtained (using the boundary conditions in z) as

$$u = (\nabla p + g\rho(0,1))\frac{z^2 - zb}{2\mu}$$

Define $\bar{u}$ the mean velocity in the z direction. Then we have that

$$\bar{u} = \frac{(\nabla p + g\rho(0,1))}{2b\mu}\int_0^b z^2 - zbdz = \frac{-b^2}{12\mu}(\nabla p + g\rho(0,1)),$$

being this new averaged velocity also incompressible.

The Hele-Shaw cell problem (see [12, 16, 29, 35]) receives its name in honour of Henry Selby Hele-Shaw (1854-1941), an english engineer.

2.3 The confined Muskat problem

The evolution of the interface between two different incompressible fluids with the same viscosity in a flat two-dimensional strip is one of the main questions adressed in this thesis. This problem has an interest because it is a model of an aquifer or an oil well (see [47]). In this phenomena, the velocity of a fluid in a porous medium satisfies Darcy's law

$$\frac{\mu}{\kappa}v = -\nabla p - g\rho e_2, \tag{2.1}$$

where μ is the dynamic viscosity, κ is the permeability of the medium, g is the acceleration due to gravity, ρ is the density of the fluid, p is the pressure of the fluid and v is the incompressible field of velocities (see [4, 48]).

Equation (2.1) has also been considered as a model of the velocity for cells in tumor growth, see for instance [30, 51] and references therein.

The motion of a fluid in a two-dimensional porous medium is analogous to the Hele-Shaw cell problem. In this case the fluid is trapped between two parallel plates. The mean velocity in the cell is described by

$$\frac{12\mu}{b^2}v = -\nabla p - g\rho e_2,$$

where b is the (small) distance between the plates.

We consider the two-dimensional flat strip $S = \mathbb{R} \times (-l, l) \subset \mathbb{R}^2$ with $l > 0$. In this strip we have two immiscible and incompressible fluids with the same viscosity and different densities, ρ^1 in $S^1(t)$ and ρ^2 in $S^2(t)$, where $S^i(t)$ denotes the domain occupied by the $i-$th fluid. The curve

$$z(\alpha, t) = \{(z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}$$

is the interface between the fluids. We suppose that the initial interface $f_0(x)$ is a graph and $|f_0(x)| < l$ for all x (see Figure 2.3). The character of being a graph is preserved at least for a short time (see Section 4.1 in Chapter 4). The Rayleigh-Taylor condition is defined as

$$RT(\alpha, t) = -(\nabla p^2(z(\alpha, t)) - \nabla p^1(z(\alpha, t))) \cdot \partial_\alpha^\perp z(\alpha, t).$$

Due to the incompressiblity of the fluids and using that the curve can be parametrized as a graph, the Rayleigh-Taylor condition reduces to the sign of the jump in the density:

$$RT = g(\rho^2 - \rho^1) > 0.$$

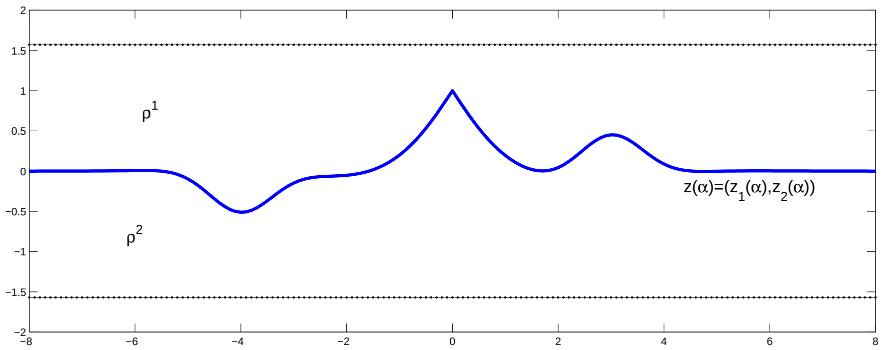

Figure 2.3: Physical situation for an interface $z(\alpha,t)$ in the strip $\mathbb{R}\times(-l,l)$.

This condition is satisfied if the denser fluid is below.

We consider the velocity field v, the pressure p and the density ρ

$$\rho(t)=\rho^1\mathbf{1}_{S^1(t)}+\rho^2\mathbf{1}_{S^2(t)}, \tag{2.2}$$

in the whole domain S. We also consider the continuity equation, so we have a weak solution to the following system of equations

$$\begin{cases} \frac{\mu}{\kappa}v(x,y,t)=-\nabla p(x,y,t)-g\rho(0,1) \quad \text{in } S,\ t>0,\\ \nabla\cdot v(x,y,t)=0 \quad \text{in } S,\ t>0,\\ \partial_t\rho(x,y,t)+v\cdot\nabla\rho(x,y,t)=0 \quad \text{in } S,\ t>0,\\ f(x,0)=f_0(x) \quad \text{in } \mathbb{R}, \end{cases} \tag{2.3}$$

with impermeable boundary conditions for the velocity (see Section 3.2 in Chapter 3). We assume that surface tension effects are negligible in the evolution.

We define the following dimensionless parameter (see [6] and references therein)

$$\mathcal{A}=\frac{\|f_0\|_{L^\infty}}{l}. \tag{2.4}$$

This parameter is called the *nonlinearity* (or *amplitude*) parameter and we have $0\leq\mathcal{A}\leq 1$. In this thesis we consider the case where $0<\mathcal{A}<1$.

The equation for the evolution of the interface in our bounded domain, which is deduced in Section 3.2, is

$$\begin{aligned}\partial_t f(x,t)=\frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\Big[&\left(\partial_x f(x)-\partial_x f(x-\eta)\right)\Xi_1(x,\eta,f)\\ &+\left(\partial_x f(x)+\partial_x f(x-\eta)\right)\Xi_2(x,\eta,f)\Big]d\eta=\frac{\rho^2-\rho^1}{4l}A[f](x),\end{aligned} \tag{2.5}$$

where the singular kernels Ξ_1 and Ξ_2 are defined as

$$\Xi_1(x,\eta)=\frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)-\cos\left(\frac{\pi}{2l}(f(x)-f(x-\eta))\right)}, \tag{2.6}$$

corresponding to the singular character of the problem, and

$$\Xi_2(x,\eta)=\frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)+\cos(\frac{\pi}{2l}(f(x)+f(x-\eta)))},\tag{2.7}$$

which becomes singular when f reaches the boundaries. The text P.V. denotes principal value. As for the whole plane case (see [13, 25]) the spatial operator $A[f](x)$ can be written as an x-derivative. Indeed,

$$A[f](x)=\frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x\left(\arctan\left(\frac{\tan\left(\frac{\pi}{2l}\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)}\right)\right)d\eta$$
$$+\frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x\left(\arctan\left(\tan\left(\frac{\pi}{2l}\frac{f(x)+f(x-\eta)}{2}\right)\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)\right)\right)d\eta.\tag{2.8}$$

When we do not parametrize the curve as a graph, *i.e.*, we consider $z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t))$, we obtain the equation

$$\partial_t z=\frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\left[\frac{(\partial_\alpha z(\alpha)-\partial_\alpha z(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))-\cos(z_2(\alpha)-z_2(\eta))}\right.$$
$$\left.+\frac{(\partial_\alpha z_1(\alpha)-\partial_\alpha z_1(\eta),\partial_\alpha z_2(\alpha)+\partial_\alpha z_2(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))+\cos(z_2(\alpha)+z_2(\eta))}\right]d\eta.\tag{2.9}$$

The case $\mathcal{A}=1$ is the case where f reaches the boundaries and we call it the *large amplitude regime*. In [39], they consider a two dimensional droplet in vacuum over a plate driven by surface tension.

The case $\mathcal{A}=0$ is the *deep water regime* for which the equation reduces to

$$\partial_t f=\frac{\rho^2-\rho^1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(x-\eta))\eta}{\eta^2+(f(x)-f(x-\eta))^2}d\eta.\tag{2.10}$$

This case has been widely studied (see [1, 7, 10, 20, 22, 23, 24, 27, 28, 38, 54] and references therein)

For the case $0<\mathcal{A}<1$, we show that the density ρ defined in (2.2) is a weak solution of the conservation of mass equation present in (2.3) if and only if the interface verifies the equation (3.1) (see Proposition 3.3 in Section 3.4 below). It also follows (see Proposition 3.2) that if we take the limit $\mathcal{A}\to 0$ we recover the equation (2.10) (see [23]).

Then, in Chapter 4 we show that for the Rayleigh-Taylor stable case and initial data in

$$H^s_l=H^s(\mathbb{R})\cap\{f:\|f\|_{L^\infty}<l\},$$

with $s\geq 3$ there is a local in time existence of classical solution in Sobolev spaces (see Theorem 4.1). This result follows the same ideas as the one in [23] for equation (2.10) but due to the boundaries the proof is more tecnical and the computations are harder.

In Chapter 5 we show that if the initial data is analytic then there is a local in time analytic solution and this result is independent of the Rayleigh-Taylor condition (see Theorem 5.1). This result is interesting because it does not assume that some integral norm of the function and its derivatives is finite.

In the second part of this thesis we study some qualitative properties and the existence of global in time solutions. Furthermore, in Chapter 6 we obtain that the previous Sobolev solutions in the Rayleigh-Taylor stable regime (see Theorem 4.1) become analytic (see Theorem 6.1). Since the solution have a *smoothing effect*, we expect backward ill-posedness (see Theorem 6.2). Both Theorems have their infinite depth counterpart (see [23, 10]). The main point of Theorem 6.2 is that we do not need a family of solutions which exits global in time. We also note that the proof in [23] relies strongly in the homogeneity of the kernel, and due to expressions (2.6) and (2.7) we can not use it. In Chapter 7 we obtain a maximum principle for the amplitude of the interface (see Theorem 7.1),

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})},$$

and a decay estimate for one signed and integrable initial data

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1(\mathbb{R})}, \|f_0\|_{L^\infty(\mathbb{R})}, \rho^2, \rho^1, l) \exp\left(-\frac{\pi\|f_0\|_{L^1(\mathbb{R})}}{l\|f(t)\|_{L^\infty(\mathbb{R})}}\right) \tag{2.11}$$

This decay estimate is very different than the one in [24]. Indeed, in the case where the depth is infinite the decay rate is given by

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1}, \|f_0\|_{L^\infty}, \rho^1, \rho^2)\|f(t)\|^2_{L^\infty(\mathbb{R})},$$

which is a faster decay. We also show a maximum principle for $\|f(t)\|_{L^2(\mathbb{R})}$ (see Theorem 7.2)

$$\|f(t)\|^2_{L^2(\mathbb{R})} + \frac{2}{\rho^2-\rho^1}\int_0^t \|v(s)\|^2_{L^2(S)} ds = \|f_0\|^2_{L^2(\mathbb{R})}.$$

Using this result, we can prove that, for every initial data in H^3_l, the decay is given by

$$\frac{d}{dt}\|f(t)\|_{L^\infty} \leq -c(\|f_0\|_{L^2}, \|f_0\|_{L^\infty}, \rho^2, \rho^1, l) \exp\left(-\frac{2\pi}{l}\frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\left(1+\frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\right)\right).$$

Moreover, the previous decay estimate gives us that there is not non-trivial steady solutions.

In Chapter 7, we show a maximum principle for $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ for initial data satisfying some hypotheses related to the amplitude, the slope and the depth (see Theorem 7.4). This result is very significant because in the whole plane case the condition for the initial data is only on the size of the slope (see [24]). Moreover, we obtain that if the initial datum is such that $\|f_0\|_{L^\infty(\mathbb{R})} > l/2$ then the previous hypotheses can not hold even if the derivative is very small. We also obtain a region where the slope initially can grow but it remains uniformly bounded by one. This region does not appear in the infinite depth case.

In Chapter 8, we show that for large initial datum in C^1 there are *turning waves*, *i.e* a blow up for $\|\partial_x f\|_{L^\infty}$ (see [10]) (see Theorem 8.1). For other results see [1, 7, 10, 22, 38, 54]. In order to prove this result we construct an appropriate family of piecewise smooth curves such that the velocity has the appropriate sign. We approximate these curves with analytical ones and we use a Cauchy-Kowalevsky Theorem (see Theorem 8.3). We close the argument using the backward and forward well-posedness. We notice that the size of $\|f(t)\|_{L^\infty(\mathbb{R})}$ does not play any role. Furthermore, we can construct curves with an arbitrary small amplitude and such that they turn (see Theorem 8.2).

In Chapter 9, we obtain global in time existence of Lipschitz continuous solutions of equation (2.5) for initial data satisfying the previous smallness hypotheses related to the amplitude, the slope and the depth. For these initial data we show that the amplitude and the slope remain bounded and the unique singularity can be a blow up of the curvature with finite first derivative , *i.e.* the singularity can be *a corner*. This result excludes the formation of cups (blow up of the first and second derivatives) and turning waves for these initial data, remaining open the existence (or non-existence) of corners (blow up of the curvature with finite first derivative) during the evolution. Notice that in the limit $l \to \infty$ we recover the result contained in [13].

All these results have been published in [26, 34]. Notice that the previous results have a fixed permeability $\kappa = \kappa^1 \mathbf{1}_{|y|<l}$. Thus, a natural question arises: what happen if $\kappa = \kappa^1\mathbf{1}_{\Omega^1} + \kappa^2\mathbf{1}_{\Omega^2}$?

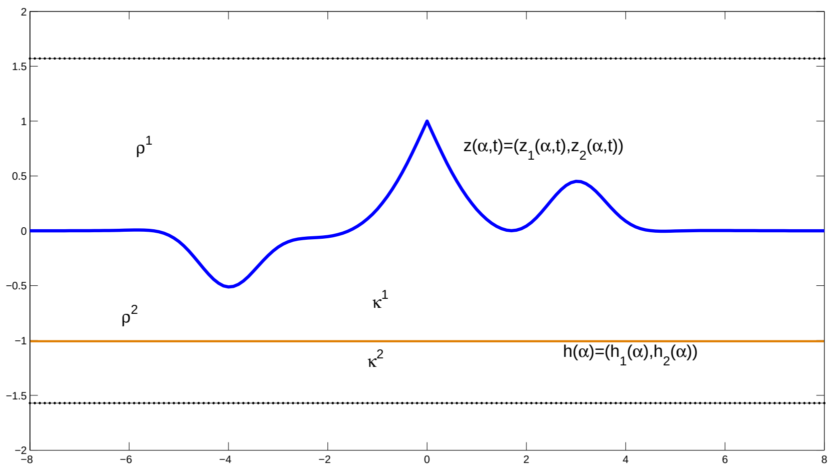

Figure 2.4: The physical situation.

2.4 The inhomogeneous Muskat problem

We study the evolution of the interface between two different incompressible fluids with the same viscosity coefficient in a porous medium with two different permeabilities. This problem is of practical importance because it is used as a model for a geothermal reservoir (see [11] and references therein). The velocity of a fluid flowing in a porous medium satisfies Darcy's law (see [4, 47, 48])

$$\frac{\mu}{\kappa(\vec{x})}v = -\nabla p - g\rho(\vec{x})(0,1),$$

where μ is the dynamic viscosity, $\kappa(\vec{x})$ is the permeability of the medium, g is the acceleration due to gravity, $\rho(\vec{x})$ is the density of the fluid, $p(\vec{x})$ is the pressure of the fluid and $v(\vec{x})$ is the incompressible velocity field. In our favourite units, we can assume $g = \mu = 1$.

The spatial domains considered in this part of the thesis are $S = \mathbb{R}^2, \mathbb{T}\times\mathbb{R}$ (infinite depth) and $\mathbb{R}\times(-\pi/2,\pi/2)$ (finite depth). We have two immiscible and incompressible fluids with the same viscosity and different densities; ρ^1 fill the upper domain $S^1(t)$ and ρ^2 fill the lower domain $S^2(t)$. The curve

$$z(\alpha,t) = \{(z_1(\alpha,t), z_2(\alpha,t)) : \alpha\in\mathbb{R}\}$$

is the interface between the fluids. In particular we are making the ansatz that S^1 and S^2 are a partition of S and they are separated by a curve z (see Figure 2.4).

As before, the system is in the Rayleigh-Taylor stable regime if the denser fluid is below the lighter one, *i.e.* $\rho^2 > \rho^1$ (see Section 2.3).

If in the first part of the thesis the permeability was a fixed constant inside the porous medium, in this part we study the case where permeability $\kappa(\vec{x})$ is a step function, more precisely, we have a curve

$$h(\alpha) = \{(h_1(\alpha), h_2(\alpha)) : \alpha\in\mathbb{R}\}$$

separating two regions with different values for the permeability (see Figure 2.4). We study the regime with infinite depth, for periodic and for "flat at infinity" initial datum, but also the case where the depth is finite and equal to $\frac{\pi}{2}$. In the region above the curve $h(\alpha)$ the permeability is

$\kappa(\vec{x}) \equiv \kappa^1$, while in the region below the curve $h(\alpha)$ the permeability is $\kappa(\vec{x}) \equiv \kappa^2 \neq \kappa^1$. Note that the curve $h(\alpha)$ is known and fixed. We write

$$\mathcal{K} = \frac{\kappa^1 - \kappa^2}{\kappa^1 + \kappa^2}.$$

This is an adimensional parameter akin to the Atwood's number.

Then it follows from Darcy's law that the vorticity is

$$\omega(\vec{x}) = \varpi_1(\alpha,t)\delta(\vec{x} - z(\alpha,t)) + \varpi_2(\alpha,t)\delta(\vec{x} - h(\alpha)),$$

where ϖ_1 corresponds to the difference of the densities, ϖ_2 corresponding to the difference of permeabilities and δ is the usual Dirac's distribution. In fact both amplitudes for the vorticity are quite different, while ϖ_1 is a derivative, the amplitude ϖ_2 has a nonlocal character (see (2.13), (2.15) and Section 10.2). The equation for the interface, when $h(x) = (x, -h_2)$ and the fluid fill the whole plane, is

$$\partial_t f(x) = \frac{\kappa^1(\rho^2 - \rho^1)}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x f(x) - \partial_x f(\beta))(x-\beta)}{(x-\beta)^2 + (f(x) - f(\beta))^2} d\beta + \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varpi_2(\beta)(x - \beta + \partial_x f(x)(f(x) + h_2))}{(x-\beta)^2 + (f(x) + h_2)^2} d\beta, \quad (2.12)$$

with

$$\varpi_2(x) = \frac{\kappa^1 - \kappa^2}{\kappa^2 + \kappa^1} \frac{\kappa^1(\rho^2 - \rho^1)}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\partial_x f(\beta)(h_2 + f(\beta))}{(x-\beta)^2 + (-h_2 - f(\beta))^2} d\beta \quad (2.13)$$

If the fluids fill the whole space but the initial curve is periodic the equation reduces to

$$\partial_t f(x) = \frac{\kappa^1(\rho^2 - \rho^1)}{4\pi} \text{P.V.} \int_{\mathbb{T}} \frac{\sin(x-\beta)(\partial_x f(x) - \partial_x f(\beta)) d\beta}{\cosh(f(x) - f(\beta)) - \cos(x-\beta)} + \frac{1}{4\pi} \text{P.V.} \int_{\mathbb{T}} \frac{(\partial_x f(x) \sinh(f(x) + h_2) + \sin(x-\beta))\varpi_2(\beta) d\beta}{\cosh(f(x) + h_2) - \cos(x-\beta)}, \quad (2.14)$$

where the second vorticity amplitude can be written as

$$\varpi_2(x) = \frac{\kappa^1(\rho^2 - \rho^1)}{2\pi} \frac{\kappa^1 - \kappa^2}{\kappa^1 + \kappa^2} \text{P.V.} \int_{\mathbb{T}} \frac{\sinh(h_2 + f(\beta)) \partial_x f(\beta) d\beta}{\cosh(h_2 + f(\beta)) - \cos(x-\beta)}. \quad (2.15)$$

If we consider the regime where the amplitude of the wave and the depth of the medium are of the same order then the equation for the interface, when the depth is chosen to be $\pi/2$, is

$$\begin{aligned}
\partial_t f(x) &= \frac{\kappa^1(\rho^2 - \rho^1)}{4\pi} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x f(x) - \partial_x f(\beta)) \sinh(x-\beta)}{\cosh(x-\beta) - \cos(f(x) - f(\beta))} d\beta \\
&+ \frac{\kappa^1(\rho^2 - \rho^1)}{4\pi} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x f(x) + \partial_x f(\beta)) \sinh(x-\beta)}{\cosh(x-\beta) + \cos(f(x) + f(\beta))} d\beta \\
&+ \frac{1}{4\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varpi_2(\beta)(\sinh(x-\beta) + \partial_x f(x) \sin(f(x) + h_2))}{\cosh(x-\beta) - \cos(f(x) + h_2)} d\beta \\
&+ \frac{1}{4\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varpi_2(\beta)(-\sinh(x-\beta) + \partial_x f(x) \sin(f(x) - h_2))}{\cosh(x-\beta) + \cos(f(x) - h_2)} d\beta, \quad (2.16)
\end{aligned}$$

where

$$
\begin{aligned}
\varpi_2(x) &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\beta)\frac{\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \\
&\quad -\mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\beta)\frac{\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta \\
&\quad +\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}}*\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \\
&\quad -\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}}*\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta, \qquad (2.17)
\end{aligned}
$$

with

$$
G_{h_2,\mathcal{K}}(x)=\mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}{1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}\right)
$$

a Schwartz function.

For these equations, in Chapter 11, we obtain a maximum principle for $\|f(t)\|^2_{L^2(\mathbb{R})}$ (see Theorem 11.1)

$$
\|f(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\frac{\|v\|^2_{L^2(\mathbb{R}\times(-h_2,\pi/2))}}{\kappa^1(\rho^2-\rho^1)}+\frac{\|v\|^2_{L^2(\mathbb{R}\times(-\pi/2,-h_2))}}{\kappa^2(\rho^2-\rho^1)}ds=\|f_0\|^2_{L^2(\mathbb{R})}.
$$

We also show local in time solvability in Sobolev spaces (see Theorems 11.2 and 11.3). These proofs follows the same ideas as in Chapter 4. It is interesting that we do not need a condition on the sign of $\mathcal{K}$ to ensure the existence of solution.

In Chapter 12, we obtain a family of initial data, depending on the physycal parameters h_2 and $\mathcal{K}$, such that they turn in finite time (see Theorems 12.1 and 12.2). These singularities are analogous the singularities in Chapter 8.

These result have been published in [5].

2.5 A comparison between the models

In order to better understand the evolution of equations (2.5), (2.12) and (2.14), we perform some numerics following the ideas in [25] (see Chapter 13). We consider the same initial datum evolving with (2.10) and (2.5) and we compare the maximum. Then, these numerics show that the confined case is more singular than the case where the depth is infinite, *i.e.* the decay in the confined case is slower. These simulations agree with the result in Chapter 7 and the estimate (2.11).

We also compare the case homogeneous (2.10) and the case with different permeabilities (2.12). In these simulations we observe that $\|f\|_{C^1}$ decays but rather differently depending on $\mathcal{K}$. If $\mathcal{K}<0$ the decay of $\|f\|_{L^\infty(\mathbb{R})}$ is faster than the case $\mathcal{K}=0$. When the evolution of $\|\partial_x f\|_{L^\infty}$ is considered, the situation is reversed. Now the simulations corresponding to $\mathcal{K}>0$ have the faster decay. With these result we can not define a *stable* regime for $\mathcal{K}$ in which the evolution would be *smoother*. Recall that we know that there is not any hypothesis on the sign or size of $\mathcal{K}$ to ensure the existence (see Theorem 11.2 and 11.3).

In Section 13.4 we obtain curves such that they turn only in the finite depth case, while, if the same curves evolve in the whole plane they become graphs (see Theorem 13.1). The proof of this fact is computer assited and, in this section we only outline the analytical part of the proof. For the details about the computation of the integrals using interval arithmetic and its background, the codes of the proof and some other technical issues see [32].

As a corollary (see Corollary 13.1) we obtain that there exist solutions to the water wave problem such that they turn if the depth is finite but the same curves if the depth is infinite become graphs.

As we said before, the case $\mathcal{K} > 0$ seems to be more stable (from the viewpoint of singularity formation). However, this is not true (see Numerical evidence 13.2 and Theorem 13.2). This latter result means that the different permeabilities can not prevent the turning.

These results are contained in [33].

2.6 Conclusions

In this thesis we have continued the study of internal waves in a porous medium contained in the works [1, 7, 10, 13, 22, 23, 24, 25, 38, 54] and the references therein. In particular we have researched the effect of boundaries and regions with different permeabilities in the evolution of an internal wave in a porous medium. This analysis resulted in the works [5, 26, 33, 34].

When the results for the confined case (see Chapters 3-9 and Chapter 13) are compared with the known results in the case where the depth is infinite (see [10, 13, 23, 24]) three main differences appear:

1. the decay of the maximum amplitude is slower in the confined case.
2. there are smooth curves with finite energy that turn over in the confined case but do not show this behaviour when the fluids fill the whole plane.
3. to avoid the turning effect in the confined case you need to have smallness conditions in $\|f_0\|_{L^\infty(\mathbb{R})}$ and $\|\partial_x f_0\|_{L^\infty(\mathbb{R})}$. However, in the unconfined case, only the condition in the slope is required. Moreover, in the confined case a new region without turning effect appears: a region without a maximum principle for the slope but with an uniform bound. In both cases (the region with the maximum principle and the region with the uniform bound), Theorem 9.1 ensures the existence of a global Lipschitz continuous solution.

Keeping these results in mind, we can say that the boundaries make the problem more singular.

Concerning the homogeneous and confined case, there are some questions that remain open. For instance,

1. the existence of a wave whose maximum slope grows but remains uniformly bounded,
2. the existence of a wave with small slope such that, due to the distance to the boundaries, it grows,
3. the existence (or non-existence) of corner-like singularities when the initial data considered is small in $W^{1,\infty}(\mathbb{R})$.

In the inhomogeneous case (see Chapters 10-12 and Chapter 13) only few preliminary results are known, being the lack of stability condition for the sign of the permeability jump one of the most intriguing. The absence of maximum principles and other conserved quantities for this system and the role of the sign of $\mathcal{K}$ are very important questions that should be addressed in future works. Another open problem is the global existence of some special class of solutions and the existence (even locally in time) when the interface reach the curve where the permeability changes.

Part I

Local solvability for the confined Muskat problem

Chapter 3

The confined Muskat problem

3.1 Foreword

In this Chapter we obtain the equation for the dynamics of an interface in a bounded porous medium and we study some properties of the system

$$\begin{cases} \frac{\mu}{\kappa}v(x,y,t) = -\nabla p(x,y,t) - g\rho(0,1) \quad \text{in } S,\ t>0, \\ \nabla\cdot v(x,y,t) = 0 \quad \text{in } S,\ t>0, \\ \partial_t\rho(x,y,t) + v\cdot\nabla\rho(x,y,t) = 0 \quad \text{in } S,\ t>0, \\ v_2(x,l,t) = v_2(x,-l,t) = 0 \quad \text{in } \mathbb{R},\ t>0, \\ f(x,0) = f_0(x) \quad \text{in } \mathbb{R}, \end{cases}$$

We denote by $v^1(x,y,t)$ the velocity field in $S^1(t)$ and by $v^2(x,y,t)$ the velocity field in $S^2(t)$. The interface moves along with the fluids. Therefore, if initially we have an interface which is the graph of a function, we have the following equation for the interface:

$$\partial_t f(x,t) = (-\partial_x f(x,t),1)\cdot v^i(x,f(x,t),t) = \sqrt{1+(\partial_x f(x,t))^2}\,n\cdot v^i, \tag{3.1}$$

where n denotes the unit normal to the interface.

In each subdomain $S^i(t)$ the fluids satisfy Darcy's law,

$$\frac{\mu}{\kappa}v^i(x,y,t) = -\nabla p^i(x,y,t) - g\rho^i(0,1) \quad \text{in } S^i(t), \tag{3.2}$$

and the incompressibility condition

$$\nabla\cdot v^i(x,y,t) = 0 \quad \text{in } S^i(t). \tag{3.3}$$

Thus, we obtain that the velocities v^i are smooth in S^i, their normal components are continuous along the interface but, due to the definition of the density ρ (see (2.2)), its tangential components are not. We neglect surface tension effects.

In the following section we translate the *local* equation (3.1), which involves f and $v\cdot n$, into a nonlocal equation involving only f. Then, in Section 3.3, we prove that, taking the limit $l\to\infty$, we recover the contour equation studied in [23]. Finally, in Section 3.4, we obtain some properties of the solution to the interface equation f, and we study the continuity equation present in (2.3).

In particular, we show that if the initial datum is even, odd or periodic this symmetry propagates (see Lemma 3.2), we obtain the linearized equation and we show that the density given by (2.2) is a weak solution of the continuity equation if and only if the interface satisfies (3.1).

Remark 3.1. *In order to simplify the notation we take $\mu/\kappa = g = 1$ and we sometimes suppress the dependence on t. We denote v_i the component $i-th$ of the vector v. We remark that v^i is the velocity field in $S^i(t)$. We write n for the unitary normal to the curve Γ vector and $\bar{n}$ for the non-unitary normal vector. We denote $\bar{\rho} = \frac{\rho^2-\rho^1}{4l}$. We define*

$$H^s_l = H^s(\mathbb{R}) \cap \{f : \|f\|_{L^\infty} < l\}.$$

3.2 The equation for the internal wave

Now, we obtain the contour equation in an explicit formula following two different approaches. Given $l > 0$ we consider the system (2.3) where $g = \frac{\mu}{\kappa} = 1$. We also drop the dependence in t of the interface curve.

We have to add boundary conditions to this equation. We consider impermeable boundary conditions for v, *i.e.* $v(x, \pm l, t) \cdot n = 0$.

Using the incompressibility condition we have that there exists a scalar function Ψ such that $v = \nabla^\perp \Psi$. The function Ψ is the stream function. Then,

$$\Delta\Psi = -\text{curl}(0, \rho) = \omega,$$

where the vorticity ω is supported on the curve z,

$$\omega(x, y) = \varpi(\alpha)\delta((x, y) - z(\alpha, t)),$$

with amplitude

$$\varpi(\alpha) = -(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha).$$

Here we have used that ρ, v, p is a weak solution of the system (2.3) (see Section 3.4).

3.2.1 Using the Fundamental Solution for the Poisson equation

Our first approach is based on the Fundamental solution and in the use of complex variables. In this domain we need to obtain the Biot-Savart Law. Using the images method[1] we obtain that the Green function for the equation $\Delta u = f$ in the strip $\mathbb{R} \times (0, 2l)$ with homogeneous Dirichlet conditions is given by

$$u(x, y) = \int_{\mathbb{R}^2} G(x, y, \mu, \nu) f(\mu, \nu) d\mu d\nu,$$

with

$$G(x, y, \mu, \nu) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\log\left(\sqrt{(x-\mu)^2 + (y - (4nl + \nu))^2}\right) - \log\left(\sqrt{(x-\mu)^2 + (y - (4nl - \nu))^2}\right) \right].$$

[1] Equivalently, we can use the conformal mapping method

3.2. The equation for the internal wave

The Biot-Savart Law in this strip is given by the kernel

$$BS(x,y,\mu,\nu)=\nabla^{\perp}_{x,y}G(x,y,\mu,\nu)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(\Gamma_n^+)^{\perp}}{|\Gamma_n^+|^2}-\frac{(\Gamma_n^-)^{\perp}}{|\Gamma_n^-|^2}\right],$$

where

$$\Gamma_n^+=(x-\mu,y-(4nl+\nu)),$$

and

$$\Gamma_n^-=(x-\mu,y-(4nl-\nu)).$$

It is useful to consider complex variables notation. Then

$$\overline{BS}(x,y,\mu,\nu)=\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\left[\frac{1}{\Gamma_n^+}-\frac{1}{\Gamma_n^-}\right].$$

Fixed n, we compute the following

$$\frac{1}{\Gamma_n^+}+\frac{1}{\Gamma_{-n}^+}=\frac{2(x-\mu+i(y-\nu))}{(x-\mu+i(y-\nu))^2+(4nl)^2},$$

$$\frac{1}{\Gamma_n^-}+\frac{1}{\Gamma_{-n}^-}=\frac{2(x-\mu+i(y+\nu))}{(x-\mu+i(y+\nu))^2+(4nl)^2}.$$

We change variables ($y-l=y,\nu-l=\nu$) to recover the initial strip $S=\mathbb{R}\times(-l,l)$, moreover, without lossing generality we take $l=\pi/2$. Due to the formula

$$\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2+(2n\pi)^2}=\frac{1}{2}\coth\left(\frac{z}{2}\right), \tag{3.4}$$

we obtain

$$BS(x,y,\mu,\nu)=\frac{1}{4\pi}\left(-\frac{\sin(y-\nu)}{\cosh(x-\mu)-\cos(y-\nu)}-\frac{\sin(y+\nu)}{\cosh(x-\mu)+\cos(y+\nu)},\right.$$
$$\left.\frac{\sinh(x-\mu)}{\cosh(x-\mu)-\cos(y-\nu)}-\frac{\sinh(x-\mu)}{\cosh(x-\mu)+\cos(y+\nu)}\right). \tag{3.5}$$

Using that $\omega=\varpi(\alpha)\delta((x,y)-(z_1(\alpha),z_2(\alpha)))$, we have that the velocity is

$$v(\vec{x})=\frac{1}{4\pi}\int_{\mathbb{R}}\varpi(\beta)\left(\frac{-\sin(y-z_2(\beta))}{\cosh(x-z_1(\beta))-\cos(y-z_2(\beta))}+\frac{-\sin(y+z_2(\beta))}{\cosh(x-z_1(\beta))+\cos(y+z_2(\beta))},\right.$$
$$\left.\frac{\sinh(x-z_1(\beta))}{\cosh(x-z_1(\beta))-\cos(y-z_2(\beta))}-\frac{\sinh(x-z_1(\beta))}{\cosh(x-z_1(\beta))+\cos(y+z_2(\beta))}\right).$$

We use the identity

$$\int_{\mathbb{R}}\partial_\eta\log(\cosh(z_1(\alpha)-z_1(\eta))\pm\cos(z_2(\alpha)\pm z_2(\eta)))=0$$

to obtain that the average velocity in the curve is

$$v(z(\alpha))=-\frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_1(\eta)\left[\frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))-\cos(z_2(\alpha)-z_2(\eta))}\right.$$
$$\left.+\frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))+\cos(z_2(\alpha)+z_2(\eta))}\right]d\eta$$
$$-i\frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_2(\eta)\left[\frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))-\cos(z_2(\alpha)-z_2(\eta))}\right.$$
$$\left.-\frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))+\cos(z_2(\alpha)+z_2(\eta))}\right]d\eta.$$

The interface is convected by this velocity but we can add any velocity in the tangential direction without altering the shape of the curve. The tangential velocity in a curve only changes the parametrization. We consider then the following equation with the redefined velocity

$$\partial_t z(\alpha) = v(z(\alpha)) + c(\alpha)\partial_\alpha z(\alpha),$$

where

$$c(\alpha) = \frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))-\cos(z_2(\alpha)-z_2(\eta))} + \frac{\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))+\cos(z_2(\alpha)+z_2(\eta))}d\eta.$$

Following this approach we obtain

$$\partial_t z = \frac{\rho^2-\rho^1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \left[\frac{(\partial_\alpha z(\alpha)-\partial_\alpha z(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))-\cos(z_2(\alpha)-z_2(\eta))} + \frac{(\partial_\alpha z_1(\alpha)-\partial_\alpha z_1(\eta), \partial_\alpha z_2(\alpha)+\partial_\alpha z_2(\eta))\sinh(z_1(\alpha)-z_1(\eta))}{\cosh(z_1(\alpha)-z_1(\eta))+\cos(z_2(\alpha)+z_2(\eta))}\right]d\eta.$$

Due to the choice of $c(\alpha)$ we obtain that, if initially the curve can be parametrized as a graph, *i.e.*, $z(x,0)=(x,f_0(x))$, we have that the velocity v_1 on the curve is zero, thus, our curve is parametrized as a graph for $t>0$ and we recover the contour equation (2.5).

3.2.2 Using Fourier series

We can follow an approach based on Fourier series. The impermeable boundary conditions for v gives us homogeneus Dirichlet boundary conditions for Ψ. Recalling (3.2) and assuming that the curve can be parametrized as a graph, $f(x,t)$, we have the following espression

$$\Psi = \sum_{n=1}^{\infty}\psi_n(x)\sin\left(\frac{n\pi(y+l)}{2l}\right).$$

We denote $\ell_n(y) = n\pi(y+l)/2l$.

Using Fourier method and the equation for Ψ we obtain the equation for ψ_n:

$$\begin{aligned}\partial_x^2\psi_n(x) - \frac{n^2\pi^2}{4l^2}\psi_n(x) &= -\frac{1}{l}\int_{-l}^{l}\partial_x\rho\sin(\ell_n(y))\,dy\\ &= -\frac{1}{l}\int_{-l}^{l}\left(\rho^2-\rho^1\right)\partial_x f(x)\,\delta_{y=f(x)}\sin(\ell_n(y))\,dy\\ &= -\frac{\rho^2-\rho^1}{l}\partial_x f(x)\sin(\ell_n(f(x)))\\ &= \frac{2(\rho^2-\rho^1)}{n\pi}\partial_x(\cos(\ell_n(f(x)))).\end{aligned}$$

Taking the Fourier transform

$$\mathcal{F}(\psi_n)(\xi) = \frac{\rho^2-\rho^1}{l}\mathcal{F}(\sin(\ell_n(f(x)))\partial_x f(x))\frac{1}{\xi^2+\frac{n^2\pi^2}{4l^2}},$$

and we obtain the expression

$$\psi_n(x) = (\rho^2-\rho^1)\int_{\mathbb{R}} \partial_x f(\eta)\sin\left(\ell_n(f(\eta))\right)\frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi}d\eta$$
$$= (\rho^2-\rho^1)\int_{\mathbb{R}}\left(-\frac{2l}{n\pi}\right)\partial_\eta\cos(\ell_n(f(\eta)))\frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi}d\eta. \quad (3.6)$$

The velocity is given by

$$\begin{aligned} v_1 &= -\partial_y\Psi = (\rho^2-\rho^1)\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\partial_\eta\cos\left(\ell_n(f(\eta))\right)\cos\left(\ell_n(y)\right)\frac{e^{-\frac{n\pi}{2l}|x-\eta|}}{n\pi}d\eta \\ &= -2\bar{\rho}\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\cos\left(\ell_n(f(\eta))\right)\cos\left(\ell_n(y)\right)e^{-\frac{n\pi}{2l}|x-\eta|}\left(\frac{x-\eta}{|x-\eta|}\right)d\eta \\ &= -\bar{\rho}\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\left(\cos\left(\frac{n\pi}{2l}(y-f(x-\eta))\right)+\cos\left(\frac{n\pi}{2l}(y+f(x-\eta))\right)(-1)^n\right) \\ &\qquad \cdot e^{-\frac{n\pi}{2l}|\eta|}\frac{\eta}{|\eta|}d\eta, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} v_2 &= \partial_x\Psi = 2\bar{\rho}\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\eta)\sin\left(\ell_n(f(\eta))\right)\sin\left(\ell_n(y)\right)e^{-\frac{n\pi}{2l}|x-\eta|}\left(-\frac{x-\eta}{|x-\eta|}\right)d\eta \\ &= -\bar{\rho}\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\left(\cos\left(\frac{n\pi}{2l}(y-f(x-\eta))\right)-\cos\left(\frac{n\pi}{2l}(y+f(x-\eta))\right)(-1)^n\right) \\ &\qquad \cdot\partial_x f(x-\eta)e^{-\frac{n\pi}{2l}|\eta|}\frac{\eta}{|\eta|}d\eta \end{aligned} \quad (3.8)$$

The tangent vector to the curve $z(x,t)=(x,f(x))$ is $\tau=(1,\partial_x f(x))$. In addition, the non-unitary upper normal to the curve is $\bar{n}=(-\partial_x f(x),1)$. The contour equation satisfies the following kinematic condition

$$\partial_t f(x) = \bar{n}\cdot v|_{y=f(x)}.$$

By the incompressibility condition, v is continuous in the normal direction through the interface. On the other hand, because of the concentration of vorticity along the interface, the velocity is discontinuous in the tangential direction at the interface. In order to prove this claim, we have to take a double limit carefully.

Proposition 3.1. *Let $f\in H^3_l(\mathbb{R})$ and v be the velocity field of (2.3). We consider the following limits on the interface $f(x,t)$:*

$$\begin{aligned} v^+(x,f(x,t),t) &= \lim_{\epsilon\to 0^+} v(x-\epsilon\partial_x f(x,t), f(x,t)+\epsilon), \\ v^-(x,f(x,t),t) &= \lim_{\epsilon\to 0^+} v(x+\epsilon\partial_x f(x,t), f(x,t)-\epsilon). \end{aligned}$$

Then,

$$v^{\pm} = \pm\frac{\rho^2-\rho^1}{2}\frac{\partial_x f(x)}{1+(\partial_x f(x))^2}(1,\partial_x f(x)) + \frac{\bar{\rho}}{2}\left(-P.V.\int_{\mathbb{R}}\left[\Xi_1+\Xi_2\right]d\eta, -P.V.\int_{\mathbb{R}}\partial_x f(\eta)\left[\Xi_1-\Xi_2\right]d\eta\right).$$

i.e., v is discontinuous along the interface in the tangential direction $(1,\partial_x f(x))$.

Proof. The velocity field of (2.3), $v(x,y,t)$, is defined in (3.7) and (3.8). We do for the first component v_1, being analogous for the second component v_2. Let $\delta > 0$ be a parameter. We decompose our domain in its *'in'* and *'out'* parts:

$$v_1^{\pm} = \lim_{\epsilon\to 0^+} -\bar{\rho}\sum_{n=1}^{\infty} \text{P.V.}\left(\int_{B(x,\delta)} + \int_{\mathbb{R}-B(x,\delta)}\right)\left(\cos\left(\frac{n\pi}{2l}(f(x)\pm\epsilon - f(\eta))\right)\right.$$
$$\left. + \cos\left(\frac{n\pi}{2l}(f(x)\pm\epsilon + f(\eta)) + n\pi\right)\right)e^{-\frac{n\pi}{2l}|x\mp\epsilon\partial_x f(x)-\eta|}\frac{x\mp\epsilon\partial_x f(x)-\eta}{|x\mp\epsilon\partial_x f(x)-\eta|}d\eta$$

We write S_{in}^1 for the series, corresponding to $v_1^{\pm}$, with the integral domain a neighbourhood of the point x and S_{out}^1 the series with the outer domain and we take the double limit.

For S_{out}^1 we take the limit $\epsilon \to 0$ and then the limit $\delta \to 0$. For the ϵ limit the expression is continuous so we obtain

$$\lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+} S_{out}^1 = -\bar{\rho}\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\left(\cos\left(\frac{n\pi}{2l}(f(x)-f(\eta))\right)\right.$$
$$\left. + \cos\left(\frac{n\pi}{2l}(f(x)+f(\eta)) + n\pi\right)\right)e^{-\frac{n\pi}{2l}|x-\eta|}\frac{x-\eta}{|x-\eta|}d\eta.$$

In order to study S_{in}^1 we have to take $\epsilon \to 0$ faster than δ, i.e., $\epsilon = o(\delta)$. We do it to get $x \pm \epsilon\partial_x f(x)$ inside $B(x,\delta)$. We want to consider the series as an integral. Let $\alpha = \frac{n\pi\epsilon}{2l}$, then since α is a continuous variable as $\epsilon \to 0^+$ and $d\alpha = \frac{\pi\epsilon}{2l}$, we have

$$\lim_{\epsilon\to 0^+} S_{in}^1 = -\lim_{\delta,\epsilon\to 0^+}\int\limits_{B(x,\delta)}\frac{\rho^2-\rho^1}{2\pi\epsilon}\int_0^{\infty}\frac{x\mp\epsilon\partial_x f(x)-\eta}{|x\mp\epsilon\partial_x f(x)-\eta|}\exp\left(-\alpha\frac{|x\mp\epsilon\partial_x f(x)-\eta|}{\epsilon}\right)$$
$$\left(\cos\left(\alpha\left(\frac{f(x)-f(\eta)}{\epsilon}\pm 1\right)\right) + \cos\left(\alpha\left(\frac{f(x)+f(\eta)+2l}{\epsilon}\pm 1\right)\right)\right)d\alpha d\eta.$$

Using

$$\int_0^{\infty} e^{-\frac{\alpha}{\varepsilon}|\eta|}\cos(\alpha\mu)\,d\alpha = \frac{\varepsilon|\eta|}{\eta^2+\varepsilon^2\mu^2} \tag{3.9}$$

in the previous limit, we obtain

$$\lim_{\epsilon\to 0^+} S_{in}^1 = -\lim_{\delta,\epsilon\to 0^+}\frac{\rho^2-\rho^1}{2\pi}\int_{x-\delta}^{x+\delta}\frac{x-\eta\mp\epsilon\partial_x f(x)}{(x-\eta\mp\epsilon\partial_x f(x))^2 + (f(x)+f(\eta)+2l\pm\epsilon)^2}$$
$$+\frac{x-\eta\mp\epsilon\partial_x f(x)}{(x-\eta\mp\epsilon\partial_x f(x))^2+(f(x)-f(\eta)\pm\epsilon)^2}d\eta.$$

We remark that the first term in this integral is not singular, so this part vanishes in the limit. Thus,

$$\lim_{\epsilon\to 0^+} S_{in}^1 = -\lim_{\epsilon\to 0^+}\frac{\rho^2-\rho^1}{2\pi}\int_{x-\delta}^{x+\delta}\frac{(x-\eta\mp\epsilon\partial_x f(x))d\eta}{(x-\eta\mp\epsilon\partial_x f(x))^2+(f(x)-f(\eta)\pm\epsilon)^2}.$$

Supposing that $f \in H^3$, we approximate $f(x)-f(\eta)$ using the Mean Value Theorem and we sustitute $\partial_x f(\xi)$ by $\partial_x f(x)$. We have

$$\lim_{\delta,\epsilon\to 0^+} S_{in}^1 = -\lim_{\delta,\epsilon\to 0^+}\frac{\rho^2-\rho^1}{2\pi}\int_{x-\delta}^{x+\delta}\frac{(x-\eta\mp\epsilon\partial_x f(x))d\eta}{(x-\eta\mp\epsilon\partial_x f(x))^2+(\partial_x f(x)(x-\eta)\pm\epsilon)^2}.$$

Changing variables $\beta = x - \eta$, we obtain

$$\int_{x-\delta}^{x+\delta} \frac{(x-\eta \mp \epsilon\partial_x f(x))d\eta}{(x-\eta\mp\epsilon\partial_x f(x))^2 + (\partial_x f(x)(x-\eta)\pm\epsilon)^2} = \frac{1}{1+(\partial_x f(x))^2}\int_{-\delta}^{\delta}\frac{\beta\mp\epsilon\partial_x f(x)}{\beta^2+\epsilon^2}d\beta$$
$$= \frac{1}{1+(\partial_x f(x))^2}\left(\frac{1}{2}\log(\beta^2+\epsilon^2)\mp\partial_x f(x)\arctan\left(\frac{\beta}{\epsilon}\right)\right)\Bigg|_{\beta=-\delta}^{\beta=\delta}.$$

Taking the limits in the $\epsilon = o(\delta)$ regime, we get

$$\lim_{\delta,\epsilon\to 0^+} S^1_{in} = \frac{\rho^2-\rho^1}{2\pi}\frac{\pm\partial_x f(x)}{1+(\partial_x f(x))^2}\left(\arctan(\infty)-\arctan(-\infty)\right).$$

We notice that the way to carry out the limits for S^2_{out} is completely analogous to the limits for S^1_{out}. In order to obtain the result for S^2_{in} the unique difference with S^1_{in} is that we have to substitute $\partial_x f(\eta)$ with $\partial_x f(x)$. The remainder of this substitution goes to zero with δ. Summing up the series S^i_{out} we conclude the result.

□

Using the expressions (3.7) and (3.8) we obtain the formula

$$\partial_t f(x) = \bar{\rho}A[f](x), \tag{3.10}$$

where the nonlinear and nonlocal operator is as follows:

$$\begin{aligned} A[f](x) = 2\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}\Big[&\cos(\ell_n(f(x)))\cos(\ell_n(f(x-\eta)))\,\partial_x f(x) \\ &-\sin(\ell_n(f(x)))\sin(\ell_n(f(x-\eta)))\,\partial_x f(x-\eta)\Big]d\eta \\ = \sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}\Big[&\cos\left(\frac{n\pi}{2l}(f(x)-f(x-\eta))\right)(\partial_x f(x)-\partial_x f(x-\eta)) \\ &+(-1)^n\cos\left(\frac{n\pi}{2l}(f(x)+f(x-\eta))\right)(\partial_x f(x)+\partial_x f(x-\eta))\Big]d\eta. \end{aligned} \tag{3.11}$$

We have the following Lemma concerning the finiteness of this expression:

Lemma 3.1. *Let $f \in H^3_l$ and $A[f]$ be the operator defined by (3.11). Then, $A[f] \in L^\infty(\mathbb{R})$ and*

$$|A[f](x)| \le c(\|f\|_{C^2}, l),\ \forall x\in\mathbb{R}.$$

Proof. We decompose $A[f]$ in two terms, S^1 and S^2:

$$S^1 = \sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}\cos\left(\frac{n\pi}{2l}(f(x)-f(x-\eta))\right)(\partial_x f(x)-\partial_x f(x-\eta))d\eta,$$

and

$$S^2 = \sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}(-1)^n\cos\left(\frac{n\pi}{2l}(f(x)+f(x-\eta))\right)(\partial_x f(x)+\partial_x f(x-\eta))d\eta.$$

We split in the *'in'* and *'out'* parts:

$$\left|S^1\right| \leq \sum_{n=1}^{\infty}\left(\text{P.V.}\int_{|\eta|<1}\left|e^{-\frac{n\pi}{2l}|\eta|}(\partial_x f(x)-\partial_x f(x-\eta))\right|d\eta \right.$$
$$\left. +\int_{|\eta|\geq 1}\left|e^{-\frac{n\pi}{2l}|\eta|}(\partial_x f(x)-\partial_x f(x-\eta))\right|d\eta\right)=\sum_{n=1}^{\infty} I^n_{in}+I^n_{out}.$$

These integrals are bounded as follows

$$I^n_{out}\leq 8l\|\partial_x f\|_{L^\infty}\frac{1}{n\pi}e^{-\frac{n\pi}{2l}},$$

and

$$I^n_{in}\leq 2\|\partial_x^2 f\|_{L^\infty}\left(\frac{4l^2}{n^2\pi^2}(1-e^{-\frac{n\pi}{2l}})-\frac{2l}{n\pi}e^{-\frac{n\pi}{2l}}\right)\leq 2\|\partial_x^2 f\|_{L^\infty}\left(\frac{4l^2}{n^2\pi^2}\right),$$

using the Mean Value Theorem, so the series is finite if $\|\partial_x f\|_{L^\infty}$ and $\|\partial_x^2 f\|_{L^\infty}$ are finite. By the Sobolev embedding this occurs if $f\in H^3$.

Now we study the second series. We write

$$S^2=\sum_{n=1}^{\infty}\text{P.V.}\int_{\mathbb{R}}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}(-1)^n\cos\left(\frac{n\pi}{2l}(f(x)+f(x-\eta))\right)$$
$$\cdot(2\partial_x f(x)-(\partial_x f(x)-\partial_x f(x-\eta))d\eta=S^2_1+S^2_2.$$

The second term S^2_2 can be bounded as the previous series S^1. The *'out'* term in S^2_1 also is analogous, so we need to study the term

$$S^2_{1,in}=2\partial_x f(x)\sum_{n=1}^{\infty}\text{P.V.}\int_{B(0,1)}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}(-1)^n\cos\left(\frac{n\pi}{2l}(2f(x)-(f(x)-f(x-\eta)))\right)d\eta,$$

and we use the classical trigonometric formulas in order to obtain

$$S^2_{1,in}=2\partial_x f(x)\sum_{n=1}^{\infty}\text{P.V.}\int_{B(0,1)}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}(-1)^n\left[\cos\left(\frac{n\pi}{2l}2f(x)\right)\cos\left(\frac{n\pi}{2l}(f(x)-f(x-\eta))\right)\right.$$
$$\left.+\sin\left(\frac{n\pi}{2l}2f(x)\right)\sin\left(\frac{n\pi}{2l}(f(x)-f(x-\eta))\right)\right]d\eta=A+B.$$

The first term can be easily bounded if we use an extra cancellation. We have

$$A=2\partial_x f(x)\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{n\pi}{2l}2f(x)\right)$$
$$\cdot\text{P.V.}\int_{B(0,1)}\frac{\eta}{|\eta|}e^{-\frac{n\pi}{2l}|\eta|}\left[\cos\left(\frac{n\pi}{2l}(f(x)-f(x-\eta))\right)-\cos\left(\frac{n\pi}{2l}\eta\partial_x f(x)\right)\right].$$

Now we use the sum-to-product trigonometric identity to obtain

$$A\leq c(\|f\|_{C^2},l)\sum_{n=1}^{\infty}n\text{P.V.}\int_{B(0,1)}e^{-\frac{n\pi}{2l}|\eta|}\eta^2 d\eta\leq c(\|f\|_{C^2},l).$$

The term B can be written as

$$B = 2\partial_x f(x) \sum_{k=1}^{\infty} \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} \Bigg[e^{-\frac{(2k-1)\pi}{2l}|\eta|} (-1) \sin\left(\frac{(2k-1)\pi}{2l} 2f(x)\right) \sin\left(\frac{(2k-1)\pi}{2l}(f(x)-f(x-\eta))\right) + e^{-\frac{2k\pi}{2l}|\eta|} \sin\left(\frac{2k\pi}{2l}2f(x)\right) \sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right) \Bigg] d\eta.$$

We add the crossed term

$$\pm e^{-\frac{(2k-1)\pi}{2l}|\eta|} \sin\left(\frac{(2k-1)\pi}{2l}2f(x)\right) \sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right),$$

thus $B = B_1 + B_2$ where

$$B_1 = 2\partial_x f(x) \sum_{k=1}^{\infty} \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} e^{-\frac{(2k-1)\pi}{2l}|\eta|} \sin\left(\frac{(2k-1)\pi}{2l}2f(x)\right) \left(\sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right) - \sin\left(\frac{(2k-1)\pi}{2l}(f(x)-f(x-\eta))\right) \right) d\eta,$$

and

$$B_2 = 2\partial_x f(x) \sum_{k=1}^{\infty} \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} \left(e^{-\frac{2k\pi}{2l}|\eta|} \sin\left(\frac{2k\pi}{2l}2f(x)\right) - e^{-\frac{(2k-1)\pi}{2l}|\eta|} \sin\left(\frac{(2k-1)\pi}{2l}2f(x)\right) \right) \sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right) d\eta.$$

Using the sum-to-product formula for the difference of the sines we conclude

$$B_1 \leq c(\|f\|_{C^1}, l).$$

We have to bound the B_2 term. In order to do this we add

$$\pm e^{-\frac{(2k-1)\pi}{2l}|\eta|} \sin\left(\frac{(2k)\pi}{2l}2f(x)\right),$$

obtaining $B_2 = B_2^1 + B_2^2$ with

$$B_2^1 = 2\partial_x f(x) \sum_{k=1}^{\infty} \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} \sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right) \left(e^{-\frac{2k\pi}{2l}|\eta|} - e^{-\frac{(2k-1)\pi}{2l}|\eta|} \right) \sin\left(\frac{2k\pi}{2l}2f(x)\right) d\eta,$$

and

$$B_2^2 = 2\partial_x f(x) \sum_{k=1}^{\infty} \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} \sin\left(\frac{2k\pi}{2l}(f(x)-f(x-\eta))\right) e^{-\frac{(2k-1)\pi}{2l}|\eta|} \left(\sin\left(\frac{2k\pi}{2l}2f(x)\right) - \sin\left(\frac{(2k-1)\pi}{2l}2f(x)\right) \right) d\eta.$$

We have

$$B_2^1 \leq c(\|f\|_{C^1}) \sum_{k=1}^{\infty} \int_0^1 e^{-\frac{(2k-1)\pi}{2l}\eta} \eta d\eta \leq c(\|f\|_{L^\infty}, l).$$

We use that the series

$$\sum_{k=1}^{\infty} \cos\left(\frac{4k-1}{l}\pi f(x)\right) \int_{B(0,1)} \frac{\eta}{|\eta|} e^{-\frac{2k-1}{2l}\pi|\eta|} \sin\left(\frac{2k}{2l}\eta\partial_x f(x)\right) d\eta$$

converges in order to bound the last term B_2^2. We write

$$B_2^2 = c(\|f\|_{C^1}, l) + 2\partial_x f(x) \sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2l} f(x)\right) \cos\left(\frac{(4k-1)\pi}{2l} 2f(x)\right)$$

$$\cdot \text{P.V.} \int_{B(0,1)} \frac{\eta}{|\eta|} e^{-\frac{(2k-1)\pi}{2l}|\eta|} \left(\sin\left(\frac{2k\pi}{2l}(f(x) - f(x-\eta))\right) - \sin\left(\frac{2k\pi}{2l}\eta\partial_x f(x)\right)\right)$$

Now using the sum-to-product formulas for the sines and integrating by parts twice we conclude the result. □

Using the formulas below

$$\sum_{n=1}^{\infty} e^{-\frac{n\pi|\eta|}{2l}} \cos\left(\frac{n\pi}{2l}(f(x) - f(x-\eta))\right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[e^{\frac{n\pi}{2l}[(f(x)-f(x-\eta))i-|\eta|]} + e^{\frac{n\pi}{2l}[(f(x)-f(x-\eta))(-i)-|\eta|]}\right]$$

$$= \frac{1}{2}\left(\frac{e^{\frac{\pi}{2l}[(f(x)-f(x-\eta))i-|\eta|]}}{1-e^{\frac{\pi}{2l}[(f(x)-f(x-\eta))i-|\eta|]}} + \frac{e^{\frac{\pi}{2l}[(f(x)-f(x-\eta))(-i)-|\eta|]}}{1-e^{\frac{\pi}{2l}[(f(x)-f(x-\eta))(-i)-|\eta|]}}\right)$$

$$= \frac{1}{2}\left(-1 + \frac{\sinh\left(\frac{\pi}{2l}|\eta|\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) - \cos(\frac{\pi}{2l}(f(x) - f(x-\eta)))}\right),$$

and

$$\sum_{n=1}^{\infty} (-1)^n e^{-\frac{n\pi|\eta|}{2l}} \cos\left(\frac{n\pi}{2l}(f(x) + f(x-\eta))\right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[e^{\frac{n\pi}{2l}[(f(x)+f(x-\eta))i-|\eta|]}(-1)^n + e^{\frac{n\pi}{2l}[(f(x)+f(x-\eta))(-i)-|\eta|](-1)^n}\right]$$

$$= -\frac{1}{2}\left(\frac{e^{\frac{\pi}{2l}[(f(x)+f(x-\eta))i-|\eta|]}}{1+e^{\frac{\pi}{2l}[(f(x)+f(x-\eta))i-|\eta|]}} + \frac{e^{\frac{\pi}{2l}[(f(x)+f(x-\eta))(-i)-|\eta|]}}{1+e^{\frac{\pi}{2l}[(f(x)-f(x-\eta))(-i)-|\eta|]}}\right)$$

$$= \frac{1}{2}\left(-1 + \frac{\sinh\left(\frac{\pi}{2l}|\eta|\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) + \cos(\frac{\pi}{2l}(f(x) + f(x-\eta)))}\right),$$

in (3.11) we obtain the integral expressions for $A[f]$:

$$A[f](x) = \frac{1}{2}\text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x-\eta))\Xi_1 + (\partial_x f(x) + \partial_x f(x-\eta))\Xi_2 d\eta, \tag{3.12}$$

where the kernels Ξ_1 and Ξ_2 are given by

$$\Xi_1(x,\eta) = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) - \cos(\frac{\pi}{2l}(f(x) - f(x-\eta)))},$$

and

$$\Xi_2(x,\eta) = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) + \cos(\frac{\pi}{2l}(f(x)+f(x-\eta)))}.$$

We remark that the spatial operator is a derivative in x and we can write also (2.8). Thus, we have

$$\int_{\mathbb{R}} f(x,t)dx = \int_{\mathbb{R}} f_0(x)dx. \tag{3.13}$$

As a conclusion, we obtain the contour equation (2.5) as a model for the evolution of a interfacial wave between two different fluids in a porous media.

3.3 On the connection with the infinite depth equation

We are interested in the limit case when $l \to \infty$ in the equation (2.5) in order to recover the equation for the whole plane (2.10). The equation for the whole plane case has been studied in [23]. Indeed, we prove the following proposition:

Proposition 3.2. *Let $f \in H^3(\mathbb{R})$ and $A[f]$ be the operator defined in (3.11). Then,*

$$\lim_{l\to\infty} \frac{1}{l} A[f](x) = \frac{2}{\pi} P.V. \int_{\mathbb{R}} \frac{\eta(\partial_x f(x) - \partial_x f(x-\eta))}{\eta^2 + (f(x)-f(x-\eta))^2} d\eta.$$

Moreover, the corresponding expressions for the velocity tends to the classical expresion using the Biot-Savart Law in the whole plane.

Proof. Given the explicit kernels Ξ_1 and Ξ_2 the proof is straightforward. If we use the expression with the series (3.11) we recover the Riemann sums when $l \to \infty$ so we change the series by an integral.

We have

$$\partial_t f(x) = S_1 + S_2,$$

where

$$S_1 = \frac{\rho_2-\rho_1}{4l} \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{n\pi|\eta|}{2l}} \cos\left(\frac{n\pi}{2l}\left(f(x) - f(x-\eta)\right)\right) \left(\partial_x f(x) - \partial_x f(x-\eta)\right) d\eta,$$

$$S_2 = \frac{\rho_2-\rho_1}{4l} \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{n\pi|\eta|}{2l}} \cos\left(\frac{n\pi}{2l}\left(f(x) + f(x-\eta)\right)\right) \cos(n\pi) \left(\partial_x f(x) + \partial_x f(x-\eta)\right) d\eta.$$

Firstly we study the behaviour of S_2: we obtain

$$S_2 = -\frac{\rho_2-\rho_1}{4l} \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{2k-1}{2l}\pi|\eta|} \cos\left(\frac{2k-1}{2l}\pi(f(x)+f(x-\eta))\right) (\partial_x f(x) + \partial_x f(x-\eta)) d\eta$$
$$+ \frac{\rho_2-\rho_1}{4l} \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{2k}{2l}\pi|\eta|} \cos\left(\frac{2k}{2l}\pi(f(x)+f(x-\eta))\right) (\partial_x f(x) + \partial_x f(x-\eta)) d\eta.$$

We write $\beta = \frac{k\pi}{l}$ and $\gamma = \frac{2k-1}{2l}\pi$. In the limit we have the Riemann integral with $d\beta = d\gamma = \frac{\pi}{l}$. Thus,

$$S_2 = \frac{\rho_2-\rho_1}{4\pi} \int_0^\infty \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\gamma|\eta|} \cos(\gamma(f(x)+f(x-\eta))) (\partial_x f(x) + \partial_x f(x-\eta)) d\eta d\gamma$$
$$- \frac{\rho_2-\rho_1}{4\pi} \int_0^\infty \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\beta|\eta|} \cos(\beta(f(x)+f(x-\eta))) (\partial_x f(x) + \partial_x f(x-\eta)) d\eta d\beta$$

and therefore

$$S_2 = 0.$$

Now we consider $\frac{n\pi}{2l} = \alpha$ and $\frac{\pi}{2l} = d\alpha$ in S_1 and, using (3.9), we obtain (2.10). The proof of the claim about the velocity's expressions follows the same steps. This concludes the result. $\square$

As a consequence of this Proposition, our equation (2.5) as $l \to \infty$ is the equation in the whole plane case, i.e., $S^1 \cup S^2 = \mathbb{R}^2$.

3.4 The continuity equation and some properties

In this section we prove that the odd, even and periodic symmetry propagates, we obtain the linearized equation and we study the equivalence between the conservation of mass equation in (2.3) and the initial value problem for the interface (2.5).

First, we show that if the initial datum is even, odd or periodic, then the evolution preserves this property:

Lemma 3.2. *Given an initial data f_0 which is odd, even or periodic, then the solution corresponding to this initial data $f(x)$ is odd, even or periodic.*

Proof. To simplify the exposition we take $\pi/2 = l$. Let f_0 be an odd initial data. Then we define $g_1(x,t) = -f(-x,t)$. In order to prove the result we want to show that the equation for $g_1(x,t)$ is the same equation as for $f(x,t)$. We have that $\partial_x g_1(x,t) = \partial_x f(-x,t)$, $\partial_t g_1(x,t) = -\partial_t f(-x,t)$ and that the nonlocal term is given by $\partial_x g_1(x-\xi) = \partial_x f(-x+\xi)$. For the evolution of g_1 we have

$$\partial_t g_1(x,t) = -\text{P.V.}\int_{-\infty}^{\infty} \frac{(\partial_x f(-x) - \partial_x f(-x-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(f(-x) - f(-x-\eta))}d\eta \\ - \int_{-\infty}^{\infty} \frac{(\partial_x f(-x) + \partial_x f(-x-\eta))\sinh(\eta)}{\cosh(\eta) + \cos(f(-x) + f(-x-\eta))}d\eta$$

Changing variables $\eta = -\xi$ in this expression and using the definition of g_1 we conclude the result.

To show the case where f_0 is even we define $g_2(x,t) = f(-x,t)$. In the same manner we show that the equation for $g_2(x,t)$ and for $f(x,t)$ is the same. For the periodic case, define $g_3(x,t) = f(x+2k\pi,t)$. In the same way we conclude the proof. $\square$

Now we obtain the linear equation associated to (2.5). In the plane it is a well-known fact (see [23]) that the linear operator is $\Lambda = (-\Delta)^{1/2}$, i.e., the linear equation associated to (2.10) is

$$\partial_t g\,(x) = \frac{\rho^1 - \rho^2}{2}\Lambda g. \tag{3.14}$$

We note

$$\Lambda g = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{1}{\eta^2}\partial_x g\,(t, x-\eta)\,d\eta = H\,(\partial_x g)\,,$$

where H is the Hilbert transform. In our domain we obtain a very similar linear operator.

Lemma 3.3 (Linearized equation). *The linearized around the rest state equation associated to (2.5) is*

$$\partial_t g = -\bar{\rho}\frac{\pi}{2l}P.V.\int_{\mathbb{R}} \frac{(g(x) - g(x-\eta))\cosh\left(\frac{\pi}{2l}\eta\right)}{\sinh^2\left(\frac{\pi}{2l}\eta\right)}d\eta$$

Proof. We take f a perturbation of the rest state, $f(t,x) = \epsilon g(t,x)$. As ϵ is small enough we can consider that

$$\Xi_1 \approx \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)-1},$$

and

$$\Xi_2 \approx \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)+1}.$$

Because of these linearized kernels we have that

$$\text{P.V.}\int_{\mathbb{R}} \partial_x g(x) \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right)\pm 1} = 0.$$

Putting the two terms together we obtain a 'Hilbert-like' transform. We have that

$$-\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x g(x-\eta)}{\sinh\left(\frac{\pi}{2l}\eta\right)} = -\text{P.V.}\int_{\mathbb{R}} \frac{\partial_\eta (g(x)-g(x-\eta))}{\sinh\left(\frac{\pi}{2l}\eta\right)},$$

so, integrating by parts, we obtain the result. □

Remark 3.2. *If we take the limit $l \to \infty$, we recover the operator $\Lambda = \sqrt{-\Delta}$. Indeed, looking at the Taylor expansion we have that*

$$\frac{1}{l^2}\frac{\cosh\left(\frac{\pi}{2l}\eta\right)}{\sinh^2\left(\frac{\pi}{2l}\eta\right)} \to \frac{1}{\left(\frac{\pi}{2}\eta\right)^2}.$$

Now we study the equivalence between the full system (2.3) and the initial value problem of the interface (2.5). Furthermore, we obtain the pressure p (up to a constant) solving the equation

$$-\Delta p = g\partial_y \rho,$$

with Neumann boundary conditions

$$\partial_n p|_{y=l} = -g\rho^1, \qquad \partial_n p|_{y=-l} = g\rho^2.$$

In this way we obtain v, p satisfying Darcy's Law and the incompressibility condition. We need to check that $\rho(x,y,t)$ is a weak solution of the conservation of mass equation.

Definition 3.1 (Weak solution of the continuity equation)**.** *Let v be an incompressible field of velocities following Darcy's Law. We define the weak solution of the conservation of mass equation present in (2.3) as a function satisfying*

$$\int_0^T\int_{\mathbb{R}}\int_{-l}^{l} \rho(x,y,t)\partial_t\phi(x,y,t) + v(x,y,t)\rho(x,y,t)\nabla_{x,y}\phi(x,y,t)dydxdt = 0 \qquad (3.15)$$

for all $\phi \in C_c^\infty(\mathbb{R}\times(-l,l)\times(0,T))$.

We conclude this chapter with the following result.

Proposition 3.3. *Let ρ be the function defined in (2.2). Then ρ is a weak solution, in the sense of Definition 3.1, of the conservation of mass equation if and only if f is a solution of (2.5).*

Proof. By hypothesis, given $\phi(x,y,t) \in C_c^\infty(S\times(0,T))$ we have

$$\int_0^T\int_{\mathbb{R}}\int_{-l}^{l} (\rho\partial_t\phi + v\rho\nabla\phi)dy\,dx\,dt = 0.$$

By the compact support in time

$$0=\int_0^T \partial_t \int_{f(x,t)}^{l} \phi\left(x,y,t\right) dy\, dt = \int_0^T \left(\int_{f(x,t)}^{l} \partial_t\phi\, dy - \phi\left(x,f\left(x,t\right),t\right)\partial_t f\left(x\right)\right) dt,$$

and

$$0=\int_0^T \partial_t \int_{-l}^{f(x,t)} \phi\left(x,y,t\right) dy\, dt = \int_0^T \left(\int_{-l}^{f(x,t)} \partial_t\phi\, dy + \phi\left(x,f\left(x,t\right),t\right)\partial_t f\left(x\right)\right) dt.$$

So

$$\int_0^T \int_{\mathbb{R}} \int_{-l}^{l} \rho\partial_t\phi\, dy\, dx\, dt = \left(\rho^1-\rho^2\right)\int_0^T \int_{\mathbb{R}} \phi\left(x,f\left(x,t\right),t\right)\partial_t f\left(x\right) dx\, dt.$$

Integrating by parts and using the incompressibility of v we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \int_{-l}^{l} v\rho\nabla\phi\, dy\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}} \phi\left(x,f\left(x\right),t\right)\left(\rho^1 v\left(x,f\left(x\right),t\right)\cdot\left(-\bar{n}\right) - \rho^2 v\left(x,f\left(x\right),t\right)\cdot\left(-\bar{n}\right)\right) dx\, dt \\ &= \left(\rho^1-\rho^2\right)\int_0^T \int_{\mathbb{R}} \phi\left(x,f\left(x\right),t\right)\left(v\left(x,f\left(x\right),t\right)\cdot\left(-\bar{n}\right)\right) dx\, dt\end{aligned}$$

Taking $\phi(x,y,t)=\phi(x,t)$ for $|y|\leq\|f\|_{L^\infty}$ we conclude that f satisfies the equation (2.5).

In the same manner we multiply (2.5) by a test function evaluated at the interface $\left(\rho^1-\rho^2\right)\phi\left(x,f\left(x\right),t\right)$ and we integrate

$$\left(\rho^1-\rho^2\right)\int_0^T \int_{\mathbb{R}} \phi\left(x,f\left(x\right),t\right)\left(\partial_t f\left(x\right) - v\left(x,f\left(x\right),t\right)\cdot\bar{n}\right) dx\, dt = 0$$

Using the incompressibility of v, we conclude the proof. □

Chapter 4

Local solvability in Sobolev spaces

4.1 Well-posedness in Sobolev spaces

In this Chapter we prove the existence of local solutions for the equation

$$\partial_t f(x,t) = \frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\Big[(\partial_x f(x) - \partial_x f(x-\eta))\,\Xi_1(x,\eta,f) \\ + (\partial_x f(x) + \partial_x f(x-\eta)\,\Xi_2(x,\eta,f)\Big]d\eta = \frac{\rho^2-\rho^1}{4l}A[f](x), \quad (4.1)$$

when the initial data is in Sobolev spaces. In particular, we prove the following Theorem:

Theorem 4.1 (Well-posedness)**.** *If the Rayleigh-Taylor condition is satisfied,* i.e. $\rho^2-\rho^1>0$*, and the initial data* $f_0(x)=f(x,0)\in H^k_l(\mathbb{R})$*,* $k\geq 3$*, then there exists an unique classical solution of* (4.1) $f\in C([0,T],H^k_l(\mathbb{R}))$ *where* $T=T(\|f_0\|_{H^k},\|f_0\|_{L^\infty})$*. Moreover, we have* $f\in C^1([0,T],C(\mathbb{R}))\cap C([0,T],C^2(\mathbb{R}))$.

Without loss of generality we take $2l=\pi$ and $\bar{\rho}=2$. We indicate the constants with a dependency on l as $c(l)$. In order to show this result we have to define the *'energy'*

$$E[f](t) = \|f(t)\|^2_{H^3} + \|d[f]\|_{L^\infty}(t), \quad (4.2)$$

where $d[f]:\mathbb{R}^2\times\mathbb{R}^+\mapsto\mathbb{R}^+$ is defined as

$$d[f](x,\eta,t) = \frac{1}{\cosh(\eta)+\cos(f(x)+f(x-\eta))}. \quad (4.3)$$

The function (4.3) measures the distance between f and the top and floor $\pm l$ (recall that, without loss of generality, we are supposing that $l=\frac{\pi}{2}$). In other words, $\|d[f]\|_{L^\infty}<\infty$ implies that $\|f\|_{L^\infty}<\frac{\pi}{2}$. So this is the natural *'energy'* associated to the space $H^3_l(\mathbb{R})$.

We obtain some *a priori* bounds for the energy (4.2) and we conclude the proof using classical energy methods, see Chapter 3 in [44]. The proof is similar to the infinitely deep case [23] but, due to the presence of boundaries, there is the new kernel Ξ_2, defined in (2.7). The terms with this second kernel are not singular and will be bounded using (4.3).

4.2 Starting with *a priori* estimates

We divide the proof in different lemmas. We start with $\|f\|_{L^2}$.

Lemma 4.1. *Let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$, and let $E[f]$ be the energy functional defined in (4.2). Then, we have*

$$\frac{d}{dt}\|f\|_{L^2}^2 \leq c(l)(E[f]+1)^2.$$

Proof. We have that

$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 = \int_{\mathbb{R}} f(x)\text{P.V.}\int_{\mathbb{R}}(\partial_x f(x) - \partial_x f(x-\eta))\Xi_1 + (\partial_x f(x) + \partial_x f(x-\eta))\Xi_2 d\eta dx.$$

We separate this integral in its '*in*' and '*out*' parts for η:

$$A_1 = \int_{\mathbb{R}} f(x)\text{P.V.}\int_{B(0,1)}(\partial_x f(x) - \partial_x f(x-\eta))\Xi_1 d\eta dx \\ + \int_{\mathbb{R}} f(x)\text{P.V.}\int_{B(0,1)}(\partial_x f(x) + \partial_x f(x-\eta))\Xi_2 d\eta dx,$$

and

$$A_2 = \int_{\mathbb{R}} f(x)\text{P.V.}\int_{B^c(0,1)}(\partial_x f(x) - \partial_x f(x-\eta))\Xi_1 d\eta dx \\ + \int_{\mathbb{R}} f(x)\text{P.V.}\int_{B^c(0,1)}(\partial_x f(x) + \partial_x f(x-\eta))\Xi_2 d\eta dx.$$

We use te following identity

$$\partial_x f(x) - \partial_x f(x-\eta) = \int_0^1 \partial_x^2 f(x+(s-1)\eta)\eta ds, \tag{4.4}$$

and the definition of $d[f]$ to bound the A_1 term in the following manner

$$A_1 \leq \|f\|_{L^2}\|\partial_x^2 f\|_{L^2}\int_{B(0,1)}\frac{\eta\sinh(\eta)}{\cosh(\eta)-1}d\eta \\ + \|d[f]\|_{L^\infty}\sinh(1)\int_{\mathbb{R}}\int_{B(0,1)}|f(x)|(|\partial_x f(x)| + |\partial_x f(x-\eta)|)d\eta dx \\ \leq c(l)\|f\|_{H^3}^2(1+\|d[f]\|) \leq c(l)E[f](1+E[f]).$$

We calculate

$$\partial_x \Xi_1(x,\eta) = -\sinh(\eta)\frac{\sin(f(x)-f(x-\eta))(\partial_x f(x) - \partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2}, \tag{4.5}$$

$$\partial_x \Xi_2(x,\eta) = \sinh(\eta)\frac{\sin(f(x)+f(x-\eta))(\partial_x f(x) + \partial_x f(x-\eta))}{(\cosh(\eta) + \cos(f(x)+f(x-\eta)))^2}, \tag{4.6}$$

$$\partial_\eta \Xi_1(x,\eta) = \frac{\cosh(\eta)(\cosh(\eta) - \cos(f(x)-f(x-\eta)))}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2} \\ - \frac{\sinh(\eta)(\sinh(\eta) + \sin(f(x)-f(x-\eta))\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2}, \tag{4.7}$$

and

$$\partial_\eta \Xi_2(x,\eta) = \frac{\cosh(\eta)(\cosh(\eta)+\cos(f(x)+f(x-\eta)))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} - \frac{\sinh(\eta)(\sinh(\eta)+\sin(f(x)+f(x-\eta))\partial_x f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2}. \quad (4.8)$$

We rearrange A_2 obtaining the expression

$$A_2 = \frac{1}{2}\int_{\mathbb{R}} \text{P.V.}\int_{B^c(0,1)} \partial_x(f^2(x))(\Xi_1+\Xi_2)d\eta dx + \int_{\mathbb{R}} \text{P.V.}\int_{B^c(0,1)} f(x)\partial_\eta f(x-\eta)(\Xi_1-\Xi_2)d\eta dx = B_1 + B_2.$$

Integrating by parts in B_1 in the x variable and using (4.5), (4.6) and the Sobolev embedding we obtain the following bound

$$B_1 \leq c(l)\|f\|_{C^1}\|f\|_{L^2}^2 \leq c(l)(E[f]^2+1).$$

Now we integrate by parts in B_2 in the η variable, use (4.7) and (4.8) and apply the Cauchy–Schwarz inequality and the Sobolev embedding to obtain

$$B_2 \leq c(l)\|f\|_{L^2}^2(1+\|f\|_{C^1}) \leq c(l)E[f](1+E[f]).$$

Putting all these bounds together we conclude the result. □

Now, we study the behaviour of $\|\partial_x^3 f\|_{L^2}^2$. Using the definition (3.12), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x^3 f\|_{L^2}^2 &= \int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} \partial_x(\partial_x^3 f(x)-\partial_x^3 f(x-\eta))\Xi_1(x,\eta) \\
&\qquad + \partial_x(\partial_x^3 f(x)+\partial_x^3 f(x-\eta))\Xi_2(x,\eta)d\eta dx \qquad (4.9)\\
&\quad + 3\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} (\partial_x^3 f(x)-\partial_x^3 f(x-\eta))\partial_x\Xi_1(x,\eta) \\
&\qquad + (\partial_x^3 f(x)+\partial_x^3 f(x-\eta))\partial_x\Xi_2(x,\eta)d\eta dx \qquad (4.10)\\
&\quad + 3\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} (\partial_x^2 f(x)-\partial_x^2 f(x-\eta))\partial_x^2\Xi_1(x,\eta) \\
&\qquad + (\partial_x^2 f(x)+\partial_x^2 f(x-\eta))\partial_x^2\Xi_2(x,\eta)d\eta dx \qquad (4.11)\\
&\quad + \int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} (\partial_x f(x)-\partial_x f(x-\eta))\partial_x^3\Xi_1(x,\eta) \\
&\qquad + (\partial_x f(x)+\partial_x f(x-\eta))\partial_x^3\Xi_2(x,\eta)d\eta dx \qquad (4.12)\\
&= I_1 + 3I_2 + 3I_3 + I_4,
\end{aligned}$$

where Ξ_1, Ξ_2 are defined in (2.6) and (2.7). We estimate these integrals in different lemmas. First, we start with I_1.

4.3 Bound for I_1: The singular terms

Lemma 4.2. *Let I_1 be defined as in (4.9), let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then, if the Rayleigh-Taylor condition is satisfied the following bound holds*

$$I_1 \leq c(l)(E[f]+1)^5.$$

Proof. We split into the *'more local'* and *'less local'* part and integrate by parts in the first term

$$I_1 = -\frac{1}{2}\int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{\mathbb{R}} \partial_x(\Xi_1(x,\eta)+\Xi_2(x,\eta))d\eta dx$$
$$-\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} \partial_x(\partial_x^3 f(x-\eta))(\Xi_1(x,\eta)-\Xi_2(x,\eta))d\eta dx = J_1^1 + J_2^1.$$

According to (4.5), the term $\partial_x \Xi_1$ is singular when $\eta = 0$. Thus, *'out'* of $\eta = 0$, J_1^1 can be bounded easily:

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \int_{B^c(0,1)} \partial_x(\Xi_1(x,\eta)+\Xi_2(x,\eta))d\eta dx$$
$$\leq c(l)\|f\|_{C^1}\|f\|_{H^3}^2 \int_{B^c(0,1)} \frac{\sinh(|\eta|)}{(\cosh(\eta)-1)^2}d\eta \leq c(l)\|f\|_{H^3}^3. \quad (4.13)$$

Checking the singularity order of $\partial_x(\Xi_1+\Xi_2)$, we write

$$\partial_x(\Xi_1+\Xi_2) = -\sinh(\eta)\left[\frac{(\sin(f(x)-f(x-\eta)) - (f(x)-f(x-\eta))(\partial_x f(x)-\partial_x f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2}\right.$$
$$\left. + \frac{(f(x)-f(x-\eta))(\partial_x f(x)-\partial_x f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} - \frac{\sin(f(x)+f(x-\eta))(\partial_x f(x)+\partial_x f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2}\right]$$

Thus, the *'in'* term in J_1^1 is decomposed in three terms

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \partial_x(\Xi_1+\Xi_2)d\eta dx = K_1^1 + K_2^1 + K_3^1. \quad (4.14)$$

We use Taylor theorem for K_1^1

$$\begin{aligned}
K_1^1 &= \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \sinh(\eta)\frac{-\sin(\xi)(f(x)-f(x-\eta))^3(\partial_x f(x)-\partial_x f(x-\eta))}{(\cosh(\eta)-\cos((f(x)-f(x-\eta))))^2} \\
&\leq c(l)\|f\|_{C^1}^3\|f\|_{C^2}\int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \int_{B(0,1)} \frac{\sinh(|\eta|)\eta^4}{(\cosh(\eta)-\cos((f(x)-f(x-\eta))))^2} \\
&\leq c(l)\|f\|_{C^1}^3\|f\|_{C^2}\|\partial_x^3 f\|_{L^2}^2 \int_{B(0,1)} \frac{\sinh(|\eta|)\eta^4}{(\cosh(\eta)-1)^2} = c(l)\|f\|_{H^3}^6. \quad (4.15)
\end{aligned}$$

We decompose K_2^1 in the following terms:

$$\begin{aligned}
K_2^1 &= \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \sinh(\eta)\frac{(f(x)-f(x-\eta)-\partial_x f(x)\eta)(\partial_x f(x)-\partial_x f(x-\eta))}{(\cosh(\eta)-\cos((f(x)-f(x-\eta))))^2}d\eta dx \\
&+ \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \sinh(\eta)\frac{\partial_x f(x)\eta(\partial_x f(x)-\partial_x f(x-\eta)-\partial_x^2 f(x)\eta)}{(\cosh(\eta)-\cos((f(x)-f(x-\eta))))^2}d\eta dx \\
&+ \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \left(\frac{\sinh(\eta)\partial_x f(x)\eta^2\partial_x^2 f(x)}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2}\right. \\
&- \left.\frac{\sinh(\eta)\partial_x f(x)\eta^2\partial_x^2 f(x)}{4\sinh^4\left(\frac{\eta}{2}\right)(1+(\partial_x f(x))^2)^2}\right)d\eta dx \\
&+ \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.}\int_{B(0,1)} \partial_x f(x)\partial_x^2 f(x)\frac{\sinh(\eta)\eta^2 d\eta}{4\sinh^4\left(\frac{\eta}{2}\right)(1+(\partial_x f(x))^2)^2}d\eta dx.
\end{aligned}$$

4.3. Bound for I_1: The singular terms

We note that

$$\text{P.V.}\int_{B(0,1)}\frac{\sinh(\eta)\,\eta^2 d\eta}{4\sinh^4\left(\frac{\eta}{2}\right)}d\eta=0.$$

Moreover,

$$\frac{1}{\left(\cosh(\eta)-\cos(f(x)-f(x-\eta))\right)^2}-\frac{1}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+(\partial_x f(x))^2\right)^2}$$
$$=\frac{1}{4\left(\sinh^2\left(\frac{\eta}{2}\right)+\sin^2\left(\frac{f(x)-f(x-\eta)}{2}\right)\right)^2}-\frac{1}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+(\partial_x f(x))^2\right)^2}$$
$$=\frac{1}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+\sinh^{-2}\left(\frac{\eta}{2}\right)\sin^2\left(\frac{f(x)-f(x-\eta)}{2}\right)\right)^2}-\frac{1}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+(\partial_x f(x))^2\right)^2}$$
$$=\frac{\left(1+(\partial_x f(x))^2\right)^2-\left(1+\sinh^{-2}\left(\frac{\eta}{2}\right)\sin^2\left(\frac{f(x)-f(x-\eta)}{2}\right)\right)^2}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+\sinh^{-2}\left(\frac{\eta}{2}\right)\sin^2\left(\frac{f(x)-f(x-\eta)}{2}\right)\right)^2\left(1+(\partial_x f)^2\right)^2}=O(\eta^{-3}).$$

Indeed, we have

$$\left(1+(\partial_x f(x))^2\right)^2-\left(1+\sinh^{-2}\left(\frac{\eta}{2}\right)\sin^2\left(\frac{f(x)-f(x-\eta)}{2}\right)\right)^2=$$
$$\left(2+(\partial_x f(x))^2+\left(\frac{\sin(f(x)-f(x-\eta))}{\sinh(\eta)}\right)^2\right)\left(\partial_x f(x)+\frac{\sin(f(x)-f(x-\eta))}{\sinh(\eta)}\right)$$
$$\times\left(\partial_x f(x)-\frac{\sin(f(x)-f(x-\eta))}{\sinh(\eta)}\right).$$

Using that

$$\left|\frac{\sin(f(x)-f(x-\eta))}{\sinh(\eta)}\right|\leq\left|\frac{f(x)-f(x-\eta)}{\eta}\right|,$$

we can bound the last term in K_2^1 :

$$\int_{\mathbb{R}}|\partial_x^3 f(x)|^2\text{P.V.}\int_{B(0,1)}\left[\frac{\sinh(\eta)\,\partial_x f(x)\eta^2\partial_x^2 f(x)}{\left(\cosh(\eta)-\cos(f(x)-f(x-\eta))\right)^2}-\frac{\sinh(\eta)\,\partial_x f(x)\eta^2\partial_x^2 f(x)}{4\sinh^4\left(\frac{\eta}{2}\right)\left(1+(\partial_x f)^2\right)^2}\right]d\eta dx$$
$$\leq c(l)\|\partial_x^3 f\|_{L^2}^2\|f\|_{C^1}\|f\|_{C^2}\left(\|f\|_{C^1}^6+\|f\|_{C^2}^4+\|f\|_{C^2}^2\right).$$

Using the Mean Value Theorem, Taylor Theorem and Hölder norms, we get

$$\begin{aligned}|K_2^1| \leq\ & c_1(l)\|\partial_x^3 f\|_{L^2}^2\|f\|_{C^2}^2\int_{B(0,1)}\frac{\sinh(|\eta|)\,|\eta|^3}{(\cosh(\eta)-1)^2}d\eta\\ &+c_2(l)\|\partial_x^3 f\|_{L^2}^2\|f\|_{C^{2,\delta}}\|f\|_{C^1}\int_{B(0,1)}\frac{\sinh(|\eta|)\,|\eta|^{2+\delta}}{(\cosh(\eta)-1)^2}d\eta\\ &+c_3(l)\|\partial_x^3 f\|_{L^2}^2\|f\|_{C^1}\|f\|_{C^2}\left(\|f\|_{C^1}^6+\|f\|_{C^2}^4+\|f\|_{C^2}^2\right).\end{aligned}$$

We conclude that

$$|K_2^1|\leq c(l)\|f\|_{H^3}^4(\|f\|_{H^3}^6+1). \tag{4.16}$$

Using the definition (4.3) we have

$$\begin{aligned}|K_3^1| \leq\ & \int_{\mathbb{R}}|\partial_x^3 f(x)|^2\text{P.V.}\int_{B(0,1)}\sinh(|\eta|)\left|\frac{\sin\left(f(x)+f(x-\eta)\right)\left(\partial_x f(x)+\partial_x f(x-\eta)\right)}{\left(\cosh(\eta)+\cos((f(x)+f(x-\eta)))\right)^2}\right|d\eta dx\\ \leq\ & c(l)\|f\|_{C^1}\|\partial_x^3 f\|_{L^2}^2\|d[f]\|_{L^\infty}^2\leq c(l)\|f\|_{H^3}^3\|d[f]\|_{L^\infty}^2.\end{aligned} \tag{4.17}$$

Since (4.13)-(4.17), we obtain

$$|J_1^1| \leq c(l)\left(\|f\|_{H^3}^{10} + 1 + \|f\|_{H^3}^3\|d[f]\|_{L^\infty}^2\right). \tag{4.18}$$

We have to bound J_2^1 to conclude with I_1. By means of an integration by parts, we have

$$\begin{aligned} J_2^1 &= -\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} \partial_\eta(\partial_x^3 f(x) - \partial_x^3 f(x-\eta))(\Xi_1 - \Xi_2)d\eta dx \\ &= \int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta))\partial_\eta(\Xi_1 - \Xi_2)d\eta dx. \end{aligned}$$

We now use the expressions (4.7) and (4.8). First, we estimate the Ξ_2 term. Using (4.3) and splitting in the *'in'* and *'out'* parts, we get

$$\left|\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta))\partial_\eta \Xi_2(x,\eta)d\eta dx\right| \leq$$
$$c(l)\|\partial_x^3 f\|_{L^2}^2(\|d[f]\|_{L^\infty}^2 + \|f\|_{C^1}\|d[f]\|_{L^\infty}^2 + \|f\|_{C^1} + 1). \tag{4.19}$$

Now, we estimate the Ξ_1 term of J_2^1 using the Rayleigh-Taylor condition. If the Rayleigh-Taylor condition has the oposite sign, this term cannot be bounded in terms of the H^3 norm. We decompose this term in *'in'* and *'out'* terms. The *'out'* term can be bounded as follows:

$$\begin{aligned} &\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{B^c(0,1)} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta))\partial_\eta \Xi_1(x,\eta)d\eta dx \\ &\qquad \leq c(l)\|\partial_x^3 f\|_{L^2}^2 \int_{B^c(0,1)} \left(\frac{1+\cosh(\eta)}{(\cosh(\eta)-1)^2} + \|f\|_{C^1}\frac{\sinh(\eta)}{(\cosh(\eta)-1)^2}\right)d\eta \\ &\qquad \leq c(l)\|\partial_x^3 f\|_{L^2}^2(1+\|f\|_{C^1}). \end{aligned} \tag{4.20}$$

To estimate the *'in'* term, we use Taylor theorem for the function $\cosh(s)$ and $\cos(s)$ in $\partial_x \Xi_1$ with $\alpha_i = \alpha(x,\eta)$ are the points for the remainder:

$$\begin{aligned} \partial_\eta \Xi_1(x,\eta) = &\frac{\cosh(\eta)\left(\frac{\eta^2}{2} + \cosh(\alpha_1)\frac{\eta^4}{4!} + \frac{(f(x)-f(x-\eta))^2}{2} - \cos(\alpha_2)\frac{(f(x)-f(x-\eta))^4}{4!}\right)}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2} \\ &- \frac{\sinh(\eta)(\sinh(\eta) + \left((f(x)-f(x-\eta)) - \sin(\alpha_3)\frac{(f(x)-f(x-\eta))^3}{3!}\right)\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2}. \end{aligned}$$

Therefore, we have terms which can be integrated ($O(1)$) and terms which cannot be integrated ($O(\eta^{-2})$). Thus,

$$\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{B(0,1)} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta))\partial_\eta \Xi_1(x,\eta)d\eta dx = L_1^1 + L_2^1. \tag{4.21}$$

The terms which can be integrated are

$$\begin{aligned} L_1^1 = &\int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{B(0,1)} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \\ &\cdot\left[\frac{\cosh(\eta)\left(\cosh(\alpha_1)\frac{\eta^4}{4!} - \cos(\alpha_2)\frac{(f(x)-f(x-\eta))^4}{4!}\right)}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2}\right. \\ &\left. - \frac{\sinh(\eta)(-\sin(\alpha_3)\frac{(f(x)-f(x-\eta))^3}{3!}\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x)-f(x-\eta)))^2}\right]d\eta dx, \end{aligned}$$

and so

$$L_1^1 \leq c(l)\|\partial_x^3 f\|_{L^2}^2(\|f\|_{C^1}^4 + 1). \tag{4.22}$$

The terms which cannot be integrated are

$$\begin{aligned} L_2^1 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} &(\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \\ &\Bigg[\frac{\frac{\cosh(\eta)}{2}\left(\eta^2 + (f(x) - f(x-\eta))^2\right)}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} \\ &- \frac{\sinh(\eta)(\sinh(\eta) + (f(x) - f(x-\eta))\,\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2}\Bigg] d\eta dx. \end{aligned}$$

We change (up to integrable terms) some $\sinh(\eta)$ by η and $\cosh(\eta)$ by 1:

$$\begin{aligned} L_2^1 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} &(\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \\ &\Bigg[\frac{\frac{1}{2}\left(\eta^2 + (f(x) - f(x-\eta))^2\right)}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} \\ &- \frac{\eta(\eta + (f(x) - f(x-\eta))\,\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2}\Bigg] d\eta dx. \end{aligned}$$

We have

$$\begin{aligned} &\frac{-\eta(\eta + (f(x) - f(x-\eta))\,\partial_x f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} \\ &\quad = \frac{-(f(x) - f(x-\eta))\,(\eta\partial_x f(x-\eta) - (f(x) - f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} \\ &\qquad - \frac{\eta^2 + (f(x) - f(x-\eta))^2}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2}. \end{aligned}$$

Thus, we obtain the following expression for L_2^1:

$$\begin{aligned} L_2^1 &= \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \\ &\quad \Bigg[\frac{\frac{1}{2}\left(\eta^2 + (f(x) - f(x-\eta))^2\right) - \left(\eta^2 + (f(x) - f(x-\eta))^2\right)}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} \\ &\quad - \frac{(f(x) - f(x-\eta))\,(\eta\partial_x f(x-\eta) - (f(x) - f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2}\Bigg] d\eta dx \\ &= M_1^1 + M_2^1. \end{aligned}$$

In this expression, the term M_2^1 can be integrated in principal value and can be estimated as K_2^1 in J_1^1. We complete M_1^1 in η and decompose the integral in two terms in order to check the sign of one of them:

$$\begin{aligned} M_1^1 = \int_{\mathbb{R}} \partial_x^3 f(x) \left(\text{P.V.} \int_{\mathbb{R}} - \int_{B^c(0,1)}\right) &(\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \\ &\cdot \frac{-\frac{1}{2}\left(\eta^2 + (f(x) - f(x-\eta))^2\right)}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^2} d\eta dx = N_1^1 + N_2^1. \end{aligned}$$

As we claim the term N_1^1 has a sign. Indeed, if we change variables, first in one dimension $x-\eta=\eta$, and then in two dimension $x=\eta$ and $\eta=x$ we obtain

$$\begin{aligned} N_1^1 &= \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} (\partial_x^3 f(x) - \partial_x^3 f(\eta)) \frac{-\frac{1}{2}\left((x-\eta)^2 + (f(x)-f(\eta))^2\right)}{(\cosh(x-\eta) - \cos(f(x)-f(\eta)))^2} d\eta dx \\ &= -\int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(\eta)(\partial_x^3 f(x) - \partial_x^3 f(\eta)) \frac{-\frac{1}{2}\left((x-\eta)^2 + (f(x)-f(\eta))^2\right)}{(\cosh(x-\eta) - \cos(f(x)-f(\eta)))^2} d\eta dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} (\partial_x^3 f(x) - \partial_x^3 f(\eta))^2 \frac{-\frac{1}{2}\left((x-\eta)^2 + (f(x)-f(\eta))^2\right)}{(\cosh(x-\eta) - \cos(f(x)-f(\eta)))^2} d\eta dx \leq 0. \end{aligned}$$

Therefore, estimating N_2^1 as (4.20), we obtain

$$L_2^1 \leq c(l)(1 + \|f\|_{H^3}^{10}). \tag{4.23}$$

Thus, by (4.19)-(4.23), we get

$$J_2^1 \leq c(l)\left(\|f\|_{H^3}^{10} + 1 + \|f\|_{H^3}^2(\|d[f]\|_{L^\infty}^2 + \|f\|_{H^3}\|d[f]\|_{L^\infty}^2)\right). \tag{4.24}$$

Using (4.18), (4.24) and the definition of energy (4.2) we conclude the proof of the lemma. □

4.4 Bound for I_2

Lemma 4.3. *Let I_2 be defined as in (4.10), let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then,*

$$I_2 \leq c(l)(E[f]+1)^5.$$

Proof. We decompose the integral I_2 in a similar manner of I_1:

$$\begin{aligned} I_2 = \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \text{P.V.} \int_{\mathbb{R}} \partial_x(\Xi_1(x,\eta) + \Xi_2(x,\eta)) d\eta dx \\ - \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x-\eta)\partial_x(\Xi_1(x,\eta) - \Xi_2(x,\eta)) d\eta dx = J_1^2 + J_2^2. \end{aligned}$$

The term J_1^2 can be estimated as the term J_1^1 and we obtain the bound

$$J_1^2 \leq c(l)\left(\|f\|_{H^3}^{10} + 1 + \|f\|_{H^3}^3\|d[f]\|_{L^\infty}^2\right). \tag{4.25}$$

To estimate J_2^2 we split in two integrals $J_2^2 = K_1^2 + K_2^2$, one of them with the kernel $\partial_x \Xi_1$ and the other with the kernel $\partial_x \Xi_2$. Thus, in the same manner as (4.17), using (4.5), (4.6) and the Sobolev embedding,

$$K_2^2 \leq \left| \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x-\eta)\partial_x \Xi_2(x,\eta) d\eta dx \right| \leq c(l)\|f\|_{H^3}^3(\|d[f]\|_{L^\infty}^2 + 1). \tag{4.26}$$

The term K_1^2 vanishes,

$$K_1^2 = -\int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x-\eta)\partial_x \Xi_1(x,\eta) d\eta dx = 0. \tag{4.27}$$

Indeed, if we change variables (in one dimension) $x-\eta=\eta$ and we change variables again (in two dimensions) $x=\eta$ and $\eta=x$, we obtain

$$\begin{aligned} K_1^2 &= \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x)\partial_x^3 f(\eta)\sinh(x-\eta)\frac{\sin(f(x)-f(\eta))(\partial_x f(x)-\partial_x f(\eta))}{(\cosh(x-\eta)-\cos(f(x)-f(\eta)))^2}d\eta dx \\ &= \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(\eta)\partial_x^3 f(x)\sinh(\eta-x)\frac{\sin(f(\eta)-f(x))(\partial_x f(\eta)-\partial_x f(x))}{(\cosh(\eta-x)-\cos(f(\eta)-f(x)))^2}d\eta dx \\ &= -\int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x)\partial_x^3 f(\eta)\sinh(x-\eta)\frac{\sin(f(x)-f(\eta))(\partial_x f(x)-\partial_x f(\eta))}{(\cosh(x-\eta)-\cos(f(x)-f(\eta)))^2}d\eta dx \\ &= 0. \end{aligned}$$

Finally, since we have defined (4.27) and (4.26), we estimate J_2^2. With this and (4.25), using the definition of the energy $E[f]$, we conclude the proof of this Lemma. □

4.5 Bound for I_3

Lemma 4.4. *Let I_3 be defined as in (4.11), let $f(t)\in H^3(\mathbb{R})$ be a solution of (4.1) for any $t\geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then,*

$$I_3 \leq c(l)(E[f]+1)^5.$$

Proof. We split I_3 in two integrals, $I_3=H_1^3+H_2^3$ where H_1^3 correspond to $\partial_x^2\Xi_1$ and H_2^3 correspond to $\partial_x^2\Xi_2$. Thus,

$$\begin{aligned} H_1^3 = \int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}} &(\partial_x^2 f(x)-\partial_x^2 f(x-\eta))\partial_x^2\Xi_1(x,\eta)d\eta dx \\ &= \int_{\mathbb{R}} \partial_x^3 f(x)\text{P.V.}\int_{B(0,1)} (\partial_x^2 f(x)-\partial_x^2 f(x-\eta))\partial_x^2\Xi_1(x,\eta)d\eta dx \\ &\quad + \int_{\mathbb{R}} \partial_x^3 f(x)\int_{B^c(0,1)} (\partial_x^2 f(x)-\partial_x^2 f(x-\eta))\partial_x^2\Xi_1(x,\eta)d\eta dx = J_1^3+J_2^3, \end{aligned}$$

where

$$\begin{aligned} \partial_x^2\Xi_1(x,\eta) = -\sinh(\eta)\Bigg[&\frac{\cos(f(x)-f(x-\eta))(\partial_x f(x)-\partial_x f(x-\eta))^2}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\ &+\frac{\sin(f(x)-f(x-\eta))(\partial_x^2 f(x)-\partial_x^2 f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\ &-\frac{2\sin^2(f(x)-f(x-\eta))(\partial_x f(x)-\partial_x f(x-\eta))^2}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^3}\Bigg]. \end{aligned}$$

Following the procedure of previous lemmas, we obtain the following *a priori* bounds for J_1^3 and J_2^3

$$J_1^3 \leq c(l)\|\partial_x^3 f\|_{L^2}^2\big(\|f\|_{C^2}^2+\|f\|_{C^{2,\delta}}(\|f\|_{C^1}^3+\|f\|_{C^1})+\|f\|_{C^1}^6\|f\|_{C^2}^2+\|f\|_{C^1}^2\|f\|_{C^2}^2\big)$$

$$J_2^3 \leq c(l)\|\partial_x^3 f\|_{L^2}^2(\|f\|_{C^1}^2+\|f\|_{C^2}).$$

Analogously,

$$H_2^3 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} (\partial_x^2 f(x) + \partial_x^2 f(x-\eta)) \partial_x^2 \Xi_2(x,\eta) d\eta dx$$
$$\int_{\mathbb{R}} \partial_x^3 f(x) \int_{B(0,1)} (\partial_x^2 f(x) + \partial_x^2 f(x-\eta)) \partial_x^2 \Xi_2(x,\eta) d\eta dx$$
$$+ \int_{\mathbb{R}} \partial_x^3 f(x) \int_{B^c(0,1)} (\partial_x^2 f(x) + \partial_x^2 f(x-\eta)) \partial_x^2 \Xi_2(x,\eta) d\eta dx = J_3^3 + J_4^3,$$

where

$$\partial_x^2 \Xi_2(x,\eta) = \sinh(\eta) \left[\frac{\cos(f(x)+f(x-\eta))(\partial_x f(x) + \partial_x f(x-\eta))^2}{(\cosh(\eta) + \cos(f(x)+f(x-\eta)))^2} + \frac{\sin(f(x)+f(x-\eta))(\partial_x^2 f(x) + \partial_x^2 f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} + \frac{2(\sin(f(x)+f(x-\eta))(\partial_x f(x)+\partial_x f(x-\eta)))^2}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^3} \right].$$

With the same ideas of the previous lemmas we obtain

$$\begin{aligned} J_3^3 &\leq c(l) \|\partial_x^3 f\|_{L^2} \|\partial_x^2 f\|_{L^2} \left(\|f\|_{C^1}^2 \|d[f]\|_{L^\infty}^2 + \|f\|_{C^2} \|d[f]\|_{L^\infty}^2 + \|f\|_{C^1}^2 \|d[f]\|_{L^\infty}^3 \right) \\ J_4^3 &\leq c(l) \|\partial_x^3 f\|_{L^2} \|\partial_x^2 f\|_{L^2} (\|f\|_{C^1}^2 + \|f\|_{C^2}). \end{aligned}$$

Using our bounds for J_i^3, by Sobolev embedding and our definition of energy (4.2), we conclude with the proof of this lemma. □

4.6 Bound for I_4

Lemma 4.5. *Let I_4 be defined as in (4.12), let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then,*

$$I_4 \leq c(l)(E[f] + 1)^5.$$

Proof. To do this we have to know $\partial_x^3\Xi_1$ and $\partial_x^3\Xi_2$. We calculate

$$\begin{aligned}
\partial_x^3\Xi_1(x,\eta) \quad = \quad & -\sinh(\eta)\Bigg[\frac{-\sin(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))^3}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\
&+\frac{3\cos(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))(\partial_x^2 f(x)-\partial_x^2 f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\
&-\frac{2\cos(f(x)-f(x-\eta))\sin(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))^3}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\
&+\frac{\sin(f(x)-f(x-\eta))\,(\partial_x^3 f(x)-\partial_x^3 f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\
&-\frac{2\sin^2(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))(\partial_x^2 f(x)-\partial_x^2 f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^2} \\
&-\frac{4\sin^2(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))(\partial_x^2 f(x)-\partial_x^2 f(x-\eta))}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^3} \\
&-\frac{4\sin(f(x)-f(x-\eta))\cos(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))^2}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^3} \\
&+\frac{6(\sin(f(x)-f(x-\eta))\,(\partial_x f(x)-\partial_x f(x-\eta))^3}{(\cosh(\eta)-\cos(f(x)-f(x-\eta)))^4}\Bigg],
\end{aligned}$$

$$\begin{aligned}
\partial_x^3\Xi_2(x,\eta) \quad = \quad & \sinh(\eta)\Bigg[\frac{-\sin(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))^3}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} \\
&+\frac{3\cos(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))(\partial_x^2 f(x)+\partial_x^2 f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} \\
&+\frac{2\cos(f(x)+f(x-\eta))\sin(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))^3}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} \\
&+\frac{\sin(f(x)+f(x-\eta))\,(\partial_x^3 f(x)+\partial_x^3 f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} \\
&+\frac{2\sin^2(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))(\partial_x^2 f(x)+\partial_x^2 f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^2} \\
&+\frac{4\sin^2(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))(\partial_x^2 f(x)+\partial_x^2 f(x-\eta))}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^3} \\
&+\frac{4\sin(f(x)+f(x-\eta))\cos(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))^2}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^3} \\
&+\frac{6(\sin(f(x)+f(x-\eta))\,(\partial_x f(x)+\partial_x f(x-\eta))^3}{(\cosh(\eta)+\cos(f(x)+f(x-\eta)))^4}\Bigg].
\end{aligned}$$

We decompose in the *'in'* and *'out'* parts, and the *'out'* part is estimated as previous lemmas.

$$I_4 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))\partial_x^3 \Xi_1(x,\eta)$$
$$(\partial_x f(x) + \partial_x f(x-\eta))\partial_x^3 \Xi_2(x,\eta) d\eta dx$$
$$+ \int_{\mathbb{R}} \partial_x^3 f(x) \int_{B^c(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))\partial_x^3 \Xi_1(x,\eta)$$
$$(\partial_x f(x) + \partial_x f(x-\eta))\partial_x^3 \Xi_2(x,\eta) d\eta dx = J_1^4 + J_2^4.$$

Also, the component J_2^4, corresponding to the kernel $\partial_x^3 \Xi_2$ can easily be bounded using (4.2) and (4.3). We only calculate the bounds for the most singular terms of $\partial_x^3 \Xi_1$ in the *'in'* part. In particular, we study the following integrals:

$$K_1^4 = -\int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} (\partial_x f(x) - \partial_x f(x-\eta)) \sinh(\eta)$$
$$\frac{\sin(f(x) - f(x-\eta))\left(\partial_x^3 f(x) - \partial_x^3 f(x-\eta)\right)}{\left(\cosh(\eta) - \cos(f(x) - f(x-\eta))\right)^2} d\eta dx, \quad (4.28)$$

$$K_2^4 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))^2 \sinh(\eta)$$
$$\frac{4\sin^2(f(x) - f(x-\eta))\left(\partial_x^2 f(x) - \partial_x^2 f(x-\eta)\right)}{\left(\cosh(\eta) - \cos(f(x) - f(x-\eta))\right)^3} d\eta dx, \quad (4.29)$$

$$K_3^4 = \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{B(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))^3 \sinh(\eta)$$
$$\frac{4\sin(f(x) - f(x-\eta))\cos(f(x) - f(x-\eta))}{\left(\cosh(\eta) - \cos(f(x) - f(x-\eta))\right)^3} d\eta dx. \quad (4.30)$$

K_1^4 is the same as in Lemma 4.3. On the other hand, from (4.29), we obtain as previous estimates that the K_2^4 term can be bounded as

$$\begin{aligned} K_2^4 &= \int_{\mathbb{R}} \text{P.V.} \int_{B(0,1)} \partial_x^3 f(x)(\partial_x f(x) - \partial_x f(x-\eta))^2 \sinh(\eta)\left(\partial_x^2 f(x) - \partial_x^2 f(x-\eta)\right) \\ &\quad \frac{\left(\sin^2(f(x) - f(x-\eta)) - (f(x) - f(x-\eta))^2\right)}{\left(\cosh(\eta) - \cos(f(x) - f(x-\eta))\right)^3} d\eta dx \\ &\quad + \int_{\mathbb{R}} \text{P.V.} \int_{B(0,1)} \partial_x^3 f(x)(\partial_x f(x) - \partial_x f(x-\eta))^2 \sinh(\eta) \\ &\quad \frac{(f(x) - f(x-\eta))^2\left(\partial_x^2 f(x) - \partial_x^2 f(x-\eta)\right)}{\left(\cosh(\eta) - \cos(f(x) - f(x-\eta))\right)^3} d\eta dx \\ &\leq c(l)\|\partial_x^3 f\|_{L^2}^2 (\|f\|_{C^1}^5 \|f\|_{C^2}^2 + \|f\|_{C^1}^2 \|f\|_{C^2}^2). \end{aligned}$$

Finally, from (4.30), we have

$$\begin{aligned} K_3^4 &= \int_{\mathbb{R}} \text{P.V.} \int_{B(0,1)} \partial_x^3 f(x)(\partial_x f(x) - \partial_x f(x-\eta))^3 \sinh(\eta) \cos(f(x) - f(x-\eta)) \\ &\qquad \frac{4(\sin(f(x) - f(x-\eta)) - (f(x) - f(x-\eta)))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^3} d\eta dx \\ &+ \int_{\mathbb{R}} \text{P.V.} \int_{B(0,1)} \partial_x^3 f(x)(\partial_x f(x) - \partial_x f(x-\eta))^3 \sinh(\eta) \\ &\qquad \frac{4(f(x) - f(x-\eta)) \cos(f(x) - f(x-\eta))}{(\cosh(\eta) - \cos(f(x) - f(x-\eta)))^3} d\eta dx. \end{aligned}$$

The first integral is estimated as the previous one, but the second is more singular. In particular, this integral is as J_1^1 in Lemma 4.2. Therefore, K_3^4 is bounded by

$$K_3^4 \leq c(l)\|\partial_x^3 f\|_{L^2}\|\partial_x^2 f\|_{L^2}(\|f\|_{C^2}\|f\|_{C^{2,\delta}}\|f\|_{C^1} + \|f\|_{C^2}^3 + \|f\|_{C^1}^3\|f\|_{C^2}^2 + \|f\|_{C^1}\|f\|_{C^2}^2).$$

The other terms in I_4 gives us bounds with lower powers of $\|f\|_{H^3}$. Therefore the following bound for J_1^4 holds

$$J_1^4 \leq c(l)\|f\|_{H^3}^3(\|f\|_{H^3}^6 + \|f\|_{H^3}^4 + \|f\|_{H^3}^2 + \|f\|_{H^3} + 1).$$

We conclude the result using Sobolev embedding. □

4.7 Bound for $\|d[f]\|_{L^\infty}$

In order to use classical energy methods we have to bound the evolution of $\|d[f]\|_{L^\infty}$ in term of the energy $E[f]$. With our method we need a bound on $\|\partial_t f\|_{L^\infty}$. We have the following lemma:

Lemma 4.6. *Let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then, we have*

$$\|\partial_t f\|_{L^\infty} \leq c(l)(E[f] + 1)^2.$$

Proof. We split $\|\partial_t f\|_{L^\infty}$ in two terms, one for each kernel. So $\|\partial_t f\|_{L^\infty} = A_1 + A_2$. We give the proof for the term corresponding to the first kernel

$$A_1 = \left\| \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x-\eta))\Xi_1(x,\eta) d\eta \right\|_{L^\infty}.$$

For the term A_2 the procedure is analogous.

We split A_1 in its *'in'* and *'out'* parts, $A_1 = B_1 + B_2$, where

$$B_1 = \left\| \text{P.V.} \int_{B(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))\Xi_1(x,\eta) d\eta \right\|_{L^\infty} \leq c(l)\|\partial_x^2 f\|_{L^\infty},$$

where we use the equality (4.4), and

$$B_2 = \left\| \text{P.V.} \int_{B^c(0,1)} (\partial_x f(x) - \partial_x f(x-\eta))\Xi_1(x,\eta)d\eta \right\|_{L^\infty} \leq \left\| \text{P.V.} \int_{B^c(0,1)} \partial_x f(x)\Xi_1(x,\eta)d\eta \right\|_{L^\infty} + \left\| \text{P.V.} \int_{B^c(0,1)} -\partial_x f(x-\eta)\Xi_1(x,\eta)d\eta \right\|_{L^\infty} = C_1 + C_2.$$

We have that the integral

$$\text{P.V.} \int_{B^c(0,1)} \frac{\sinh(\eta)}{\sinh^2\left(\frac{\eta}{2}\right)} = 0.$$

Using this fact and the classical and hyperbolic trigonometric identities we can write

$$\text{P.V.} \int_{B^c(0,1)} \Xi_1(x,\eta)d\eta = \text{P.V.} \int_{B^c(0,1)} \frac{\sinh(\eta)}{2\sinh^2\left(\frac{\eta}{2}\right)} \left(\frac{1}{1 + \frac{\sin^2((f(x)-f(x-\eta))/2)}{\sinh^2(\eta/2)}} - 1 \right) d\eta.$$

We compute

$$\text{P.V.} \int_{B^c(0,1)} \Xi_1(x,\eta)d\eta = \text{P.V.} \int_{B^c(0,1)} \frac{\sinh(\eta)}{2\sinh^2\left(\frac{\eta}{2}\right)} \cdot \left(\frac{\frac{-\sin^2((f(x)-f(x-\eta))/2)}{\sinh^2(\eta/2)}}{1 + \frac{\sin^2((f(x)-f(x-\eta))/2)}{\sinh^2(\eta/2)}} \right) d\eta,$$

and we obtain

$$C_1 \leq c(l)\|\partial_x f\|_{L^\infty}.$$

To bound

$$\text{P.V.} \int_{B^c(0,1)} -\partial_x f(x-\eta)\Xi_1(x,\eta)d\eta = \text{P.V.} \int_{B^c(0,1)} \partial_\eta f(x-\eta)\Xi_1(x,\eta)d\eta$$

we integrate by parts. Using (4.7) we concluded

$$C_2 \leq c(l)\|f\|_{L^\infty}(1 + \|\partial_x f\|_{L^\infty}).$$

□

Now we can prove the last Lemma concerning the existence of classical solution to the equation (4.1):

Lemma 4.7. *Let $f(t) \in H^3(\mathbb{R})$ be a solution of (4.1) for any $t \geq 0$ and let $E[f]$ be the energy functional defined in (4.2). Then, for the function $d[f]$ defined in (4.3), the following inequality holds*

$$\frac{d}{dt}\|d[f]\|_{L^\infty} \leq c(l)(E[f]+1)^4.$$

Proof. We have that

$$\frac{d}{dt}d[f] = d[f]^2 \sin(f(x) + f(x-\eta))(\partial_t f(x) + \partial_t f(x-\eta)) \leq c(l)d[f]^2\|\partial_t f\|_{L^\infty}.$$

Due to the Lemma 4.6 we obtain

$$\frac{d}{dt}d[f] \leq c(l)d[f]\|d[f]\|_{L^\infty}(E[f]+1)^2,$$

and integrating in time we conclude that

$$d[f](t+h) \leq d[f](t)e^{\int_t^{t+h} c(l)\|d[f]\|_{L^\infty}(E[f](s)+1)^2 ds}.$$

Finally we have that

$$\frac{d}{dt}\|d[f]\|_{L^\infty} = \lim_{h\to 0} \frac{\|d[f]\|_{L^\infty}(t+h) - \|d[f]\|_{L^\infty}(t)}{h} \leq c(l)(E[f]+1)^4.$$

□

4.8 Proof of Theorem 4.1

Now, we have the conditions to finish the proof of Theorem 4.1.

Due to the Lemmas 4.1-4.7, we obtain the following bound

$$\frac{d}{dt}E[f] \leq c(l)(E[f]+1)^5.$$

Therefore

$$E[f](t) \leq \frac{E[f_0]}{\sqrt[4]{-4E[f_0]^4 c(l)t + 1}}.$$

With this *'a priori'* bound we can obtain the local existence of classical solutions using the energy methods (see [44]).

In order to show the existence of such solutions we consider a mollifier

$$\mathcal{J} \in C_c^\infty,\ \mathcal{J}(x) = \mathcal{J}(|x|),\ \mathcal{J} \geq 0 \text{ and } \int_{\mathbb{R}} \mathcal{J} = 1.$$

For instance we can take

$$\mathcal{J}(x) = \frac{e^{-\frac{1}{1-x^2}}}{\int_{-1}^1 e^{-\frac{1}{1-y^2}}dy}.$$

Now, for $\epsilon > 0$, we define

$$\mathcal{J}_\epsilon(x) = \frac{1}{\epsilon}\mathcal{J}\left(\frac{x}{\epsilon}\right).$$

We consider the following regularized system

$$\partial_t f^\epsilon = F^\epsilon(f^\epsilon);\ \ f^\epsilon(x,0) = f_0(x), \tag{4.31}$$

where

$$F^\epsilon(f^\epsilon) = \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon}d\eta\right)$$
$$+ \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) + \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))\sinh(\eta)}{\cosh(\eta) + \cos(\mathcal{J}_\epsilon * f^\epsilon(x) + \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon}d\eta\right). \tag{4.32}$$

We have to show that (4.31) has solutions for $\epsilon << 1$ in H^3 and that this sequence (in ϵ) of regularized solutions has a limit in H^3_l when $\epsilon \to 0$. We remark that the *'a priori'* estimates in previous lemmas are true for these regularized problems. We use Picard's Theorem on the Banach space H^3 to ensure the existence for the regularized problems.

Step 1: functional framework First, we get the following result:

Lemma 4.8. *Let $0 < \tau < l$ and $0 < \varsigma < \infty$ be some fixed constant and define*

$$O_\varsigma^\tau = \{f, f \in H_l^3, \|f\|_{L^\infty} < \tau, \|f\|_{H^3} < \varsigma\}.$$

Then, the set O_ς^τ is a non-empty open set in H^3.

Proof. It is obvious that it is non-empty. We have to check that the L^∞ and H^3 norms are continuous functionals on $\mathbb{R}$. The H^3 norm is obvious. Due to de reverse triangle inequality we obtain

$$\left|\|f\|_{H^3(\mathbb{R})} - \|g\|_{H^3(\mathbb{R})}\right| \leq \|f - g\|_{H^3(\mathbb{R})}.$$

The similar inequality for the L^∞ norm is due to the Sobolev Theorem:

$$\left|\|f\|_{L^\infty(\mathbb{R})} - \|g\|_{L^\infty(\mathbb{R})}\right| \leq \|f - g\|_{L^\infty(\mathbb{R})} \leq c\|f - g\|_{H^3(\mathbb{R})}.$$

As a conclusion the inverse images of the sets $(-1, \tau)$ and $(-1, \varsigma)$ are open sets and so O_ς^τ is an open set. □

Now we take τ and ς such that $f_0 \in O_\varsigma^\tau$.

We also have that in this open subset O_ς^τ the functional $\|d[\cdot]\|_{L^\infty(\mathbb{R}^2)}$, where $d[\cdot]$ is defined in (4.3), is continuous. We prove this fact in the following Lemma:

Lemma 4.9. *Let d be the functional defined in (4.3) and let E be the energy functional defined in (4.2). Then, $\|d[\cdot]\|_{L^\infty(\mathbb{R}^2)} : O_\varsigma^\tau \mapsto \mathbb{R}^+$ is continuous with respect to the H^3 norm. Moreover, we have that for $f \in O_\varsigma^\tau$ we can write $E[f] < \lambda = \lambda(\tau, \varsigma)$.*

Proof. We fix a function $f \in O_\varsigma^\tau$ and consider the ball

$$B(f, \delta_1)_{L^\infty} = \{g \in O_\varsigma^\tau, \|f - g\|_{L^\infty(\mathbb{R})} \leq c\|f - g\|_{H^3} < \delta_1\},$$

with $\delta_1 < \tau - \|f\|_{L^\infty(\mathbb{R})}$ and where c denotes the Sobolev constant. This particular choice of δ_1 is to guarantee $B(f, \delta_1)_{L^\infty} \subset O_\varsigma^\tau$. For $g \in B(f, \delta_1)_{L^\infty} = B(f, \delta_1/c)_{H^3}$, due to de reverse triangle inequality, we have

$$\left|\|d[f]\|_{L^\infty(\mathbb{R}^2)} - \|d[g]\|_{L^\infty(\mathbb{R}^2)}\right| \leq \|d[f] - d[g]\|_{L^\infty(\mathbb{R}^2)} \leq c(\|f\|_{L^\infty(\mathbb{R})}, \delta_1)\|f - g\|_{H^3(\mathbb{R})}.$$

Now, given $\epsilon > 0$ we consider

$$B(f, \delta_2)_{H^3} = \{g \in O_\varsigma^\tau, \|f - g\|_{H^3} < \delta_2\},$$

and take

$$\delta_2 = \frac{\epsilon}{c(\|f\|_{L^\infty(\mathbb{R})}, \delta_1)}.$$

Thus we only have to take $\delta = \min\{\delta_1/c, \delta_2\}$. Then we have that $\|d[f]\|_{L^\infty(\mathbb{R}^2)}$ is continuous (not uniformly) for all $f \in O_\varsigma^\tau$. □

Step 2: applying Picard's Theorem It is obvious that F^ϵ maps O_ς^τ to H^3, so in order to apply Picard's Theorem we have to check the Lipschitz property. We consider $f, g \in O_\varsigma^\tau$ such that $E[f], E[g] < \lambda$. Then we have

$$\|F^\epsilon(f) - F^\epsilon(g)\|_{L^2} \leq \|F_1^\epsilon(f) - F_1^\epsilon(g)\|_{L^2} + \|F_2^\epsilon(f) - F_2^\epsilon(g)\|_{L^2},$$

where F_i^ϵ is the operator corresponding to the kernel Ξ_i. Using Young inequality to eliminate the first mollifier we obtain

$$
\begin{aligned}
\|F_1^\epsilon(f)-F_1^\epsilon(g)\|_{L^2}^2 \quad \leq \quad & \Bigg\|\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x f(x-\eta))\sinh(\eta)}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\\
&\pm \frac{(\mathcal{J}_\epsilon * \partial_x g(x) - \mathcal{J}_\epsilon * \partial_x g(x-\eta))\sinh(\eta)}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\\
&-\frac{(\mathcal{J}_\epsilon * \partial_x g(x) - \mathcal{J}_\epsilon * \partial_x g(x-\eta))\sinh(\eta)}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * g(x) - \mathcal{J}_\epsilon * g(x-\eta))+\epsilon}d\eta\Bigg\|_{L^2}^2.
\end{aligned}
$$

For the first term we have

$$
\begin{aligned}
p_1 \quad = \quad & \left\|(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x g(x))\text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\eta)d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\right\|_{L^2}^2\\
\leq \quad & \left\|(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x g(x))\int_{B(0,2)} \frac{\sinh(\eta)d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\right\|_{L^2}^2\\
&+\left\|(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x g(x))\text{P.V.}\int_{B(0,2)^c} \frac{\sinh(\eta)d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\right\|_{L^2}^2.
\end{aligned}
$$

Now we use that $\cosh(\eta)-1+\epsilon \leq \cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon$ in order to bound the *inner* term. In order to bound the *outer* term we change variables $\eta=-\eta$. Putting all together, we obtain

$$
\begin{aligned}
p_1 \quad \leq \quad & \left\|(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x g(x))\int_2^\infty \frac{2\sinh(\eta)d\eta}{\cosh^2(\eta)-2\cosh(\eta)-2\epsilon-1}\right\|_{L^2}^2\\
&+\left\|(\mathcal{J}_\epsilon * \partial_x f(x) - \mathcal{J}_\epsilon * \partial_x g(x))\int_{B(0,2)} \frac{\sinh(\eta)d\eta}{\cosh(\eta)-1+\epsilon}\right\|_{L^2}^2\\
\leq \quad & c(l,\epsilon)\|f-g\|_{H^3}^2
\end{aligned}
$$

We remark that the *outer* integral can be bounded uniformly in ϵ if we take $0<\epsilon\leq 1/2$. The second term can be bounded using integration by parts, (4.7), and Cauchy-Schwarz and Sobolev inequalities:

$$
\begin{aligned}
p_2 \quad = \quad & \left\|\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f(x-\eta) - \mathcal{J}_\epsilon * \partial_x g(x-\eta))\sinh(\eta)d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon * f(x) - \mathcal{J}_\epsilon * f(x-\eta))+\epsilon}\right\|_{L^2}^2,\\
\leq \quad & \left\|\int_{\mathbb{R}} |\mathcal{J}_\epsilon * f(x-\eta) - \mathcal{J}_\epsilon * g(x-\eta)|\left[\frac{\cosh(\eta)+\sinh(|\eta|)\|\partial_x f\|_{L^\infty}d\eta}{(\cosh(\eta)-1+\epsilon)^2}\right]\right\|_{L^2}^2,\\
\leq \quad & \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{J}_\epsilon * f(x-\eta) - \mathcal{J}_\epsilon * g(x-\eta)|\left[\frac{\cosh(\eta)+\sinh(|\eta|)\|\partial_x f\|_{L^\infty}d\eta}{(\cosh(\eta)-1+\epsilon)^2}\right]\\
&\times\int_{\mathbb{R}} |\mathcal{J}_\epsilon * f(x-\zeta) - \mathcal{J}_\epsilon * g(x-\zeta)|\left[\frac{\cosh(\zeta)+\sinh(|\zeta|)\|\partial_x f\|_{L^\infty}d\zeta}{(\cosh(\zeta)-1+\epsilon)^2}\right]dx,\\
\leq \quad & c(l,\epsilon,\lambda)\|f-g\|_{H^3}^2.
\end{aligned}
$$

For the third term we can do analogously if we use that $|\sin(x)|\leq|x|$ and the Cauchy–Schwarz inequality. Indeed,

$$
p_3 \leq 2\|f-g\|_{L^\infty}^2\left\|\int_{\mathbb{R}} \frac{|\mathcal{J}_\epsilon * \partial_x g(x) - \mathcal{J}_\epsilon * \partial_x g(x-\eta)|\sinh(|\eta|)d\eta}{(\cosh(\eta)-1+\epsilon)(\cosh(\eta)-1+\epsilon)}\right\|_{L^2}^2,
$$

and using Sobolev inequality we conclude that

$$\|F_1^\epsilon(f)-F_1^\epsilon(g)\|_{L^2}\leq c(l,\epsilon,\lambda)\|f-g\|_{H^3}.$$

The second term involves Ξ_2, but with the same ideas we can obtain

$$\|F_2^\epsilon(f)-F_2^\epsilon(g)\|_{L^2}\leq c(l,\epsilon,\lambda)\|f-g\|_{H^3}.$$

Now, we use the following property for the mollifiers:

$$\|\mathcal{J}_\epsilon * v\|_{H^{m+k}}\leq c(\epsilon)\|v\|_{H^m}.$$

Taking $m=0,k=3$ and using the previous estimates we conclude that

$$\|F^\epsilon(f)-F^\epsilon(g)\|_{H^3}\leq c(l,\epsilon,\lambda)\|f-g\|_{H^3},$$

so the spatial operator F^ϵ is locally Lipschitz. Then, using Picard's Theorem, we conclude the existence of classical solutions $f^\epsilon\in C^1([0,T^\epsilon],O_\varsigma^\tau)$ for the regularized equations (4.32). Due to the *'a priori'* energy estimates for f^ϵ, we can choose a time of existence $0<T^*(f_0)$ independently of ϵ.

Step 3: showing that the sequence $\{f^\epsilon\}$ has a limit We have to show that the sequence $\{f^\epsilon\}$ of solutions of the regularized problems (4.31) is Cauchy in $C([0,T],L^2)$, with $T<T^*=1/4E[f_0]^4c(l)$. We recall that for the solutions f^ϵ we have that

$$E[f^\epsilon](t)\leq c(\|f_0\|_{H^3},\|f_0\|_{L^\infty}),$$

because we can use the *'a priori'* bounds in previous Lemmas. Given two solutions f^ϵ and f^δ, we have to study te following quantity

$$\frac{1}{2}\frac{d}{dt}\|f^\epsilon-f^\delta\|_{L^2}^2=\int_{\mathbb{R}}(f^\epsilon-f^\delta)(F^\epsilon(f^\epsilon)-F^\delta(f^\delta))dx.$$

For brevity, we only bound some terms. The terms with Ξ_1 are

$$\begin{aligned}
q_1 &= \int_{\mathbb{R}}(f^\epsilon-f^\delta)(F_1^\epsilon(f^\epsilon)-F_1^\delta(f^\delta))dx,\\
&= \int_{\mathbb{R}}(f^\epsilon-f^\delta)\left(\mathcal{J}_\epsilon*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\epsilon*\partial_x f^\epsilon(x)-\mathcal{J}_\epsilon*\partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon*f^\epsilon(x)-\mathcal{J}_\epsilon*f^\epsilon(x-\eta))+\epsilon}\right)dx\\
&\quad -\int_{\mathbb{R}}(f^\epsilon-f^\delta)\left(\mathcal{J}_\delta*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\delta*\partial_x f^\delta(x)-\mathcal{J}_\delta*\partial_x f^\delta(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\delta*f^\delta(x)-\mathcal{J}_\delta*f^\delta(x-\eta))+\delta}\right)dx\\
&\quad \pm\int_{\mathbb{R}}(f^\epsilon-f^\delta)\left(\mathcal{J}_\delta*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\epsilon*\partial_x f^\epsilon(x)-\mathcal{J}_\epsilon*\partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon*f^\epsilon(x)-\mathcal{J}_\epsilon*f^\epsilon(x-\eta))+\epsilon}\right)dx\\
&= \int_{\mathbb{R}}(f^\epsilon-f^\delta)\left((\mathcal{J}_\epsilon-\mathcal{J}_\delta)*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\epsilon*\partial_x f^\epsilon(x)-\mathcal{J}_\epsilon*\partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon*f^\epsilon(x)-\mathcal{J}_\epsilon*f^\epsilon(x-\eta))+\epsilon}\right)dx\\
&\quad +\int_{\mathbb{R}}(f^\epsilon-f^\delta)\left(\mathcal{J}_\delta*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\epsilon*\partial_x f^\epsilon(x)-\mathcal{J}_\epsilon*\partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\epsilon*f^\epsilon(x)-\mathcal{J}_\epsilon*f^\epsilon(x-\eta))+\epsilon}\right)dx\\
&\quad -\int_{\mathbb{R}}(f^\epsilon-f^\delta)\left(\mathcal{J}_\delta*\int_{\mathbb{R}}\frac{\sinh(\eta)(\mathcal{J}_\delta*\partial_x f^\delta(x)-\mathcal{J}_\delta*\partial_x f^\delta(x-\eta))d\eta}{\cosh(\eta)-\cos(\mathcal{J}_\delta*f^\delta(x)-\mathcal{J}_\delta*f^\delta(x-\eta))+\delta}\right)dx\\
&= r_1+r_2.
\end{aligned}$$

The term q_1 can be bounded using the Cauchy–Schwarz inequality and the properties of mollifiers:

$$\begin{aligned} r_1 &= \int_{\mathbb{R}} (f^\epsilon - f^\delta) \left((\mathcal{J}_\epsilon - \mathcal{J}_\delta) * \int_{\mathbb{R}} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right) dx \\ &\pm \int_{\mathbb{R}} (f^\epsilon - f^\delta) \int_{\mathbb{R}} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} dx \\ &\leq 2\max\{\epsilon, \delta\} \|f^\epsilon - f^\delta\|_{L^2} \left\| \int_{\mathbb{R}} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{H^1}. \end{aligned}$$

We have to bound

$$\begin{aligned} s_1 &= \left\| \int_{\mathbb{R}} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{L^2}^2 \\ &\leq \left\| \int_{B(0,2)} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{L^2}^2 \\ &\quad + \left\| \int_{B(0,2)^c} \frac{\sinh(\eta)\mathcal{J}_\epsilon * \partial_x f^\epsilon(x)d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{L^2}^2 \\ &\quad + \left\| \int_{B(0,2)^c} \frac{\sinh(\eta)(-\mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{L^2}^2 \\ &\leq c(l)\|\mathcal{J}_\epsilon * \partial_x^2 f^\epsilon\|_{L^2}^2 + c(l)\|\mathcal{J}_\epsilon * \partial_x f^\epsilon\|_{L^2}^2 + c(l)(1 + \|\mathcal{J}_\epsilon * \partial_x f^\epsilon\|_{L^\infty})\|\mathcal{J}_\epsilon * f^\epsilon\|_{L^2}^2. \end{aligned}$$

Using that the energy is uniformly bounded only in terms of the initial data we conclude $s_1 \leq c(l,T)$. For the second term we have

$$s_2 = \left\| \partial_x \int_{\mathbb{R}} \frac{\sinh(\eta)(\mathcal{J}_\epsilon * \partial_x f^\epsilon(x) - \mathcal{J}_\epsilon * \partial_x f^\epsilon(x-\eta))d\eta}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^\epsilon(x) - \mathcal{J}_\epsilon * f^\epsilon(x-\eta)) + \epsilon} \right\|_{L^2}^2 \leq c(l,T).$$

Finally we have

$$r_1 \leq c(l,T)\max\{\epsilon,\delta\}\|f^\epsilon - f^\delta\|_{L^2}.$$

For r_2 the following bound can be obtained

$$r_2 \leq c(l,T)(\max\{\epsilon,\delta\}\|f^\epsilon - f^\delta\|_{L^2} + \|f^\epsilon - f^\delta\|_{L^2}^2).$$

For the part corresponding to the second kernel, *i.e.* the terms corresponding to F_2^ϵ the same bound is achieved. Thus, we obtain

$$\frac{d}{dt}\|f^\epsilon - f^\delta\|_{L^2} \leq c(l,T)(\max\{\epsilon,\delta\} + \|f^\epsilon - f^\delta\|_{L^2}),$$

so

$$\sup_{0\leq t\leq T} \|f^\epsilon(t) - f^\delta(t)\|_{L^2} \leq c(l,T)\max\{\epsilon,\delta\}.$$

Consequently, we have that the sequence f^ϵ is Cauchy in $C([0,T],L^2)$. Now, using the energy bound as is exposed in [44], we interpolate between $H^3(\mathbb{R})$ and $L^2(\mathbb{R})$ establishing the existence of a strong limit $f \in C([0,T],H_l^s(\mathbb{R}))$ with $s < 3$ when ϵ tends to zero. For $5/2 < s < 3$ this implies $f \in C([0,T],C^2(\mathbb{R}))$.

In order to obtain the temporal regularity we remark that $\partial_t f^\epsilon$ tends in a distributional way to the weak time derivative of f, we denote it by $\partial_t f$. Indeed, for all test function ϕ, we have that

$$\int_{\mathbb{R}}\int_0^T \phi\partial_t f^\epsilon dtdx = -\int_{\mathbb{R}}\int_0^T \partial_t\phi f^\epsilon dtdx \to -\int_{\mathbb{R}}\int_0^T \partial_t\phi f dtdx = \int_{\mathbb{R}}\int_0^T \phi\partial_t f dtdx.$$

Now, using the equation we can obtain that $\partial_t f \in C([0,T],C(\mathbb{R}))$ and so we have a classical solution $f \in C^1([0,T],C(\mathbb{R})) \cap C([0,T],C^2(\mathbb{R}))$.

As we have an uniform bound for the $H^3(\mathbb{R})$ norm we have that, for any t, there exists a weak limit $h_t \in H^3(\mathbb{R})$. We claim that $h_t = f(t)$. Indeed, fixed $t < T$, we have

$$\langle f^\epsilon(t), \tilde{h}\rangle \to \langle h_t, \tilde{h}\rangle,\ \forall \tilde{h} \in H^{-3}(\mathbb{R}),$$

for some $h_t \in H^3(\mathbb{R})$. Using the inclusion

$$H^3 \subset H^2 \subset L^2 \subset H^{-2} \subset H^{-3},$$

and the dual pairing $\langle,\rangle_{H\times H'}$ defined through the L^2 inner product we have

$$\langle f^\epsilon - h_t, \phi\rangle_{H^3\times H^{-3}} = \int_{\mathbb{R}} (f^\epsilon - h_t)\phi dx \to 0, \quad \forall \phi \in L^2.$$

The strong convergence in H^2 gives us

$$\langle f^\epsilon - f, \phi\rangle_{H^2\times H^{-2}} = \int_{\mathbb{R}} (f^\epsilon - f)\phi dx \to 0, \quad \forall \phi \in L^2,$$

which is contradiction if $h_t \neq f(t)$. We have, using the *'a priori'* bound, that $f \in L^\infty([0,T],H^3(\mathbb{R}))$.

Step 4: continuity in the higher norm We start showing that $f \in C_w([0,T],H^k_l(\mathbb{R}))$. The regularized solutions f^ϵ to (4.32) are in $C^1([0,T],O^\tau_\varsigma)$. Let us consider $\phi \in H^{-k}(\mathbb{R})$. Due to the previous step we have that $f^\epsilon \to f$ in $C([0,T],H^s(\mathbb{R}))$ for $0 < s < k$. Using that H^{-s} is dense in H^{-k} for $0 < s < k$, we can take $\phi' \in H^{-s}(\mathbb{R})$ such that $\|\phi - \phi'\|_{H^{-k}} < \epsilon$. Then, if we denote $\langle,\rangle$ the dual pairing between H^{-3} and H^3,

$$\langle \phi, f^\epsilon - f\rangle = \langle \phi - \phi', f^\epsilon - f\rangle + \langle \phi', f^\epsilon - f\rangle \leq \epsilon(\|f\|_{H^k} + \|f^\epsilon\|_{H^k}) + \|\phi'\|_{H^{-s}}\|f^\epsilon(t) - f(t)\|_{H^s},$$

and using the strong convergence in $C([0,T],H^s)$ we get

$$\langle \phi, f^\epsilon(t) - f(t)\rangle \leq \epsilon c(f_0, T, l, \phi).$$

We conclude $\langle \phi, f^\epsilon(t)\rangle \to \langle \phi, f(t)\rangle$ uniformly in $[0,T]$ and, from the properties of f^ϵ, we obtain that $f \in C_w([0,T],H^k_l(\mathbb{R}))$. The strong continuity is equivalent to the fact that $\|f(t)\|_{H^k}$ is a continuous function on $[0,T]$. For the sake of simplicity we take $k = 3$, being the other cases analogous. From the energy estimates we obtain that

$$\limsup_{t\to 0} \|f(t)\|_{H^3(\mathbb{R})} \leq \|f_0\|_{H^3(\mathbb{R})}.$$

Using $f \in C_w([0,T],H^k_l(\mathbb{R}))$ we have

$$\lim_{t\to 0}(h, f(t)) \to (h, f_0),$$

which means that $f(t) \rightharpoonup f_0$. Thus, $\|f_0\|_{H^3(\mathbb{R})} \leq \liminf_{t\to 0} \|f(t)\|_{H^3(\mathbb{R})}$. We conclude the continuity at $t = 0$. To obtain the continuity at any other time t we use the parabolicity of the equation (4.1) in a classical way. From the previous energy estimates (see Lemma 4.2) we obtain that $\int_0^T \|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2}$ is bounded. Thus, iterating this procedure, for almost every time $0 < T_0$ $f(T_0) \in H^4(\mathbb{R})$. In particular for any $0 < \delta$ we have $f(\delta) = f_0^\delta \in H^4(\mathbb{R})$. We consider $f_0^\delta(x)$ a new initial data. We repeat the existence argument and we obtain a solution $f^\delta(x,t) \in C([0,T],H^s_l)$, with $0 < s < 4$. In particular $f^\delta \in C([0,T],H^3_l(\mathbb{R}))$. At this point we *claim* the uniqueness of such solutions, *i.e.* we claim $f(x,t) = f^\delta(x,t)$ for $\delta < t < T$ where T is the common time interval.

As δ was arbitrary we have that $f(x,t) \in C((0,T], H_l^3(\mathbb{R}))$. Combining this with the continuity at $t=0$ we obtain the desired result.

Step 5: uniqueness In order to prove the uniqueness of classical solution to (4.1) we suppose that we have two solutions in H_l^3 (recall that $l=\pi/2$) with the same initial data, f_1 and f_2. We have the following expression for the L^2 norm of $f=f_1-f_2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 &= \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f_1(x)-f_1(x-\eta))\left(\Xi_1(x,\eta,f_1)-\Xi_1(x,\eta,f_2)\right)d\eta dx\\ &\quad+\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f(x)-f(x-\eta))\Xi_1(x,\eta,f_2)d\eta dx\\ &\quad+\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f_1(x)+f_1(x-\eta))\left(\Xi_2(x,\eta,f_1)-\Xi_2(x,\eta,f_2)\right)d\eta dx\\ &\quad+\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f(x)+f(x-\eta))\Xi_2(x,\eta,f_2)d\eta dx\\ &= U_1+U_2+U_3+U_4.\end{aligned}$$

We study the first two terms, U_1 and U_2, being analogous the study of U_3 and U_4. So we have for U_1

$$U_1=\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f_1(x)-f_1(x-\eta))\left(\Xi_1(x,\eta,f_1)-\Xi_1(x,\eta,f_2)\right)d\eta dx.$$

It follows that

$$\begin{aligned}&\Xi_1(x,\eta,f_1)-\Xi_1(x,\eta,f_2)\\ &\qquad=\sinh(\eta)\left(\frac{\cos(f_2(x)-f_2(x-\eta))-\cos(f_1(x)-f_1(x-\eta))}{\left(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta))\right)\left(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta))\right)}\right),\end{aligned}$$

and using the classical trigonometric formulas we have

$$\begin{aligned}&\cos(f_2(x)-f_2(x-\eta))-\cos(f_1(x)-f_1(x-\eta))\\ &\qquad=2\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)\sin\left(\frac{f(x)-f(x-\eta)}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}U_1 &= \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f_1(x)-f_1(x-\eta))\sinh(\eta)\\ &\quad\cdot\left(\frac{2\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)\sin\left(\frac{f(x)-f(x-\eta)}{2}\right)}{\left(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta))\right)\left(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta))\right)}\right)d\eta dx\\ &\le \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x(f_1(x)-f_1(x-\eta))\sinh(\eta)\\ &\quad\cdot\left(\frac{2\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)\left(\frac{f(x)-f(x-\eta)}{2}\right)}{\left(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta))\right)\left(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta))\right)}\right)d\eta dx\\ &\quad+\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} |f(x)||\partial_x(f_1(x)-f_1(x-\eta))||\sinh(\eta)|\\ &\quad\cdot\left(\frac{2\left|\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)\right|\frac{1}{6}\left(\frac{|f(x)-f(x-\eta)|}{2}\right)^3}{\left(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta))\right)\left(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta))\right)}\right)d\eta dx\\ &\le V_1^1+V_2^1,\end{aligned}$$

with

$$
\begin{aligned}
V_1^1 &= \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} |f(x)|^2\partial_x(f_1(x)-f_1(x-\eta))\sinh(\eta)\\
&\quad\cdot\left(\frac{\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)}{(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta)))(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta)))}\right)d\eta dx\\
&\quad-\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)f(x-\eta)\partial_x(f_1(x)-f_1(x-\eta))\sinh(\eta)\\
&\quad\cdot\left(\frac{\sin\left(\frac{(f_1(x)-f_1(x-\eta))+(f_2(x)-f_2(x-\eta))}{2}\right)}{(\cosh(\eta)-\cos(f_1(x)-f_1(x-\eta)))(\cosh(\eta)-\cos(f_2(x)-f_2(x-\eta)))}\right)d\eta dx\\
&= W_1^1+W_1^2.
\end{aligned}
$$

We observe that W_1^2 vanishes. Therefore, if we change variables $x=\eta$ and $\eta=x$ we have

$$
\begin{aligned}
W_1^2 &= -\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)f(\eta)\\
&\quad\cdot\left(\frac{\partial_x(f_1(x)-f_1(\eta))\sinh(x-\eta)\sin\left(\frac{(f_1(x)-f_1(\eta))+(f_2(x)-f_2(\eta))}{2}\right)}{(\cosh(x-\eta)-\cos(f_1(x)-f_1(\eta)))(\cosh(x-\eta)-\cos(f_2(x)-f_2(\eta)))}\right)d\eta dx\\
&= \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)f(\eta)\\
&\quad\cdot\left(\frac{\partial_x(f_1(x)-f_1(\eta))\sinh(x-\eta)\sin\left(\frac{(f_1(x)-f_1(\eta))+(f_2(x)-f_2(\eta))}{2}\right)}{(\cosh(x-\eta)-\cos(f_1(x)-f_1(\eta)))(\cosh(x-\eta)-\cos(f_2(x)-f_2(\eta)))}\right)d\eta dx\\
&= 0.
\end{aligned}
$$

For W_1^1 in a similar way to the term K_2^1 (4.16) in the Lemma 4.2 we have

$$V_1^1 = W_1^1 \leq c(\|f_1\|_{H^3},\|f_2\|_{H^3})\|f\|_{L^2}^2.$$

The term V_2^1 is not singular, so we have

$$V_2^1 \leq c(\|f_1\|_{H^3},\|f_2\|_{H^3})\|f\|_{L^2}^2.$$

Finally

$$U_1 \leq c(\|f_1\|_{H^3},\|f_2\|_{H^3})\|f\|_{L^2}^2.$$

We split the second integral in two terms: $U_2 = V_1^2 + V_2^2$ with

$$V_1^2 = \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x f(x)\Xi_1(x,\eta,f_2)d\eta dx,$$

and

$$V_2^2 = -\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_x f(x-\eta)\Xi_1(x,\eta,f_2)d\eta dx.$$

Analogously to the proof of Lemma 4.2 we have

$$V_1^2 \leq c(\|f_2\|_{H^3})\|f\|_{L^2}^2.$$

With the same technique as in Lemma 4.2 we have

$$
\begin{aligned}
V_2^2 &= -\int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)\partial_\eta(f(x)-f(x-\eta))\Xi(x,\eta,f_2)d\eta dx\\
&= \int_{\mathbb{R}} \text{P.V}\int_{\mathbb{R}} f(x)(f(x)-f(x-\eta))\partial_\eta\Xi(x,\eta,f_2)d\eta dx \leq c(\|f_2\|_{H^3})\|f\|_{L^2}^2.
\end{aligned}
$$

Therefore,

$$U_2 \leq c(\|f_2\|_{H^3})\|f\|_{L^2}^2.$$

Analogously, for the other terms we prove

$$U_3, U_4 \leq c(\|f_2\|_{H^3}, \|d[f_2]\|_{L^\infty})\|f\|_{L^2}^2.$$

Thus, applying Gronwall's Inequality and the bounds for the U_i terms the uniqueness holds. We have concluded the proof of Theorem 4.1.

Remark 4.1. *From the* a priori *energy estimates, it follows that the Beale-Kato-Majda criterion for this equation involves* $\|f\|_{C^{2,\delta}}$.

Chapter 5

Local solvability in the analytic setting

5.1 Well-posednes for analytic initial data

In this Chapter we study the well-posedness of equation

$$\partial_t f(x,t) = \frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\Big[\left(\partial_x f\left(x\right)-\partial_x f\left(x-\eta\right)\right)\Xi_1(x,\eta,f) \\ +\left(\partial_x f\left(x\right)+\partial_x f\left(x-\eta\right)\Xi_2(x,\eta,f)\Big]d\eta = \frac{\rho^2-\rho^1}{4l}A[f](x), \quad (5.1)$$

when our initial data is analytic and the Rayleigh-Taylor condition is not necessarily satisfied. The proof relies in the appropriate norms and bounds for the complex extension of the equation (5.1). Let $0<\delta<1$ be a fixed parameter. Now, we consider the complex strip

$$\mathbb{B}_r = \{x+i\xi, |\xi|<r\},$$

and the following Banach space

$$X_r = \{f \text{ analytic on } \mathbb{B}_r\},$$

with norm

$$\|f\|_r = \sum_{k=0}^{2}\sup_{x+i\xi\in\mathbb{B}_r}|\partial_x^k f(x+i\xi)| + |\partial_x^2 f|_\delta,$$

where

$$|f|_\delta = \sup_{x+i\xi\in\mathbb{B}_r,\beta\in\mathbb{R}}\frac{|f(x+i\xi)-f(x+i\xi-\beta)|}{|\beta|^\delta}.$$

Notice that $\{X_r\}$ is a Banach scale, *i.e.* $X_r\subset X_{r'}$ and $\|\cdot\|_{r'}\le\|\cdot\|_r$ for $r'\le r$.

In particular, we prove the following Theorem:

Theorem 5.1 (Existence in the unstable case)**.** *Let $f_0\in X_{r_0}$, for some $r_0>0$, be the initial data and assume that this initial data does not reach the boundaries. Then there exists an analytic solution of the Muskat problem* (5.1)*, $f\in C([-T,T],X_r), r<r_0$, for a small enough $T>0$.*

Thus, we get a unique forward and backward solution. For a similar result in the case with infinite depth see [10]. Since the spatial operator has order 1, we can prove this result by a Cauchy-Kowalevski Theorem (see [10], [49], [50] and [46]).

For notational convenience we write $\gamma = x \pm ir$, $\gamma' = x \pm ir'$ and we take $\bar{\rho} = 2$ and $l = \pi/2$. We claim that, for $0 < r' < r$,

$$\|\partial_x f\|_{r'} \leq \frac{C}{r - r'}\|f\|_r. \tag{5.2}$$

Indeed, we apply Cauchy's integral formula with $\Gamma = \gamma' + (r - r')e^{i\theta}$ and we easily obtain

$$\sum_{k=0}^{2} \sup_{\gamma' \in \mathbb{B}_{r'}} |\partial_x^k \partial_x f(\gamma')| \leq \frac{C}{r - r'} \sum_{k=0}^{2} \sup_{\gamma \in \mathbb{B}_r} |\partial_x^k f(\gamma)|.$$

In order to bound the Hölder seminorm of the second derivative we apply Cauchy's integral formula twice with $\Gamma_1 = \gamma' + (r - r')e^{i\theta}$ and $\Gamma_2 = \gamma' - \beta + (r - r')e^{i\theta}$ and we use the Hölder seminorm for $\partial_x^2 f$.

In the appropriate units, equation (5.1) can be written as

$$\partial_t f(\gamma) = \text{P.V.} \int_{\mathbb{R}} \left[\frac{(\partial_x f(\gamma) - \partial_x f(\gamma - \eta)) \sinh(\eta)}{\cosh(\eta) - \cos((f(\gamma) - f(\gamma - \eta)))} + \frac{(\partial_x f(\gamma) + \partial_x f(\gamma - \eta) \sinh(\eta)}{\cosh(\eta) + \cos((f(\gamma) + f(\gamma - \eta)))} \right] d\eta. \tag{5.3}$$

We define

$$d^+[f](x + i\xi, \eta) = \frac{\cosh^2(\eta/2)}{\cosh(\eta) + \cos(f(x + i\xi) + f(x + i\xi - \eta))},$$

and

$$d^-[f](x + i\xi, \eta) = \frac{\sinh^2(\eta/2)}{\cosh(\eta) - \cos(f(x + i\xi) - f(x + i\xi - \eta))}.$$

The function d^+ controls the distance to the boundaries and the function d^- controls the arc-chord condition. To simplify notation we write

$$f^-(\gamma) = f(\gamma) - f(\gamma - \eta), \qquad f^+(\gamma) = f(\gamma) + f(\gamma - \eta), \tag{5.4}$$

$$\sin^-(\gamma) = \sin(f^-(\gamma)), \qquad \sin^+(\gamma) = \sin(f^+(\gamma)), \tag{5.5}$$

$$\cos^-(\gamma) = \cos(f^-(\gamma)), \qquad \cos^+(\gamma) = \cos(f^+(\gamma)). \tag{5.6}$$

5.2 The cornerstone

In this section we prove the keypoint of the Cauchy-Kowalevski argument:

Proposition 5.1. *Consider $0 \leq r' < r$ and the set*

$$O_R = \{f \in X_r \text{ such that } \|f\|_r < R, \quad \|d^-[f]\|_{L^\infty(\mathbb{B}_r)} < R, \quad \|d^+[f]\|_{L^\infty(\mathbb{B}_r)} < R\},$$

where $d^-[f]$ and $d^+[f]$ are defined above. Then, for $f, g \in O_R$, $F : O_R \to X_{r'}$ is continuous. Moreover, the following inequalities holds:

1. $\|F[f]\|_{r'} \leq \frac{C_R}{r-r'}\|f\|_r$,

2. $\|F[f]-F[g]\|_{r'} \leq \frac{C_R}{r-r'}\|f-g\|_r$,
3. $\sup_{(\gamma,\beta)\in\mathbb{B}_r\times\mathbb{R}} |F[f](\gamma)-F[f](\gamma-\beta)| \leq C_R|\beta|$.

Proof. To simplify notation e denote C_R a constant depending only on R that can change from one line to another. To prove the last assertion we show the following Lemma:

Lemma 5.1. *For a classical solution of equation* (5.3) *we have that*

$$\|\partial_t\partial_x f\|_{L^\infty(\mathbb{B}_r)} \leq C_R.$$

Proof. The equation for this quantity is given by

$$\begin{aligned}\partial_t\partial_x f(\gamma) = \text{P.V.}\int_{\mathbb{R}}\left[\frac{\partial_x^2 f^-(\gamma)\sinh(\eta)}{\cosh(\eta)-\cos^-(\gamma)}+\frac{\partial_x^2 f^+(\gamma)\sinh(\eta)}{\cosh(\eta)+\cos^+(\gamma)}\right]d\eta\\ -\text{P.V.}\int_{\mathbb{R}}\left[\frac{(\partial_x f^-(\gamma))^2\sin^-\sinh(\eta)}{(\cosh(\eta)-\cos^-(\gamma))^2}+\frac{(\partial_x f^+(\gamma))^2\sin^+\sinh(\eta)}{(\cosh(\eta)+\cos^+(\gamma))^2}\right]d\eta\\ = A_1+A_2+A_3+A_4.\end{aligned}$$

Splitting in the *'in'* and *'out'* parts the term A_1+A_2 we obtain

$$\begin{aligned}A_1+A_2 \leq c(l)\|f\|_{C^{2,\delta}(\mathbb{B})}\left(\|d^-[f]\|_{L^\infty(\mathbb{B})}+\|d^+[f]\|_{L^\infty(\mathbb{B})}\right)\\ +\|f\|_{C^2(\mathbb{B})}\left\|\int_{B^c(0,1)}\Xi_1+\Xi_2 d\eta\right\|_{L^\infty(\mathbb{B})}+\left|\int_{B^c(0,1)}\partial_\eta\partial_x f(\gamma-\eta)(\Xi_1-\Xi_2)d\eta\right|,\end{aligned}$$

where Ξ_i were defined in (2.6), (2.7). We can bound the last term integrating by parts, obtaining

$$A_1+A_2 \leq C_R$$

For the terms A_3+A_4, in the same way, we get

$$A_3+A_4 \leq C_R.$$

□

The last assertion follows in a straightforward way.

The first one is a special case from the second one, so we need to prove the second inequality. We write the spatial operator in (5.1) as

$$F[f] = F_1[f]+F_2[f],$$

where F_i is the integral corresponding to the kernel Ξ_i.

Step 1: the bound for $\|F[f]-F[g]\|_{L^\infty(\mathbb{B}_{r'})}$ Then, splitting the integrals and using the definition of O_R, we have

$$\begin{aligned}\text{P.V.}\int_{B(0,1)}\frac{(\partial_x f(\gamma')-\partial_x f(\gamma'-\eta)-\partial_x g(\gamma')+\partial_x g(\gamma'-\eta))\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma')-f(\gamma'-\eta)}d\eta\\ +\text{P.V.}\int_{B(0,1)}(\partial_x g(\gamma')-\partial_x g(\gamma'-\eta))\sinh(\eta)\left(\frac{1}{\cosh(\eta)-\cos(f(\gamma')-f(\gamma'-\eta)}\right.\\ \left.-\frac{1}{\cosh(\eta)-\cos(g(\gamma')-g(\gamma'-\eta)}\right)d\eta\\ \leq C_R(\|\partial_x^2(f-g)\|_{L^\infty(\mathbb{B}_{r'})}+\|\partial_x(f-g)\|_{L^\infty(\mathbb{B}_{r'})}) \leq C_R\|\partial_x(f-g)\|_{r'},\end{aligned}$$

and, due to (5.2) we obtain the desired inequality. The outer part is

$$\text{P.V.}\int_{B^c(0,1)} \frac{(\partial_x f(\gamma') - \partial_x f(\gamma'-\eta) - \partial_x g(\gamma') + \partial_x g(\gamma'-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(f(\gamma') - f(\gamma'-\eta)} d\eta$$
$$+ \text{P.V.}\int_{B^c(0,1)} (\partial_x g(\gamma') - \partial_x g(\gamma'-\eta))\sinh(\eta)\left(\frac{1}{\cosh(\eta) - \cos(f(\gamma') - f(\gamma'-\eta)}\right.$$
$$\left. - \frac{1}{\cosh(\eta) - \cos(g(\gamma') - g(\gamma'-\eta))}\right) d\eta = A_1 + A_2.$$

Using (5.2) and integrating by parts, we obtain

$$A_1 \leq \frac{C_R}{r-r'}\|f-g\|_r.$$

The term A_2 can be easily bounded using the definition of $d^-[\cdot]$ and O_R:

$$A_2 \leq \frac{C_R}{r-r'}\|f-g\|_r.$$

Using the previous bounds, we conclude

$$\|F_1[f] - F_1[g]\|_{L^\infty(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|f-g\|_r.$$

The part corresponding to the kernel Ξ_2 can be bounded in a similar way using the definition of $d^+[\cdot]$ and that the integral is no singular. Thus, we obtain

$$\|F_2[f] - F_2[g]\|_{L^\infty(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|f-g\|_r.$$

Thus, we conclude

$$\|F[f] - F[g]\|_{L^\infty(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|f-g\|_r. \tag{5.7}$$

Step 2: the bound for $\|\partial_x(F[f] - F[g])\|_{L^\infty(\mathbb{B}_{r'})}$ Moreover, with the same techniques we can prove

$$\|F[f] - F[g]\|_{L^\infty(\mathbb{B}_r)} \leq C_R\|f-g\|_r.$$

Then, using Cauchy's integral formula, we obtain

$$\|\partial_x(F[f] - F[g])\|_{L^\infty(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|f-g\|_r. \tag{5.8}$$

Step 3 the bound for $\|\partial_x^2(F[f] - F[g])\|_{L^\infty(\mathbb{B}_{r'})}$: We need to bound $\|\partial_x(F[f] - F[g])\|_{L^\infty(\mathbb{B}_r)} \leq C_R\|f-g\|_r$ in order to use Cauchy's integral formula. We change slightly the notation in (5.5) and (5.6) and write

$$\sin^-(f,\gamma) = \sin(f^-(\gamma)), \qquad \sin^+(f,\gamma) = \sin(f^+(\gamma)),$$

$$\cos^-(f,\gamma) = \cos(f^-(\gamma)) \text{ and } \cos^+(f,\gamma) = \cos(f^+(\gamma)).$$

5.2. The cornerstone

We have

$$
\begin{aligned}
\partial_x(F_1[f]-F_1[g])(\gamma) &= (\partial_x^2 f(\gamma)-\partial_x^2 g(\gamma))\text{P.V.}\int_{\mathbb{R}}\frac{\sinh(\eta)}{\cosh(\eta)-\cos^-(f,\gamma)}d\eta\\
&-\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta))\sinh(\eta)}{\cosh(\eta)-\cos^-(f,\gamma)}d\eta\\
&+\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x^2 g^-(\gamma)\sinh(\eta)}{\cosh(\eta)-\cos^-(f,\gamma)}-\frac{\partial_x^2 g^-(\gamma)\sinh(\eta)}{\cosh(\eta)-\cos^-(g,\gamma)}d\eta\\
&-\text{P.V.}\int_{\mathbb{R}}\frac{((\partial_x f^-(\gamma))^2-(\partial_x g^-(\gamma))^2)\sinh(\eta)\sin^-(f,\gamma)}{(\cosh(\eta)-\cos^-(f,\gamma))^2}d\eta\\
&-\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x g^-(\gamma))^2\sinh(\eta)(\sin^-(f,\gamma)-\sin^-(g,\gamma))}{(\cosh(\eta)-\cos^-(f,\gamma))^2}d\eta\\
&-\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x g^-(\gamma))^2\sinh(\eta)\sin^-(g)}{(\cosh(\eta)-\cos^-(f,\gamma))^2}d\eta\\
&+\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x g^-(\gamma))^2\sinh(\eta)\sin^-(g)}{(\cosh(\eta)-\cos^-(g,\gamma))^2}d\eta.
\end{aligned}
$$

We split the integrals:

$$\partial_x(F_1[f]-F_1[g]) = A_3 + A_4,$$

where A_3 are the integrals in $B(0,1)$ and A_4 are integrals in $B^c(0,1)$. The *'outer'* integral can be easily bounded with the previous techniques without lossing derivatives, obtaining

$$A_4 \leq C_R\|f-g\|_r.$$

We write

$$A_3 = B_1 + B_2 + \text{ non-singular terms.}$$

We proceed now with the singular terms, those with $\partial_x^2(f-g)$, in A_3 without using higher derivatives. We have

$$
\begin{aligned}
B_1 &= (\partial_x^2 f(\gamma)-\partial_x^2 g(\gamma))\text{P.V.}\int_{B(0,1)}\frac{\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}d\eta\\
&= (\partial_x^2 f(\gamma)-\partial_x^2 g(\gamma))\text{P.V.}\int_{B(0,1)}\frac{\sinh(\eta)}{\sinh^2(\eta/2)}(d^-[f](\gamma,\eta)-d^-[f](\gamma,0))d\eta.
\end{aligned}
$$

We can compute

$$|d^-[f](\gamma,\eta)-d^-[f](\gamma,0)| \leq C_R|\eta|,$$

and, consequently,

$$B_1 \leq C_R\|f-g\|_r.$$

The other singular term is

$$B_2 = -\text{P.V.}\int_{B(0,1)}\frac{(\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta))\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta)}d\eta.$$

We have

$$\begin{aligned} B_2 &= -\text{P.V.}\int_{B(0,1)} \frac{(\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta))(\sinh(\eta)-\eta)}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}d\eta \\ &\quad -\text{P.V.}\int_{B(0,1)} (\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta))d^-[f](\gamma,\eta)\eta\left(\frac{1}{\sinh^2(\eta/2)}-\frac{1}{(\eta/2)^2}\right)d\eta \\ &\quad -\text{P.V.}\int_{B(0,1)} (\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta))\frac{\eta}{(\eta/2)^2}(d^-[f](\gamma,\eta)-d^-[f](\gamma,0))d\eta \\ &\quad -4d^-[f](\gamma,0)\text{P.V.}\int_{B(0,1)} \frac{\partial_x^2 f(\gamma-\eta)-\partial_x^2 g(\gamma-\eta)}{\eta}d\eta \\ &= C_1+C_2+C_3+C_4. \end{aligned}$$

With this splitting we observe that C_1, C_2 and C_3 are not singular and then

$$C_1+C_2+C_3 \leq C_R\|f-g\|_r.$$

The C_4 term is the inner part of the Hilbert Transform. We have

$$C_4 = 4d^-[f](\gamma,0)\text{P.V.}\int_{B(0,1)} \frac{\partial_x^2(f-g)(\gamma)-\partial_x^2(f-g)(\gamma-\eta)}{\eta}d\eta \leq C_R|\partial_x^2(f-g)|_\delta.$$

Putting all together we obtain

$$B_2 \leq C_R\|f-g\|_r.$$

The bound for the terms corresponding to F_2 can be easily obtained, then we conclude

$$\|\partial_x^2(F[f]-F[g])\|_{L^\infty(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|\partial_x(F[f]-F[g])\|_{L^\infty(\mathbb{B}_r)} \leq \frac{C_R}{r-r'}\|f-g\|_r.$$

Step 4: the bound for $|\partial_x^2(F[f]-F[g])|_\delta$ It remains the Hölder seminorm. We need to prove that

$$\partial_x^2(F[f]-F[g])(\gamma')-\partial_x^2(F[f]-F[g])(\gamma'-\beta) \leq \frac{C_R}{r-r'}\|f-g\|_r|\beta|^\delta.$$

For the sake of brevity we bound only the more singular terms. We have the term

$$A_5 = (\partial_x^3 f(\gamma')-\partial_x^3 g(\gamma'))\text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma')-f(\gamma'-\eta))}d\eta.$$

Now we use the classical inequality

$$|XY|_\delta \leq \|X\|_{L^\infty}|Y|_\delta + \|Y\|_{L^\infty}|X|_\delta,$$

to obtain

$$|A_5|_\delta \leq \|\partial_x(f-g)\|_{r'}\left\|\text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma')-f(\gamma'-\eta))}d\eta\right\|_{C^\delta}.$$

We use the following interpolation inequality

$$\frac{|G(x)-G(x-y)|}{|y|^\delta} = \left(\frac{|G(x)-G(x-y)|}{|y|}\right)^\delta |G(x)-G(x-y)|^{1-\delta} \leq c\|G\|_{C^1}$$

and the previous techniques to obtain

$$\left\|\text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\eta)}{\cosh(\eta)-\cos(f(\gamma')-f(\gamma'-\eta))}d\eta\right\|_{C^\delta} \leq C_R.$$

We write

$$A_6 = -\left(\text{P.V.}\int_{B(0,1)} + \text{P.V.}\int_{B^c(0,1)}\right)\frac{(\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(f(\gamma') - f(\gamma'-\eta))}d\eta = B_3 + B_4.$$

We split

$$\begin{aligned}
B_3 &= -\text{P.V.}\int_{B(0,1)} \frac{(\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta))(\sinh(\eta)-\eta)}{\cosh(\eta) - \cos(f(\gamma') - f(\gamma'-\eta))}d\eta \\
&\quad -\text{P.V.}\int_{B(0,1)} (\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta))d^-[f](\gamma',\eta)\eta\left(\frac{1}{\sinh^2(\eta/2)} - \frac{1}{(\eta/2)^2}\right)d\eta \\
&\quad -\text{P.V.}\int_{B(0,1)} (\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta))\frac{\eta}{(\eta/2)^2}(d^-[f](\gamma',\eta) - d^-[f](\gamma',0))d\eta \\
&\quad -4d^-[f](\gamma',0)\text{P.V.}\int_{B(0,1)} \frac{\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta)}{\eta}d\eta \\
&= C_5 + C_6 + C_7 + C_8.
\end{aligned}$$

We have to study the variations given by

$$dC_i = C_i(\gamma') - C_i(\gamma'-\beta), \text{ for } i = 5,6,7,8.$$

The first three terms are not singular, so they can be bounded easily by

$$dC_5 + dC_6 + dC_7 \leq \|\partial_x(f-g)\|_{r'}C_R|\beta|^\delta \leq \frac{C_R}{r-r'}\|f-g\|_r|\beta|^\delta.$$

The last term is

$$\begin{aligned}
dC_8 = -4(d^-[f](\gamma',0) - d^-[f](\gamma'-\beta,0))\text{P.V.}\int_{B(0,1)} \frac{\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta)}{\eta}d\eta \\
- 4d^-[f](\gamma'-\beta,0) \\
\times \text{P.V.}\int_{B(0,1)} \frac{\partial_x^3 f(\gamma'-\eta) - \partial_x^3 g(\gamma'-\eta) - \partial_x^3 f(\gamma'-\eta-\beta) + \partial_x^3 g(\gamma'-\eta-\beta)}{\eta}d\eta \\
= dC_8^1 + dC_8^2.
\end{aligned}$$

Using the cancelation of the Principal Value integral we have

$$dC_8^1 \leq C_R|\beta|^\delta\left|\text{P.V.}\int_{B(0,1)} \frac{\partial_x^3(f-g)(\gamma') - \partial_x^3(f-g)(\gamma'-\eta)}{\eta}d\eta\right| \leq \frac{C_R}{r-r'}|\beta|^\delta\|f-g\|_r.$$

We complete the integrals in dC_8^2 as follows:

$$\begin{aligned}
dC_8^2 = -4d^-[f](\gamma'-\beta,0)\left(H\left(\partial_x^3(f-g)(\gamma')\right) - H\left(\partial_x^3(f-g)(\gamma'-\beta)\right)\right) \\
+ 4d^-[f](\gamma'-\beta,0)\text{P.V.}\int_{B^c(0,1)} \frac{\partial_x^3(f-g)(\gamma'-\eta) - \partial_x^3(f-g)(\gamma'-\beta-\eta)}{\eta}d\eta.
\end{aligned}$$

Now we observe that in order to bound the Hölder seminorm of the Hilbert Transform you don't need that the function has compact support (see [44]). Then

$$\begin{aligned}
-4d^-[f](\gamma'-\beta,0)\left(H\left(\partial_x^3(f-g)(\gamma')\right) - H\left(\partial_x^3(f-g)(\gamma'-\beta)\right)\right) \\
\leq C_R|\partial_x^3(f-g)|_\delta|\beta|^\delta \leq \frac{C_R}{r-r'}\|f-g\|_r|\beta|^\delta.
\end{aligned}$$

Now, for the remaining term we integrate by parts:

$$\text{P.V.}\int_{B^c(0,1)} \frac{\partial_\eta\partial_x^2(f-g)(\gamma'-\eta)-\partial_\eta\partial_x^2(f-g)(\gamma'-\beta-\eta)}{\eta}d\eta$$
$$\leq c|\partial_x^2(f-g)|_\delta|\beta|^\delta \leq \frac{C_R}{r-r'}\|f-g\|_r|\beta|^\delta,$$

where, in the last inequality we used the previous interpolation inequality for Hölder seminorms.

In order to bound the last term, B_4, we integrate by parts to ensure the decay at infinity. Using the previous notation (5.5) and (5.6), we have

$$\begin{aligned} B_4 &= -\int_{B^c(0,1)} \frac{\partial_x^2(f-g)(\gamma'-\eta)(1-\cosh(\eta)\cos^-(f,\gamma')-\sinh(\eta)\sin^-(f,\gamma')\partial_x f(\gamma'-\eta))}{(\cosh(\eta)-\cos^-(f,\gamma'))^2} \\ &\quad -\frac{\partial_x^2(f-g)(\gamma'+1)(1-\cosh(1)\cos^-(f,\gamma')+\sinh(1)\sin^-(f,\gamma')\partial_x f(\gamma'+1))}{(\cosh(1)-\cos^-(f,\gamma'))^2} \\ &\quad -\frac{\partial_x^2(f-g)(\gamma'-1)(1-\cosh(1)\cos^-(f,\gamma')-\sinh(1)\sin^-(f,\gamma')\partial_x f(\gamma'-1))}{(\cosh(1)-\cos^-(f,\gamma'))^2} \\ &= C_9 + \text{ boundary terms.} \end{aligned}$$

Now we split the integral coming from the difference:

$$\begin{aligned} dC_9 &= -\int_{B^c(0,1)} \frac{(\partial_x^2(f-g)(\gamma'-\eta)-\partial_x^2(f-g)(\gamma'-\eta-\beta))}{(\cosh(\eta)-\cos^-(f,\gamma'))^2} \\ &\qquad \times(1-\cosh(\eta)\cos^-(f,\gamma')-\sinh(\eta)\sin^-(f,\gamma')\partial_x f(\gamma'-\eta))d\eta \\ &\quad -\int_{B^c(0,1)} \frac{\partial_x^2(f-g)(\gamma'-\eta-\beta)}{(\cosh(\eta)-\cos^-(f,\gamma'))^2}(1-\sinh(\eta)\sin^-(f,\gamma')\partial_x f(\gamma'-\eta) \\ &\qquad -\cosh(\eta)(\cos^-(f,\gamma')-\cos(f,\gamma'-\beta)))d\eta \\ &\quad -\int_{B^c(0,1)} \frac{\partial_x^2(f-g)(\gamma'-\eta-\beta)}{(\cosh(\eta)-\cos^-(f,\gamma'))^2}(1-\cosh(\eta)\cos^-(f,\gamma'-\beta) \\ &\qquad -\sinh(\eta)(\sin^-(f,\gamma')-\sin^-(f,\gamma'-\beta)))\partial_x f(\gamma'-\eta))d\eta \\ &\quad -\int_{B^c(0,1)} \frac{\partial_x^2(f-g)(\gamma'-\eta-\beta)}{(\cosh(\eta)-\cos^-(f,\gamma'))^2}(1-\cosh(\eta)\cos^-(f,\gamma'-\beta) \\ &\qquad -\sinh(\eta)\sin^-(f,\gamma'-\beta)(\partial_x f(\gamma'-\eta)-\partial_x f(\gamma'-\eta-\beta)))d\eta \\ &\quad -\int_{B^c(0,1)} \partial_x^2(f-g)(\gamma'-\eta-\beta)(1-\cosh(\eta)\cos^-(f,\gamma'-\beta) \\ &\qquad -\sinh(\eta)\sin^-(f,\gamma'-\beta)\partial_x f(\gamma'-\eta-\beta)) \\ &\quad \times\left(\frac{1}{(\cosh(\eta)-\cos^-(f,\gamma'))^2}-\frac{1}{(\cosh(\eta)-\cos^-(f,\gamma'-\beta))^2}\right)d\eta \\ &\leq \frac{C_R}{r-r'}\|f-g\|_r|\beta|^\delta. \end{aligned}$$

The other terms and those corresponding to F_2 are easier and the same inequality follows using the same techniques. Then we have

$$\|F[f]-F[g]\|_{r'} \leq \frac{C_R}{r-r'}\|f-g\|_r,$$

and we conclude the proof of Proposition 5.1. □

5.3 Proof of Theorem 5.1

The proof follows the same argument as in [10, 46, 49, 50]. As $f_0 \in X_{r_0}$ there exists R_0 such that $f_0 \in O_{R_0}$. We take $r < r_0$ and $R > R_0$ in order to define O_R and we consider the iterates

$$f_{n+1} = f_0 + \int_0^t F[f_n]ds,$$

and assume by induction that $f_k \in O_R$ for $k \leq n$. Then, we obtain a time of existence $T_{CK} > 0$. It remains to show that

$$\|d^-[f_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R, \quad \|d^+[f_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R,$$

for some times $T_A, T_B > 0$ respectively. Then we choose $T = \min\{T_{CK}, T_A, T_B\}$ and we finish the proof.

Using the classical trigonometric formulas we obtain

$$\begin{aligned}(d^-[f_{n+1}])^{-1} = &\frac{\cosh(\eta) - \cos\left(f_0(\gamma) - f_0(\gamma-\eta)\right)}{\sinh^2(\eta/2)}\\ &+ \frac{2\cos(f_0(\gamma) - f_0(\gamma-\eta))\sin^2\left(\int_0^t F[f_n](\gamma) - F[f_n](\gamma-\eta)ds/2\right)}{\sinh^2(\eta/2)}\\ &+ \frac{\sin(f_0(\gamma) - f_0(\gamma-\eta))\sin\left(\int_0^t F[f_n](\gamma) - F[f_n](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}.\end{aligned}$$

Take $t \leq 1$ and assume that $\eta \in B(0,1)$, then, using the inequality

$$\sup_{\gamma\in\mathbb{B}_r, \beta\in\mathbb{R}} |F[f](\gamma) - F[f](\gamma-\beta)| \leq C_R|\beta|,$$

inf Proposition 5.1, we have

$$(d^-[f_{n+1}])^{-1} > \frac{1}{R_0} - C_R^1(t^2+t).$$

In the case where $\eta \in B^c(0,1)$, to ensure the decay at infinity, we use the inequality

$$\|F[f_n]\|_{L^\infty(\mathbb{B}_r)} \leq C_R,$$

to get

$$(d^-[f_{n+1}])^{-1} > \frac{1}{R_0} - C_R^2(t^2+t).$$

Thus, we can take

$$0 < T_A < \min\left\{1, \sqrt{\left(\frac{1}{R_0} - \frac{1}{R}\right)\frac{1}{4\max\{C_R^1, C_R^2\}}}\right\},$$

and then $\|d^-[f_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R$. We do in the same way for $d^+[f_{n+1}]$. Using the classical trigonometric formulas and the previous inequalities, we obtain

$$
\begin{aligned}
(d^+[f_{n+1}])^{-1} = & \frac{\cosh(\eta) + \cos\left(f_0(\gamma) + f_0(\gamma-\eta)\right)}{\cosh^2(\eta/2)} \\
& + \frac{\cos(f_0(\gamma) + f_0(\gamma-\eta))\left(\cos\left(\int_0^t F[f_n](\gamma) + F[f_n](\gamma-\eta)ds\right) - 1\right)}{\cosh^2(\eta/2)} \\
& - \frac{\sin(f_0(\gamma) + f_0(\gamma-\eta))\sin\left(\int_0^t F[f_n](\gamma) + F[f_n](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\
& > \frac{1}{R_0} - C_R^3(t^2+t),
\end{aligned}
$$

thus we can take

$$
0 < T_B < \min\left\{1, \sqrt{\left(\frac{1}{R_0} - \frac{1}{R}\right)\frac{1}{2C_R^3}}\right\},
$$

and then $\|d^+[f_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R$. This concludes the proof of Theorem 5.1.

Remark 5.1. *We observe that the previous Theorem holds for periodic, and analytic initial data.*

Part II

The qualitative properties of the confined Muskat problem

Chapter 6

Smoothing effect and Ill-posedness

6.1 Foreword

In this Chapter we prove the instant analyticity for the classical solution of equation (2.5), which exists due to the Theorem 4.1, if we start with a smooth initial data. Moreover, we use the smoothing effect to obtain the ill-posedness in Sobolev spaces for negative times or equivalently the Rayleigh-Taylor unstable case $\rho^2 < \rho^1$. In particular, we prove

Theorem 6.1 (Instant analyticity)**.** *Let $f_0 \in H^3_l(\mathbb{R})$ be the initial data and assume that the Rayleigh-Taylor condition is satisfied,* i.e. *$\rho^1 < \rho^2$. Then, the unique classical solution $f(x,t)$ of equation (2.5) continues analitically into the strip $\mathbb{B} = \{x + i\xi, |\xi| < kt, \forall\, 0 < t \leq T(f_0)\}$ with $k = k(f_0, l)$.*

Without lossing generality we consider $l = \pi/2$ and $\bar{\rho} = 2$. The proof of Theorem 6.1 relies on some estimates for the complex extension of the function f (see equation (6.5)) on the boundary of the strip

$$\mathbb{B} = \{x + i\xi, |\xi| < kt, \forall\; 0 < t \leq T(f_0)\}$$

for certain k, a universal constant that will be fixed later. This property in the infinitely deep case is proved in [10].

Since in the stable case the solution becomes analytic, one expect ill-posedness in the unstable case, *i.e.* $\rho^1 > \rho^2$. We have the following Theorem:

Theorem 6.2 (Ill-posedness)**.** *There exists a solution $\tilde{f}$ of (2.5) in the Rayleigh-Taylor unstable case,* i.e. *$\rho^2 < \rho^1$, such that $\|\tilde{f}_0\|_{H^s(\mathbb{R})} < \epsilon$ and $\|\tilde{f}(\delta)\|_{H^s(\mathbb{R})} = \infty$, for any $s \geq 4$, $\epsilon > 0$ and small enough $\delta > 0$.*

The proof of this Theorem is also valid for the periodic or flat at infinity, infinite depth case while the proof in [23] (see also [54]) it is not valid in the confined case. The main difficulty of (2.5) is that the kernels don't allow any scaling.

6.2 A useful commutator estimate

We will need a result concerning the commutator $[\Lambda^\alpha, F] = \Lambda^\alpha F - F\Lambda^\alpha$ and where $\Lambda = \sqrt{-\Delta}$. For the sake of completeness we give the result.

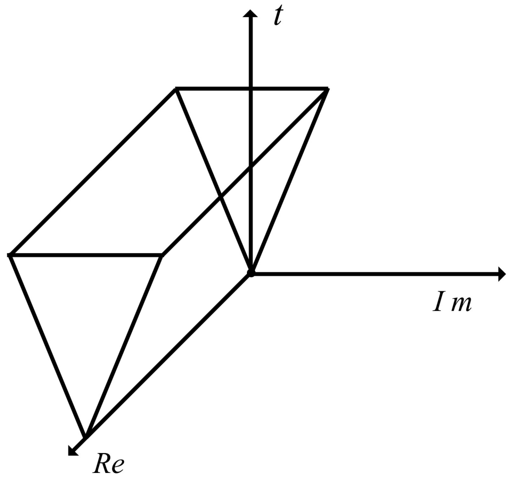

Figure 6.1: The strip of analitycity for f.

Lemma 6.1. *Let $0 < \alpha \leq 1$ be a real number and let F, G be two smooth functions decaying at infinity. Then we have the following inequalities:*

$$\|\Lambda^\alpha(FG) - F\Lambda^\alpha G\|_{L^2(\mathbb{R})} \leq c(\alpha,\beta)\|F\|_{C^\beta(\mathbb{R})}\|G\|_{L^2(\mathbb{R})}, \text{ if } 0 < \alpha < \beta < 1$$

and

$$\|\Lambda(FG) - F\Lambda G\|_{L^2(\mathbb{R})} \leq c(\beta)\|F\|_{C^{1,\beta}(\mathbb{R})}\|G\|_{L^2(\mathbb{R})}, \text{ with } 0 < \beta < 1.$$

Proof. We have

$$|\widehat{\Lambda^\alpha(FG)} - \widehat{F\Lambda^\alpha G}| = \left|\int_{\mathbb{R}} \hat{G}(\xi - s)\hat{F}(s)(|\xi|^\alpha - |\xi - s|^\alpha)ds\right| \leq c_\alpha \int_{\mathbb{R}} |\hat{G}(\xi - s)||\hat{F}(s)||s|^\alpha ds,$$

where we have used the classical inequality $|\xi|^\alpha \leq 2^{\alpha-1}(|\xi - s|^\alpha + |s|^\alpha)$. We use Plancherel Theorem and Hausdorff-Young inequality to obtain

$$\|\Lambda^\alpha(FG) - F\Lambda^\alpha G\|_{L^2(\mathbb{R})} \leq c_\alpha\|\widehat{\Lambda^\alpha F}\|_{L^1(\mathbb{R})}\|G\|_{L^2(\mathbb{R})} \leq c_\alpha\|\Lambda^\alpha F\|_{L^\infty(\mathbb{R})}\|G\|_{L^2(\mathbb{R})}.$$

We consider $0 < \alpha < 1$, then

$$\|\Lambda^\alpha F\|_{L^\infty(\mathbb{R})} \leq c_\beta\|F\|_{C^\beta(\mathbb{R})}, \ \alpha < \beta < 1.$$

In the case where $\alpha = 1$ we have

$$\|\Lambda F\|_{L^\infty(\mathbb{R})} \leq c_\beta\|F\|_{C^{1,\beta}(\mathbb{R})}, \ 0 < \beta < 1.$$

□

6.3 The appropriate energy

The proof of Theorem 6.1 relies in some *'a priori'* bounds for the appropriate energy in the complex strip $\mathbb{B}$. In this section we describe such an energy. We define

$$\|f\|^2_{L^2(\mathbb{B})} = \int_{\mathbb{R}} |f(x + ikt)|^2 dx + \int_{\mathbb{R}} |f(x - ikt)|^2 dx,$$

6.4. Bound for $\|f\|_{L^2(\mathbb{B})}$

$$\|f\|_{H^3(\mathbb{B})}^2 = \|f\|_{L^2(\mathbb{B})}^2 + \|\partial_x^3 f\|_{L^2(\mathbb{B})}^2,$$

$$d^+[f](x+i\xi,\eta) = \frac{\cosh^2(\eta/3)}{\cosh(\eta)+\cos(f(x+i\xi)+f(x+i\xi-\eta))}, \tag{6.1}$$

$$d^-[f](x+i\xi,\eta) = \frac{\sinh^2(\eta/3)}{\cosh(\eta)-\cos(f(x+i\xi)-f(x+i\xi-\eta))}. \tag{6.2}$$

We remark that $d^+[f]$, as $d[f]$ defined in (4.3), gives us the distance up to the boundary. $d^-[f]$ plays the role of the arc-chord condition (see [10]) and ensures that the singularity in the first kernel has order two. This fact for real functions f holds automatically, but this is not the case for complex valued functions. In the case of complex functions we have that if the kernel Ξ_1 is singular then the following equality holds

$$\cosh(\eta) - \cos(\mathrm{Re}f(x+ikt) - \mathrm{Re}f(x+ikt-\eta))\cosh(\mathrm{Im}f(x+ikt) - \mathrm{Im}f(x+ikt-\eta)) = 0.$$

Assume now that $|\eta| > R \geq 2l$, where R is a constant that will be fixed later, then

$$\begin{aligned}\frac{1}{2}\cosh(\eta) + \frac{1}{2}\cosh(R) - \cosh(2\|f\|_{L^\infty(\mathbb{B})}) &\leq \frac{1}{2}\cosh(\eta) + \frac{1}{2}\cosh(R) - \cosh(2\|\mathrm{Im}f\|_{L^\infty(\mathbb{B})}) \\ &\leq \cosh(\eta) - \cos(\mathrm{Re}f(x+ikt) - \mathrm{Re}f(x+ikt-\eta))\cosh(\mathrm{Im}f(x+ikt) - \mathrm{Im}f(x+ikt-\eta)),\end{aligned}$$

so we want that

$$\frac{1}{2}\cosh(R) - \cosh(2\|f\|_{L^\infty(\mathbb{B})}) \geq 0,$$

because in that case the kernel Ξ_1 is not singular and behaves as $\tanh(\eta)$. A similar analysis can be done for Ξ_2. We write

$$D[f](\gamma) = \frac{1}{\cosh(R) - 2\cosh(2|f(\gamma)|)} \tag{6.3}$$

We consider Hardy-Sobolev spaces (see [2] and references therein) on $\mathbb{B}$ so we want to obtain *'a priori'* bounds on the following energy

$$E_{\mathbb{B}}[f] = \|f\|_{H^3(\mathbb{B})}^2 + \|d^+[f]\|_{L^\infty(\mathbb{B}))} + \|d^-[f]\|_{L^\infty(\mathbb{B})} + \|D[f]\|_{L^\infty(\mathbb{B})}, \tag{6.4}$$

where

$$\|F(x+i\xi,\eta)\|_{L^\infty(\mathbb{B})} = \sup_{x+i\xi\in\mathbb{B},\eta\in\mathbb{R}} |F(x+i\xi,\eta)|.$$

Notice that, due to the Hadamard's Three Lines Theorem, it is enough to consider the supremum on $\partial\mathbb{B}$.

6.4 Bound for $\|f\|_{L^2(\mathbb{B})}$

The evolution for the complex extension of f is

$$\begin{aligned}\partial_t f(x\pm ikt) = \mathrm{P.V.}\int_{\mathbb{R}} \Bigg[&\frac{(\partial_x f(x\pm ikt) - \partial_x f(x\pm ikt-\eta))\sinh(\eta)}{\cosh(\eta) - \cos((f(x\pm ikt) - f(x\pm ikt-\eta)))} \\ &+ \frac{(\partial_x f(x\pm ikt) + \partial_x f(x\pm ikt-\eta)\sinh(\eta)}{\cosh(\eta) + \cos((f(x\pm ikt) + f(x\pm ikt-\eta)))}\Bigg] d\eta.\end{aligned} \tag{6.5}$$

For the sake of brevity we work with both boundaries $x \pm ikt$ at the same time and we write $\gamma = x \pm ikt$. We remark that $c(l)$ is an universal constant that can change from one part to another. We start with the $L^2(\mathbb{B})$ norm:

Lemma 6.2. *Let f be a $H^3(\mathbb{B})$ solution of* (6.5). *Then*

$$\frac{d}{dt}\|f\|^2_{L^2(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Proof. Using that $\int f\bar{g} = \overline{\int g\bar{f}}$ we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|f(\gamma)|^2dx = \text{Re}\int_{\mathbb{R}}\bar{f}(\gamma)(\partial_t f(\gamma)\pm ik\partial_x f(\gamma))dx.$$

The terms with the spatial derivative can be bounded as follows

$$\text{Re}\int_{\mathbb{R}}\bar{f}(\pm ik\partial_x f)dx \leq \left|\int_{\mathbb{R}}\bar{f}(\pm ik\partial_x f)dx\right| \leq k\|f\|_{L^2(\mathbb{B})}\|\partial_x f\|_{L^2(\mathbb{B})} \leq kE_{\mathbb{B}}[f].$$

For the terms with temporal derivative we split the integrals as in Lemma 4.1:

$$\text{Re}\int_{\mathbb{R}}\bar{f}\partial_t f d\eta = A_1 + A_2,$$

where

$$\begin{aligned}
A_1 &= \text{Re}\int_{\mathbb{R}}\text{P.V.}\int_{B(0,1)}\bar{f}(\gamma)\left[\frac{(\partial_x f(\gamma)-\partial_x f(\gamma-\eta))\sinh(\eta)}{\cosh(\eta)-\cos((f(\gamma)-f(\gamma-\eta)))}\right.\\
&\qquad\left.+\frac{(\partial_x f(\gamma)+\partial_x f(\gamma-\eta))\sinh(\eta)}{\cosh(\eta)+\cos((f(\gamma)+f(\gamma-\eta)))}\right]d\eta dx\\
&= \text{Re}\int_0^1\int_{\mathbb{R}}\text{P.V.}\int_{B(0,1)}\bar{f}(\gamma)\frac{(\partial_x^2 f(\gamma+(s-1)\eta))\,\eta\sinh(\eta)}{\cosh(\eta)-\cos((f(\gamma)-f(\gamma-\eta)))}d\eta dx ds\\
&\quad+\text{Re}\int_{\mathbb{R}}\text{P.V.}\int_{B(0,1)}\bar{f}(\gamma)\frac{(\partial_x f(\gamma)+\partial_x f(\gamma-\eta))\sinh(\eta)}{\cosh(\eta)+\cos((f(\gamma)+f(\gamma-\eta)))}d\eta dx\\
&\leq c(l)\|f\|_{L^2(\mathbb{B})}(\|\partial_x^2 f\|_{L^2(\mathbb{B})}\|d^-[f]\|_{L^\infty(\mathbb{B})}+\|\partial_x f\|_{L^2(\mathbb{B})}\|d^+[f]\|_{L^2(\mathbb{B})})\\
&\leq c(l)(E_{\mathbb{B}}[f])^2
\end{aligned}$$

In the first equality we used the complex extension of the formula (4.4):

$$\partial_x f(\gamma)-\partial_x f(\gamma-\eta) = \eta\int_0^1\partial_x^2 f(\gamma+(s-1)\eta)ds.$$

The term A_2 corresponds to the *'outer'* part and can be splitted as

$$A_2 = B_1 + B_2,$$

where

$$\begin{aligned}
B_1 &= \text{Re}\int_{\mathbb{R}}\bar{f}(\gamma)\partial_x f(\gamma)\int_{B^c(0,1)}(\Xi_1(\gamma,\eta)+\Xi_2(\gamma,\eta))d\eta dx\\
&\leq \|f\|_{L^2(\mathbb{B})}\|\partial_x f\|_{L^2(\mathbb{B})}\left\|\text{P.V.}\int_{B^c(0,1)}(\Xi_1(\gamma,\eta)+\Xi_2(\gamma,\eta))d\eta\right\|_{L^\infty(\mathbb{B})}.
\end{aligned}$$

Changing variables $\eta=-\eta$ we have

$$\begin{aligned}
&\text{P.V.}\int_{B^c(0,1)}\Xi_1(\gamma,\eta)d\eta\\
&\qquad=\int_1^\infty\frac{\sinh(\eta)(\cos((f(\gamma)-f(\gamma+\eta)))-\cos((f(\gamma)-f(\gamma-\eta))))d\eta}{(\cosh(\eta)-\cos((f(\gamma)-f(\gamma-\eta))))(\cosh(\eta)-\cos((f(\gamma)-f(\gamma+\eta))))},
\end{aligned}$$

and, using the definition of $d^-[f](\gamma,\eta)$ and the formulas for the difference of complex cosines, we conclude

$$\left\|\text{P.V.}\int_{B^c(0,1)}\Xi_1(\gamma,\eta)d\eta\right\|_{L^\infty(\mathbb{B})}\leq c(l)\|d^-[f]\|^2_{L^\infty(\mathbb{B})}\cosh(2\|f\|_{L^\infty(\mathbb{B})})\int_1^\infty\frac{\sinh(\eta)}{\sinh^4(\eta/3)d\eta}.$$

Following the same steps as before we obtain

$$\left\|\text{P.V.}\int_{B^c(0,1)}\Xi_2(\gamma,\eta)d\eta\right\|_{L^\infty(\mathbb{B})}\leq c(l)\|d^+[f]\|^2_{L^\infty(\mathbb{B})}\cosh(2\|f\|_{L^\infty(\mathbb{B})})\int_1^\infty\frac{\sinh(\eta)}{\cosh^4(\eta/3)d\eta}.$$

Finally, we conclude the estimate for B_1,

$$\begin{aligned}B_1&\leq c(l)\|f\|_{L^2(\mathbb{B})}\|\partial_x f\|_{L^2(\mathbb{B})}(\|d^-[f]\|^2_{L^\infty(\mathbb{B})}+\|d^+[f]\|^2_{L^\infty(\mathbb{B})})\cosh(2\|f\|_{L^\infty(\mathbb{B})})\\&\leq c(l)E_{\mathbb{B}}[f]^3\cosh(cE_{\mathbb{B}}[f])\leq c(l)\cosh^2(c(l)E_{\mathbb{B}}[f]).\end{aligned}$$

The term B_2 can be bounded as follows:

$$\begin{aligned}B_2&=\text{Re}\int_{\mathbb{R}}\bar{f}(\gamma)\int_{B^c(0,1)}\partial_\eta f(\gamma-\eta)(\Xi_1(\gamma,\eta)-\Xi_2(\gamma,\eta))d\eta dx\\&=\text{Re}\int_{\mathbb{R}}\bar{f}(\gamma)\Big[f(\gamma-\eta)(\Xi_1(\gamma,\eta)-\Xi_2(\gamma,\eta))\Big|_{\partial B^c(0,1)}\\&\quad-\int_{B^c(0,1)}f(\gamma-\eta)\partial_\eta(\Xi_1(\gamma,\eta)-\Xi_2(\gamma,\eta))d\eta\Big]dx\\&=\text{Re}\int_{\mathbb{R}}\bar{f}(\gamma)\Big[-f(\gamma-1)(c_1d^-[f](\gamma,1)-c_2d^+[f](\gamma,1))\\&\quad+f(\gamma+1)(c_3d^-[f](\gamma,-1)-c_4d^+[f](\gamma,-1))\\&\quad-\int_{B^c(0,1)}f(\gamma-\eta)\partial_\eta(\Xi_1(\gamma,\eta)-\Xi_2(\gamma,\eta))d\eta\Big]dx\\&\leq c(l)\|f\|^2_{L^2(\mathbb{B})}(\|d^-[f]\|_{L^\infty(\mathbb{B})}+\|d^+[f]\|_{L^\infty(\mathbb{B})}\\&\quad+(\|d^-[f]\|^2_{L^\infty(\mathbb{B})}+\|d^+[f]\|^2_{L^\infty(\mathbb{B})})\cosh(2\|f\|_{L^\infty(\mathbb{B})})(1+\|\partial_x f\|_{L^\infty(\mathbb{B})}))\\&\leq c(l)\cosh^2(c(l)E_{\mathbb{B}}[f]).\end{aligned}$$

Putting all the estimates together we obtain

$$\frac{d}{dt}\|f\|^2_{L^2(\mathbb{B})}\leq\exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

□

The evolution of the L^2 norm of the third derivative is given by

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^3 f\|_{L^2(\mathbb{B})}=\text{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f}(\gamma)(\partial_t\partial_x^3 f(\gamma)\pm ik\partial_x^4 f(\gamma))dx=I_1^{\mathbb{C}}+3I_2^{\mathbb{C}}+3I_3^{\mathbb{C}}+I_4^{\mathbb{C}}+I_5^{\mathbb{C}},$$

with

$$I_1^{\mathbb{C}}=\text{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}}\partial_x(\partial_x^3 f(\gamma)-\partial_x^3 f(\gamma-\eta))\Xi_1+\partial_x(\partial_x^3 f(\gamma)+\partial_x^3 f(\gamma-\eta))\Xi_2 d\eta dx,$$

$$I_2^{\mathbb{C}}=\text{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}}(\partial_x^3 f(\gamma)-\partial_x^3 f(\gamma-\eta))\partial_x\Xi_1+(\partial_x^3 f(\gamma)+\partial_x^3 f(\gamma-\eta))\partial_x\Xi_2 d\eta dx,$$

$$I_3^{\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} (\partial_x^2 f(\gamma) - \partial_x^2 f(\gamma-\eta))\partial_x^2\Xi_1 + (\partial_x^2 f(\gamma) + \partial_x^2 f(\gamma-\eta))\partial_x^2\Xi_2 d\eta dx,$$

$$I_4^{\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} (\partial_x f(\gamma) - \partial_x f(\gamma-\eta))\partial_x^3\Xi_1 + (\partial_x f(\gamma) + \partial_x f(\gamma-\eta))\partial_x^3\Xi_2 d\eta dx,$$

$$I_5^{\mathbb{C}} = \pm k\text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma) i\partial_x^4 f(\gamma)dx.$$

6.5 Bound for $I_1^{\mathbb{C}}$

In this section we bound the singular term, those with the highest derivatives. We use the notation defined in (5.4)-(5.6). We also write

$$m(t) = \min_{\gamma}\text{Re}\frac{1}{1+(\partial_x f(\gamma))^2}. \tag{6.6}$$

Lemma 6.3. *Let f be a $H^3(\mathbb{B})$ solution of (6.5) in the Rayleigh-Taylor stable case and let R be a constant such that $\|D[f]\|_{L^\infty(\mathbb{B})} < \infty$. Then, there exist $0 < K = K(l)$, a universal constant, and $0 < \epsilon$ an arbitrary constant, such that*

$$I_1^{\mathbb{C}} \leq \|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2(\mathbb{B}}\left(\frac{\epsilon}{2} + K\left\|Im\left(P.V.\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\right\|_{L^\infty(\mathbb{B})} - 2\pi m(t)\right) + \exp(c(l,\epsilon)(E_{\mathbb{B}}[f]+1)). \tag{6.7}$$

Proof. We split this integral as follows

$$I_1^{\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f(\gamma)}\partial_x^4 f(\gamma)\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta dx - \text{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f(\gamma)}\text{P.V.}\int_{\mathbb{R}}\partial_x^4 f(\gamma-\eta)(\Xi_1-\Xi_2)d\eta dx = J_1^{1,\mathbb{C}} + J_2^{1,\mathbb{C}}.$$

The first integral, after some integration by parts in the x variable, decomposes in the following way

$$\begin{aligned} J_1^{1,\mathbb{C}} &= -\int_{\mathbb{R}}\frac{1}{2}|\partial_x^3 f(\gamma)|^2\text{Re}\left(\text{P.V.}\int_{\mathbb{R}}\partial_x\Xi_1+\partial_x\Xi_2 d\eta\right)dx \\ &\quad +2\int_{\mathbb{R}}\text{Im}\partial_x^3 f(\gamma)\text{Re}\partial_x^4 f(\gamma)\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)dx \\ &\quad +\int_{\mathbb{R}}\text{Re}\partial_x^3 f(\gamma)\text{Im}\partial_x^3 f(\gamma)\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\partial_x\Xi_1+\partial_x\Xi_2 d\eta\right)dx \\ &= K_1^{1,\mathbb{C}} + K_2^{1,\mathbb{C}} + K_3^{1,\mathbb{C}}. \end{aligned}$$

We have

$$K_1^{1,\mathbb{C}} = L_1^{1,\mathbb{C}} + L_2^{1,\mathbb{C}},$$

where

$$L_1^{1,\mathbb{C}} = -\int_{\mathbb{R}}\frac{1}{2}|\partial_x^3 f(\gamma)|^2\text{Re}\left(\text{P.V.}\int_{B(0,1)}\partial_x\Xi_1+\partial_x\Xi_2 d\eta\right)dx = M_1^{1,\mathbb{C}} + M_2^{1,\mathbb{C}},$$

and

$$L_2^{1,\mathbb{C}} = -\int_{\mathbb{R}} \frac{1}{2}|\partial_x^3 f(\gamma)|^2 \text{Re}\left(\text{P.V.}\int_{B^c(0,1)} \partial_x \Xi_1 + \partial_x \Xi_2 d\eta\right) dx.$$

The outer term can be easily bounded using the definition of $d^{\pm}[f]$ in (6.4). Thus,

$$L_2^{1,\mathbb{C}} \leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\|\partial_x f\|_{L^\infty(\mathbb{B})}\cosh(2\|f\|_{L^\infty(\mathbb{B})})(\|d^-[f]\|^2_{L^\infty(\mathbb{B})} + \|d^+[f]\|^2_{L^\infty(\mathbb{B})}) \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

For the term $L_1^{1,\mathbb{C}}$ we have

$$L_1^{1,\mathbb{C}} = -\int_{\mathbb{R}} \frac{1}{2}|\partial_x^3 f(\gamma)|^2 \text{Re}\left(\text{P.V.}\int_{B(0,1)} \partial_x \Xi_1 + \partial_x \Xi_2 d\eta\right) dx = M_1^{1,\mathbb{C}} + M_2^{1,\mathbb{C}},$$

with

$$\begin{aligned} M_2^{1,\mathbb{C}} = -\int_{\mathbb{R}} \frac{1}{2}|\partial_x^3 f(\gamma)|^2 \text{Re}\left(\text{P.V.}\int_{B(0,1)} \partial_x \Xi_2 d\eta\right) dx \\ \leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\|d^+[f]\|^2_{L^\infty(\mathbb{B})}\|\partial_x f\|_{L^\infty(\mathbb{B})}\cosh(2\|f\|_{L^\infty(\mathbb{B})}) \\ \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)), \end{aligned}$$

and

$$M_1^{1,\mathbb{C}} = -\int_{\mathbb{R}} \frac{1}{2}|\partial_x^3 f(\gamma)|^2 \text{Re}\left(\text{P.V.}\int_{B(0,1)} \frac{(d^-[f](\gamma,\eta))^2 \sinh(\eta)\sin^-(\gamma)(\partial_x f^-(\gamma))}{\sinh^4(\eta/3)} d\eta\right) dx.$$

We write

$$\begin{aligned} &\frac{(d^-[f](\gamma,\eta))^2 \sinh(\eta)\sin^-(\partial_x f(\gamma) - \partial_x f(\gamma-\eta))}{\sinh^4(\eta/3)} \\ = &\frac{(d^-[f](\gamma,\eta))^2 \sinh(\eta)(\sin^- - (f(\gamma)-f(\gamma-\eta)))(\partial_x f(\gamma) - \partial_x f(\gamma-\eta))}{\sinh^4(\eta/3)} \\ + &\frac{(d^-[f](\gamma,\eta))^2 \sinh(\eta)(f(\gamma)-f(\gamma-\eta)-\partial_x f(\gamma)\eta)(\partial_x f(\gamma) - \partial_x f(\gamma-\eta))}{\sinh^4(\eta/3)} \\ + &\frac{(d^-[f](\gamma,\eta))^2 \sinh(\eta)\partial_x f(\gamma)\eta(\partial_x f(\gamma) - \partial_x f(\gamma-\eta) - \partial_x^2 f(\gamma)\eta)}{\sinh^4(\eta/3)} \\ + &\frac{((d^-[f](\gamma,\eta))^2 - (d^-[f](\gamma,0))^2)\sinh(\eta)\partial_x f(\gamma)\eta^2\partial_x^2 f(\gamma)}{\sinh^4(\eta/3)} \\ + &\frac{(d^-[f](\gamma,0))^2 \sinh(\eta)\partial_x f(\gamma)\eta^2\partial_x^2 f(\gamma)}{\sinh^4(\eta/3)}, \end{aligned}$$

and so we have $M_1^{1,\mathbb{C}} = N_1^{1,\mathbb{C}} + N_2^{1,\mathbb{C}} + N_3^{1,\mathbb{C}} + N_4^{1,\mathbb{C}} + N_5^{1,\mathbb{C}}$. Using that

$$|\sin^-(\gamma) - (f(\gamma)-f(\gamma-\eta))| \leq c(l)\cosh(2\|f\|_{L^\infty(\mathbb{B})})|\eta|^3\|\partial_x f\|^3_{L^\infty(\mathbb{B})},$$

we obtain

$$N_1^{1,\mathbb{C}} \leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\|\partial_x f\|^4_{L^\infty(\mathbb{B})}\|d^-[f]\|^2_{L^\infty(\mathbb{B})}\cosh(c(l)\|f\|_{L^\infty(\mathbb{B})}).$$

For the second term we have

$$N_2^{1,\mathbb{C}} \leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\|\partial_x^2 f\|^2_{L^\infty(\mathbb{B})}\|d^-[f]\|^2_{L^\infty(\mathbb{B})}.$$

Using the Hölder regularity we bound the third term as follows

$$N_3^{1,\mathbb{C}} \leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\|\partial_x f\|^2_{L^\infty(\mathbb{B})}\|f\|_{C^{2,\delta}}\|d^-[f]\|^2_{L^\infty(\mathbb{B})}.$$

We remark that $N_5^{1,\mathbb{C}} = 0$, so it only remains the term $N_4^{1,\mathbb{C}}$. Using the definition of $d^-[f]$ and the fact that

$$\begin{aligned}\left|\partial_x f(\gamma) - \frac{\sin((f(\gamma)-f(\gamma-\eta))/2)}{\sinh(\eta/2)}\right| &\leq \left|\partial_x f(\gamma) - \frac{f(\gamma)-f(\gamma-\eta)}{\eta}\right| \\ &+ \left|\frac{f(\gamma)-f(\gamma-\eta)}{\eta} - \frac{f(\gamma)-f(\gamma-\eta)}{\sinh(\eta)}\right| + \left|\frac{\sin^- -(f(\gamma)-f(\gamma-\eta))}{\eta}\right| \\ &\leq (\|\partial_x^2 f\|_{L^\infty(\mathbb{B})} + \|\partial_x f\|_{L^\infty(\mathbb{B})})|\eta| + |\eta|^3\|\partial_x f\|_{L^\infty(\mathbb{B})^3}\cosh(2\|f\|_{L^\infty(\mathbb{B})}),\end{aligned}$$

and we conclude

$$M_1^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Putting all the estimates together we conclude

$$K_1^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Now,

$$\begin{aligned}K_2^{1,\mathbb{C}} &= -2\int_{\mathbb{R}}[\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\mathrm{Im}\partial_x^3 f(\gamma)]\Lambda H\mathrm{Re}\partial_x^3 f(\gamma)dx \\ &= -2\int_{\mathbb{R}}\Lambda^{1/2}[\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\mathrm{Im}\partial_x^3 f(\gamma)]\Lambda^{1/2}H\mathrm{Re}\partial_x^3 f(\gamma)dx \\ &\leq 2\left\|\Lambda^{1/2}\left[\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\mathrm{Im}\partial_x^3 f(\gamma)\right]\right\|_{L^2(\mathbb{B})}\|\Lambda^{1/2}\mathrm{Re}\partial_x^3 f(\gamma)\|_{L^2(\mathbb{B})},\end{aligned}$$

and using Lemma 6.1 we obtain

$$\begin{aligned}K_2^{1,\mathbb{C}} \leq K\|\Lambda^{1/2}\partial_x^3 f(\gamma)\|_{L^2(\mathbb{B})}&\left(\left\|\partial_x \mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}(\Xi_1+\Xi_2)\,d\eta\right)\right\|_{L^\infty(\mathbb{B})}\|\partial_x^3 f\|_{L^2(\mathbb{B})}\right. \\ &\left.+\left\|\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}(\Xi_1+\Xi_2)\,d\eta\right)\right\|_{L^\infty(\mathbb{B})}\|\Lambda^{1/2}\partial_x^3 f(\gamma)\|_{L^2(\mathbb{B})}\right).\end{aligned}$$

We consider $\epsilon > 0$ such that, using Young's inequality, we obtain

$$\begin{aligned}K_2^{1,\mathbb{C}} \leq \frac{K^2}{2\epsilon}&\left\|\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}\partial_x(\Xi_1+\Xi_2)\,d\eta\right)\right\|^2_{L^\infty(\mathbb{B})}\|\partial_x^3 f\|^2_{L^2(\mathbb{B})} \\ &+\left(\frac{\epsilon}{2}+K\left\|\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}(\Xi_1+\Xi_2)\,d\eta\right)\right\|_{L^\infty(\mathbb{B})}\right)\|\Lambda^{1/2}\partial_x^3 f(\gamma)\|^2_{L^2(\mathbb{B})}.\end{aligned}$$

We get

$$\begin{aligned}\left\|\mathrm{Im}\left(\mathrm{P.V.}\int_{\mathbb{R}}\partial_x(\Xi_1+\Xi_2)\,d\eta\right)\right\|_{L^\infty(\mathbb{B})} &\leq \left\|\mathrm{P.V.}\int_{B(0,1)}\partial_x(\Xi_1+\Xi_2)\,d\eta\right\|_{L^\infty(\mathbb{B})} \\ &+\left\|\mathrm{P.V.}\int_{B^c(0,1)}\partial_x(\Xi_1+\Xi_2)\,d\eta\right\|_{L^\infty(\mathbb{B})},\end{aligned}$$

and it can be bounded as in $L_1^{1,\mathbb{C}}$ and $L_2^{1,\mathbb{C}}$. We conclude

$$K_2^{1,\mathbb{C}} \leq \|\Lambda^{1/2}\partial_x^3 f(\gamma)\|_{L^2(\mathbb{B})}^2 \left(\frac{\epsilon}{2} + K \left\|\mathrm{Im}\left(\text{P.V.}\int_{\mathbb{R}} (\Xi_1+\Xi_2)\, d\eta\right)\right\|_{L^\infty(\mathbb{B})}\right) + \exp(c(l,\epsilon)(E_{\mathbb{B}}[f]+1)).$$

The third term can be easily bounded in the same way as $L_1^{1,\mathbb{C}}$ and $L_2^{1,\mathbb{C}}$ and using Cauchy–Schwarz inequality:

$$K_3^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

To finish with $I_1^{\mathbb{C}}$ we only need to bound $J_2^{1,\mathbb{C}} = K_4^{1,\mathbb{C}} + K_5^{1,\mathbb{C}}$, where

$$K_4^{1,\mathbb{C}} = -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^4 f(\gamma-\eta)\Xi_1 d\eta dx,$$

$$K_5^{1,\mathbb{C}} = \mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^4 f(\gamma-\eta)\Xi_2 d\eta dx.$$

The term corresponding Ξ_2 can be easily bounded integrating by parts in η and using the definition of $d^+[f]$. Thus,

$$K_5^{1,\mathbb{C}} = \mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta)\partial_\eta\Xi_2 d\eta dx \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

The last term remaining gives us the needed dissipation in order to obtain *a priori* energy estimates. We have

$$\begin{aligned}
K_4^{1,\mathbb{C}} &= -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^4 f(\gamma-\eta) d^-[f](\gamma,\eta)\frac{\sinh(\eta)-\eta}{\sinh^2(\eta/3)} d\eta dx \\
&\quad -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^4 f(\gamma-\eta) d^-[f](\gamma,\eta)\eta\left(\frac{1}{\sinh^2(\eta/3)} - \frac{1}{(\eta/3)^2}\right) d\eta dx \\
&\quad -9\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^4 f(\gamma-\eta)\frac{d^-[f](\gamma,\eta)-d^-[f](\gamma,0)}{\eta} d\eta dx \\
&\quad -9\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma) d^-[f](\gamma,0)\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x^4 f(\gamma-\eta)}{\eta} d\eta dx \\
&= L_3^{1,\mathbb{C}} + L_4^{1,\mathbb{C}} + L_5^{1,\mathbb{C}} + L_6^{1,\mathbb{C}}.
\end{aligned}$$

In the first three terms, $L_3^{1,\mathbb{C}}, L_4^{1,\mathbb{C}}$ and $L_5^{1,\mathbb{C}}$, we can integrate by parts in η because we have the cancelation needed at $\eta=0$. For example:

$$L_3^{1,\mathbb{C}} = -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta) d^-[f](\gamma,\eta)\partial_\eta\left(\frac{\sinh(\eta)-\eta}{\sinh^2(\eta/3)}\right) d\eta dx,$$

and using 4.7 we can conclude $L_3^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$. The second term can be bounded using $D[f]$:

$$L_4^{1,\mathbb{C}} = -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta)\partial_\eta\left((d^-[f](\gamma,\eta)\eta\left(\frac{1}{\sinh^2(\eta/3)} - \frac{1}{(\eta/3)^2}\right)\right) d\eta dx = M_3^{1,\mathbb{C}} + M_4^{1,\mathbb{C}},$$

where

$$M_3^{1,\mathbb{C}} = -\mathrm{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{B(0,R)} \partial_x^3 f(\gamma-\eta)\partial_\eta\left((d^-[f](\gamma,\eta)\eta\left(\frac{1}{\sinh^2(\eta/3)} - \frac{1}{(\eta/3)^2}\right)\right) d\eta dx,$$

$$M_4^{1,\mathbb{C}} = -\text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{B^c(0,R)} \partial_x^3 f(\gamma-\eta)\partial_\eta\left((d^-[f](\gamma,\eta)\eta\left(\frac{1}{\sinh^2(\eta/3)}-\frac{1}{(\eta/3)^2}\right)\right)d\eta dx.$$

The inner term $M_3^{1,\mathbb{C}}$ can be bounded using the definition of $d^-[f]$. For the outer term $M_4^{1,\mathbb{C}}$, we use that in this region (and since $\|D[f]\|_{L^\infty(\mathbb{B})} < \infty$), we have

$$\left|\frac{1}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}\right| \leq \frac{4(1+2\cosh(2\|f\|_{L^\infty(\mathbb{B})}))}{\cosh(\eta)},$$

and so we conclude $L_4^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$. For the third term we can do in a similar way obtaining $L_5^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$. We remark that

$$d^-[f](\gamma,0) = \frac{2}{9}\frac{1}{1+(\partial_x f(\gamma))^2},$$

and then

$$\begin{aligned}
L_6^{1,\mathbb{C}} &= -2\pi\text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\Lambda\partial_x^3 f(\gamma)\frac{1}{1+(\partial_x f(\gamma))^2}\\
&= -2\pi\int_{\mathbb{R}}\text{Re}\left(\frac{1}{1+(\partial_x f(\gamma))^2}\right)\left(\text{Re}\partial_x^3 f(\gamma)\Lambda\text{Re}\partial_x^3 f(\gamma)+\text{Im}\partial_x^3 f(\gamma)\Lambda\text{Im}\partial_x^3 f(\gamma)\right)\\
&\quad - 2\pi\int_{\mathbb{R}}\text{Im}\left(\frac{1}{1+(\partial_x f(\gamma))^2}\right)\left(-\text{Re}\partial_x^3 f(\gamma)\Lambda\text{Im}\partial_x^3 f(\gamma)+\text{Im}\partial_x^3 f(\gamma)\Lambda\text{Re}\partial_x^3 f(\gamma)\right)\\
&= M_5^{1,\mathbb{C}} + M_6^{1,\mathbb{C}}.
\end{aligned}$$

In $M_6^{1,\mathbb{C}}$, we can find a commutator. Then, using Lemma 6.1 and the Sobolev embedding, we obtain

$$\begin{aligned}
M_6^{1,\mathbb{C}} &\leq c(l)\|\partial_x^3 f\|_{L^2(\mathbb{B})}\left\|\left[\Lambda,\text{Im}\left(\frac{1}{1+(\partial_x f(\gamma))^2}\right)\right]\text{Re}\partial_x^3 f(\gamma)\right\|_{L^2(\mathbb{B})}\\
&\leq c(l)\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}\left\|\text{Im}\left(\frac{1}{1+(\partial_x f(\gamma))^2}\right)\right\|_{C^{1,\delta}(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).
\end{aligned}$$

Now we write

$$\begin{aligned}
m(t) = \min_\gamma \text{Re}\frac{1}{1+(\partial_x f(\gamma))^2} &= \text{Re}\frac{1}{1+(\partial_x f(\gamma_t))^2}\\
&= \frac{1+(\text{Re}\partial_x f(\gamma_t))^2-(\text{Im}\partial_x f(\gamma_t))^2}{(1+(\text{Re}\partial_x f(\gamma_t))^2-(\text{Im}\partial_x f(\gamma_t))^2)^2+4(\text{Re}\partial_x f(\gamma_t))^2(\text{Im}\partial_x f(\gamma_t))^2}.
\end{aligned}$$

We remark that for classical solutions we have $0 < m(t) < \infty$ for t as long as the solution exists. Thus, we have $M_5^{1,\mathbb{C}} = N_5^{1,\mathbb{C}} + N_6^{1,\mathbb{C}}$, where

$$N_5^{1,\mathbb{C}} = -2\pi\int_{\mathbb{R}}\left(\text{Re}\left(\frac{1}{1+(\partial_x f(\gamma))^2}\right)-m(t)\right)\left(\text{Re}\partial_x^3 f(\gamma)\Lambda\text{Re}\partial_x^3 f(\gamma)+\text{Im}\partial_x^3 f(\gamma)\Lambda\text{Im}\partial_x^3 f(\gamma)\right)dx$$

$$N_6^{1,\mathbb{C}} = -m(t)2\pi\|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2(\mathbb{B})}.$$

It only remains to bound $N_5^{1,\mathbb{C}}$. This can be done using the pointwise inequality $F\Lambda F \geq \Lambda(F^2)$ (see [18]) if $0 < m(t) < \infty$ (as it is at least for a short time). Then,

$$N_5^{1,\mathbb{C}} \leq c(l)\left\|H\partial_x\text{Re}\frac{1}{1+(\partial_x f(\gamma))^2}\right\|_{L^\infty(\mathbb{B})}\|\partial_x^3 f\|^2_{L^2(\mathbb{B})}.$$

Moreover, using the Sobolev embedding and the boundedness of the Hilbert Transform in Hölder spaces, we conclude $N_5^{1,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$.

Finally, if we put all the estimates together we obtain

$$I_1^{\mathbb{C}} \leq \|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2(\mathbb{B}} \left(\frac{\epsilon}{2} + K \left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}} \Xi_1 + \Xi_2\right)\right\|_{L^\infty(\mathbb{B})} - 2\pi m(t)\right) + \exp(c(l,\epsilon)(E_{\mathbb{B}}[f]+1)).$$

□

6.6 Bound for the lower order terms

In this section we bound $I_2^{\mathbb{C}}$, $I_3^{\mathbb{C}}$, $I_4^{\mathbb{C}}$ and $I_5^{\mathbb{C}}$.

Lemma 6.4. *Given a classical solution of (6.5), we have*

$$I_2^{\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Proof. We split the integral as follows

$$I_2^{\mathbb{C}} = J_1^{2,\mathbb{C}} + J_2^{2,\mathbb{C}},$$

where

$$J_1^{2,\mathbb{C}} = \text{Re}\int_{\mathbb{R}} |\partial_x^3 f|^2(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x\Xi_1 + \partial_x\Xi_2 d\eta dx,$$

and

$$J_2^{2,\mathbb{C}} = -\text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta)(\partial_x\Xi_1 - \partial_x\Xi_2)d\eta dx = K_1^{2,\mathbb{C}} + K_2^{2,\mathbb{C}}.$$

The term $J_1^{2,\mathbb{C}}$ is the same as $K_1^{1,\mathbb{C}}$ in $J_1^{1,\mathbb{C}}$ and so $J_1^{2,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$. The term corresponding to the second kernel can be bounded by means Hölder inequality

$$\begin{aligned} K_2^{2,\mathbb{C}} &= \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\text{P.V.}\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta)\partial_x\Xi_2 d\eta dx \\ &\leq c(l)\cosh(2\|f\|_{L^\infty(\mathbb{B})})\|\partial_x f\|_{L^\infty(\mathbb{B})}\|d^+[f]\|^2_{L^\infty(\mathbb{B})}\int_{\mathbb{R}}|f(\gamma)|\int_{\mathbb{R}}|f(\gamma-\eta)|\frac{\sinh(|\eta|)}{\cosh^4(\eta/3)}d\eta dx \\ &\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)). \end{aligned}$$

The other term gives us a Hilbert transform. We follow the same idea as in $M_1^{1,\mathbb{C}}$:

$$K_1^{2,\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\int_{\mathbb{R}} \partial_x^3 f(\gamma-\eta)\frac{\sinh(\eta)\sin(f(\gamma)-f(\gamma-\eta))(\partial_x f(\gamma)-\partial_x f(\gamma-\eta))}{(\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta)))^2} = L_1^{2,\mathbb{C}} + L_2^{2,\mathbb{C}},$$

where

$$L_1^{2,\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\int_{B(0,1)} \partial_x^3 f(\gamma-\eta)\frac{\sinh(\eta)\sin(f(\gamma)-f(\gamma-\eta))(\partial_x f(\gamma)-\partial_x f(\gamma-\eta))}{(\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta)))^2},$$

$$L_2^{2,\mathbb{C}} = \text{Re}\int_{\mathbb{R}} \overline{\partial_x^3 f}(\gamma)\int_{B^c(0,1)} \partial_x^3 f(\gamma-\eta)\frac{\sinh(\eta)\sin(f(\gamma)-f(\gamma-\eta))(\partial_x f(\gamma)-\partial_x f(\gamma-\eta))}{(\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta)))^2}.$$

The *'out'* part can be easily bounded

$$L_2^{2,\mathbb{C}} \leq c(l)\|d^-[f]\|^2_{L^\infty(\mathbb{B})}\|\partial_x f\|_{L^\infty(\mathbb{B})}\cosh(2\|f\|_{L^\infty(\mathbb{B})})\|f\|^2_{L^2(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

We compute

$$\begin{aligned}
\frac{(d^-[f](\gamma,\eta))^2\sinh(\eta)\sin^-\partial_x f^-(\gamma)}{\sinh^4(\eta/3)} &= \frac{(d^-[f](\gamma,\eta))^2(\sinh(\eta)-\eta)\sin^-\partial_x f^-(\gamma)}{\sinh^4(\eta/3)}\\
&+\frac{(d^-[f](\gamma,\eta))^2\eta(\sin^- - f^-(\gamma)\partial_x f^-(\gamma)}{\sinh^4(\eta/3)} + \frac{(d^-[f](\gamma,\eta))^2\eta(f^-(\gamma)-\eta\partial_x f(\gamma))\partial_x f^-(\gamma)}{\sinh^4(\eta/3)}\\
&+(d^-[f](\gamma,0))^2\partial_x f(\gamma)\left(\frac{\partial_x^2 f(\gamma)\eta^3}{\sinh^4(\eta/3)}-\frac{\partial_x^2 f(\gamma)\eta^3}{(\eta/3)^4}\right)+(d^-[f](\gamma,0))^2\partial_x f(\gamma)\partial_x^2 f(\gamma)\frac{1}{\eta}\\
&+\frac{(d^-[f](\gamma,\eta))^2\eta^2\partial_x f(\gamma)(\partial_x f^-(\gamma)-\eta\partial_x^2 f(\gamma))}{\sinh^4(\eta/3)}\\
&+\frac{((d^-[f](\gamma,\eta))^2-(d^-[f](\gamma,0))^2)\eta^3\partial_x f(\gamma)\partial_x^2 f(\gamma)}{\sinh^4(\eta/3)}.
\end{aligned}$$

The last term is the Hilbert transform, and the other terms are not singular, so we have

$$\begin{aligned}
L_1^{2,\mathbb{C}} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)) &- \mathrm{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f}(\gamma)(d^-[f](\gamma,0))^2\partial_x f(\gamma)\partial_x^2 f(\gamma)\int_{B^c(0,1)}\frac{\partial_x^3 f(\gamma-\eta)}{\eta}d\eta dx\\
&+\mathrm{Re}\int_{\mathbb{R}}\overline{\partial_x^3 f}(\gamma)(d^-[f](\gamma,0))^2\partial_x f(\gamma)\partial_x^2 f(\gamma)H(\partial_x^3 f(\gamma-\eta))dx.
\end{aligned}$$

The term with the Hilbert transform can be easily bounded and the outer term can be bounded by means of a integration by parts in order to ensure the decay at infinity, thus we conclude the result. □

Lemma 6.5. *Given a classical solution of* (6.5)*, we have*

$$\begin{aligned}
I_3^{\mathbb{C}} &\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)),\\
I_4^{\mathbb{C}} &\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).
\end{aligned}$$

Proof. Using

$$\partial_x^2 f(\gamma)-\partial_x^2 f(\gamma-\eta)=\int_0^1\partial_x^3 f(\gamma+(s-1)\eta)\eta ds,$$

the integrals involved in $I_3^{\mathbb{C}}$ are not singular and so the proof is straightforward. In $I_4^{\mathbb{C}}$ the singular terms can be bounded as those of $I_1^{\mathbb{C}}$ in Lemma 6.3. □

Lemma 6.6. *Let f be a $H^3(\mathbb{B})$ solution of* (6.5)*. Then,*

$$I_5^{\mathbb{C}} \leq 2k\|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2(\mathbb{B})}.$$

Proof. Using $-\Lambda H=\partial_x$ and the anti-self adjointness of the Hilbert transform, we compute

$$\begin{aligned}
I_5^{\mathbb{C}} = \pm k\int_{\mathbb{R}}&\mathrm{Im}\partial_x^3 f\mathrm{Re}\partial_x^4 f - \mathrm{Re}\partial_x^3 f\mathrm{Im}\partial_x^4 f dx\\
&= \mp k\int_{\mathbb{R}}\Lambda^{1/2}\partial_x^3\mathrm{Im}f\Lambda^{1/2}H\partial_x^3\mathrm{Re}f - \Lambda^{1/2}\partial_x^3\mathrm{Re}f\Lambda^{1/2}H\partial_x^3\mathrm{Im}f dx\\
&= \mp 2k\int_{\mathbb{R}}\Lambda^{1/2}\partial_x^3\mathrm{Im}f\Lambda^{1/2}H\partial_x^3\mathrm{Re}f dx\\
&\leq 2k\|\Lambda^{1/2}\partial_x^3\mathrm{Im}f\|_{L^2(\mathbb{B})}\|\Lambda^{1/2}\partial_x^3\mathrm{Re}f\|_{L^2(\mathbb{B})},
\end{aligned}$$

and we obtain the result. $\square$

6.7 Bound for $\|D[f]\|_{L^\infty}$ and $\|d^{\pm}[f]\|_{L^\infty(\mathbb{B})}$

The following Lemma will be required in order to study the evolution of $\|D[f]\|_{L^\infty}$ and $\|d^{+}[f]\|_{L^\infty(\mathbb{B})}$.

Lemma 6.7. *Given a classical solution of* (6.5) *we have the following bound*

$$\|\partial_t f\|_{L^\infty(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Proof. In order to prove the bound we split the integrals into the *in* and *out* parts:

$$\int_{B(0,1)} (\partial_x f(\gamma) - \partial_x f(\gamma-\eta))\Xi_1 + (\partial_x f(\gamma) + \partial_x f(\gamma-\eta))\Xi_2 d\eta$$
$$\leq c(l)(\|d^{-}[f]\|_{L^\infty(\mathbb{B})} + \|d^{+}[f]\|_{L^\infty(\mathbb{B})})\|f\|_{C^2(\mathbb{B})}.$$

Now we use the bounds in Lemma 6.2 to bound the term

$$\partial_x f(\gamma)\int_{B^c(0,1)} \Xi_1 + \Xi_2 d\eta \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

The remaining term can be bounded by means of an integration by parts:

$$\int_{B^c(0,1)} \partial_\eta f(\gamma-\eta)(\Xi_1 - \Xi_2) d\eta \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

$\square$

Now we can bound the evolution of $\|D[f]\|_{L^\infty}$ and $\|d^{+}[f]\|_{L^\infty(\mathbb{B})}$.

Lemma 6.8. *Given a classical solution of* (6.5) *such that $\|f\|_{L^\infty} < R$ for a fixed constant $R \geq 2l$, we have that*

$$\begin{aligned} \frac{d}{dt}\|D[f]\|_{L^\infty(\mathbb{B})} &\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)),\\ \frac{d}{dt}\|d^{+}[f]\|_{L^\infty(\mathbb{B})} &\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)). \end{aligned}$$

Proof. Using Lemma 6.7, we have that

$$\frac{d}{dt}D[f] = \frac{4\sinh(2|f|(\gamma))\partial_t f(\gamma)\frac{f(\gamma)}{|f|(\gamma)}}{(\cosh(R) - 2\cosh(2|f|(\gamma)))^2} \leq D[f]\|D[f]\|_{L^\infty(\mathbb{B})}\exp(c(l)(E_{\mathbb{B}}[f]+1)),$$

so, integrating the ODE, we have

$$D[f](t+h) \leq D[f](t)\exp\left(\int_t^{t+h} \|D[f]\|_{L^\infty(\mathbb{B})}(s)\exp(c(l)(E_{\mathbb{B}}[f](s)1))ds\right).$$

As a conclusion we obtain

$$\frac{d}{dt}\|D[f]\|_{L^\infty(\mathbb{B})} \leq \|D[f]\|_{L^\infty(\mathbb{B})}\lim_{h\to 0}\frac{\exp\left(\int_t^{t+h}\|D[f]\|_{L^\infty(\mathbb{B})}(s)\exp(c(l)(E_{\mathbb{B}}[f](s)1))ds\right)-1}{h}$$
$$\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

For $\|d^+[f]\|_{L^\infty(\mathbb{B})}$ we do in the same way and we obtain

$$\frac{d}{dt}\|d^+[f]\|_{L^\infty(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

□

Before estimating the term $d^-[f]$, we need a Lemma concerning the boundedness of $\|\partial_t\partial_x f\|_{L^\infty(\mathbb{B})}$. This lemma will also be useful in the study of the evolution of $m(t)$ defined in (6.6).

Lemma 6.9. *For a classical solution of equation* (6.5)*, we have that*

$$\|\partial_t\partial_x f\|_{L^\infty(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1).$$

Proof. The equation for this quantity is given by

$$\begin{aligned}
\partial_t\partial_x f(x\pm ikt) &= \text{P.V.}\int_{\mathbb{R}}\left[\frac{\left(\partial_x^2 f\left(x\pm ikt\right)-\partial_x^2 f\left(x\pm ikt-\eta\right)\right)\sinh\left(\eta\right)}{\cosh\left(\eta\right)-\cos((f(x\pm ikt)-f(x\pm ikt-\eta)))}\right.\\
&\qquad\left.+\frac{\left(\partial_x^2 f\left(x\pm ikt\right)+\partial_x^2 f\left(x\pm ikt-\eta\right)\sinh\left(\eta\right)\right.}{\cosh\left(\eta\right)+\cos((f(x\pm ikt)+f(x\pm ikt-\eta)))}\right]d\eta\\
&\quad-\text{P.V.}\int_{\mathbb{R}}\left[\frac{\left(\partial_x f\left(x\pm ikt\right)-\partial_x f\left(x\pm ikt-\eta\right)\right)^2\sin^-\sinh\left(\eta\right)}{(\cosh\left(\eta\right)-\cos((f(x\pm ikt)-f(x\pm ikt-\eta))))^2}\right.\\
&\qquad\left.+\frac{\left(\partial_x f\left(x\pm ikt\right)+\partial_x f\left(x\pm ikt-\eta\right)^2\sin^+\sinh\left(\eta\right)\right.}{(\cosh\left(\eta\right)+\cos((f(x\pm ikt)+f(x\pm ikt-\eta))))^2}\right]d\eta\\
&= A_1+A_2+A_3+A_4.
\end{aligned}$$

Splitting in the *'in'* and *'out'* parts the term A_1+A_2 we obtain

$$\begin{aligned}
A_1+A_2 &\leq c(l)\|f\|_{C^{2,\delta}(\mathbb{B})}\left(\|d^-[f]\|_{L^\infty(\mathbb{B})}+\|d^+[f]\|_{L^\infty(\mathbb{B})}\right)\\
&\quad+\|f\|_{C^2(\mathbb{B})}\left\|\int_{B^c(0,1)}\Xi_1+\Xi_2 d\eta\right\|_{L^\infty(\mathbb{B})}+\int_{B^c(0,1)}\partial_\eta\partial_x f(\gamma-\eta)(\Xi_1-\Xi_2)d\eta.
\end{aligned}$$

As in Lemma 6.2 we can bound the last term integrating by parts, obtaining

$$A_1+A_2 \leq \exp(c(l)(E_{\mathbb{B}}[f]+1).$$

For the terms A_3+A_4 we can do in the same way obtaining

$$A_3+A_4 \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

□

Now we can bound the evolution of the term $\|d^-[f]\|_{L^\infty(\mathbb{B})}$:

Lemma 6.10. *For a classical solution of equation* (6.5)*, we have*

$$\frac{d}{dt}\|d^-[f]\|_{L^\infty(\mathbb{B})} \leq \exp(c(l)(E_{\mathbb{B}}[f]+1)).$$

Proof. We have that

$$\frac{d}{dt}\|d^-[f]\|^p_{L^p(\mathbb{B})} = \frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{\sinh^2(\eta/3)}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}\right|^p d\eta dx,$$

and we compute

$$\begin{aligned}\frac{d}{dt}\|d^-[f]\|^p_{L^p(\mathbb{B})} &= \text{Re}\, p\int_{\mathbb{R}}\int_{\mathbb{R}}\sinh^{2p}(\eta/3)\left|\frac{1}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}\right|^{p-2} \\ &\quad\times\overline{\left(\frac{1}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}\right)}\frac{d}{dt}\frac{d\eta dx}{\cosh(\eta)-\cos^-(\gamma)} \\ &= -\text{Re}\, p\int_{\mathbb{R}}\int_{\mathbb{R}}\sinh^{2p}(\eta/3)\left|\frac{1}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}\right|^{p} \\ &\quad\times\frac{\sin^-(\partial_t f(\gamma)-\partial_t f(\gamma-\eta))}{\cosh(\eta)-\cos(f(\gamma)-f(\gamma-\eta))}d\eta dx \\ &\leq p\|d^-[f]\|_{L^\infty(\mathbb{B})}\|d^-[f]\|^p_{L^p(\mathbb{B})}\left\|\frac{\sin^-(\partial_t f(\gamma)-\partial_t f(\gamma-\eta))}{\sinh^2(\eta/3)}\right\|_{L^\infty(\mathbb{B})}.\end{aligned}$$

So we have

$$\frac{d}{dt}\|d^-[f]\|_{L^p(\mathbb{B})}\leq\|d^-[f]\|_{L^\infty(\mathbb{B})}\|d^-[f]\|_{L^p(\mathbb{B})}\left\|\frac{\sin^-(\partial_t f(\gamma)-\partial_t f(\gamma-\eta))}{\sinh^2(\eta/3)}\right\|_{L^\infty(\mathbb{B})}.$$

Integrating, we get

$$\|d^-[f]\|_{L^p(\mathbb{B})}(t+h)\leq\|d^-[f]\|_{L^p(\mathbb{B})}(t)\exp\left(\int_t^{t+h}\|d^-[f]\|_{L^\infty(\mathbb{B})}\left\|\frac{\sin^-\partial_t f^-(\gamma)}{\sinh^2(\eta/3)}\right\|_{L^\infty(\mathbb{B})}ds\right),$$

and, taking the limit $p\to\infty$, we get

$$\begin{aligned}\frac{d}{dt}\|d^-[f]\|_{L^\infty(\mathbb{B})} &\leq \lim_{h\to 0}\frac{\|d^-[f]\|_{L^\infty(\mathbb{B})}(t)\left(\exp\left(\int_t^{t+h}\|d^-[f]\|_{L^\infty(\mathbb{B})}\left\|\frac{\sin^-\partial_t f^-(\gamma)}{\sinh^2(\eta/3)}\right\|_{L^\infty(\mathbb{B})}ds\right)-1\right)}{h} \\ &\leq\|d^-[f]\|^2_{L^\infty(\mathbb{B})}\left\|\frac{\sin^-(\partial_t f(\gamma)-\partial_t f(\gamma-\eta))}{\sinh^2(\eta/3)}\right\|_{L^\infty(\mathbb{B})}.\end{aligned}$$

Using the previous Lemma 6.9, we conclude the result. □

6.8 Proof of Theorem 6.1

With the previous lemmas we can prove the Theorem 6.1:

Fix $R>>2l$ a constant. Then, using the previous Lemmas 6.2-6.6,6.8, and 6.10 we obtain

$$\begin{aligned}\frac{d}{dt}E_{\mathbb{B}}[f] &\leq \exp(c(l,\epsilon)(E_{\mathbb{B}}[f]+1)) \\ &\quad+\|\Lambda^{1/2}\partial_x^3 f\|^2_{L^2(\mathbb{B})}\left(\frac{\epsilon}{2}+K\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2\right)\right\|_{L^\infty(\mathbb{B})}+2k-2\pi m(t)\right),\quad(6.8)\end{aligned}$$

where $\epsilon>0$ is arbitrary, $K=K(l)$ is a fixed constant and k is the width of the strip. Since ϵ is arbitrary, we take $\epsilon=4k$. Thus, at time $t=0$, (6.8) reduces to

$$4k-\frac{2\pi}{1+\|\partial_x f_0\|^2_{L^\infty(\mathbb{R})}}.$$

Now, we fix k small enough and only depending on the initial data and we get

$$\left(4k + K\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2\right)\right\|_{L^\infty(\mathbb{B})} - \frac{2\pi}{1+\|\partial_x f_0\|^2_{L^\infty(\mathbb{R})}}\right)\Big|_{t=0} = 4k - \frac{2\pi}{1+\|\partial_x f_0\|^2_{L^\infty(\mathbb{R})}} < 0. \quad (6.9)$$

Then we need to show that this quantity remains negative (at least) for a short time. To do this, we define the following new energy:

$$\mathfrak{E}_{\mathbb{B}}[f] = E_{\mathbb{B}}[f] + \frac{1}{2\pi m(t) - K\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2\right)\right\|_{L^\infty(\mathbb{B})} - 4k}. \quad (6.10)$$

If $\mathfrak{E}_{\mathbb{B}}[f] < \infty$ then $\frac{d}{dt}E_{\mathbb{B}}[f] \leq \exp(c(l)(E_{\mathbb{B}}[f]+1))$ and we have the correct *'a priori'* estimates. To prove the finiteness of $\mathfrak{E}_{\mathbb{B}}[f]$ we need to bound $m'(t)$ and

$$\frac{d}{dt}\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\right\|_{L^\infty(\mathbb{B})}.$$

Now, if we have a classical solution with $E_{\mathbb{B}}[f] < \infty$, the Sobolev embedding gives us that

$$\text{Re}\frac{1}{1+(\partial_x f(\gamma))^2} \in C^1([0,T]\times\mathbb{B})$$

and then we can apply Rademacher Theorem to the function

$$m(t) = \min_{\gamma}\text{Re}\frac{1}{1+(\partial_x f(\gamma))^2} = \text{Re}\frac{1}{1+(\partial_x f(\gamma_t))^2}.$$

Thus, using Lemma 6.9, we get

$$m'(t) = -\text{Re}\frac{2\partial_x f(\gamma_t)\partial_t\partial_x f(\gamma_t)}{(1+(\partial_x f(\gamma_t))^2)^2} \leq c(l)\|d^-[f]\|^2_{L^\infty(\mathbb{B})}\|\partial_x f\|_{L^\infty(\mathbb{B})}\|\partial_t\partial_x f\|_{L^\infty(\mathbb{B})}$$
$$\leq \exp(c(l)(E_{\mathbb{B}}[f]+1)) \leq \exp(c(l)(\mathfrak{E}_{\mathbb{B}}[f]+1)).$$

Again, if we a classical solution with finite energy $E_{\mathbb{B}}[f]$, using Lemma 6.9 we have that

$$\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right) \in C^1([0,T]\times\mathbb{B}).$$

So, for some point that we write γ_t,

$$\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\right\|_{L^\infty(\mathbb{B})} = \text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)(\gamma_t),$$

and, using Rademacher's Theorem and Lemma 6.9, we have

$$\frac{d}{dt}\left\|\text{Im}\left(\text{P.V.}\int_{\mathbb{R}}\Xi_1+\Xi_2 d\eta\right)\right\|_{L^\infty(\mathbb{B})} \leq \exp(c(l)(\mathfrak{E}_{\mathbb{B}}[f]+1)).$$

We conclude

$$\frac{d}{dt}\mathfrak{E}_{\mathbb{B}}[f] \leq \exp(c(l)(\mathfrak{E}_{\mathbb{B}}[f]+1)).$$

Integrating, we have

$$\mathfrak{E}_{\mathbb{B}}[f](t) \leq -\frac{1}{c(l)}\log(\exp(-c(l)\mathfrak{E}_{\mathbb{B}}[f_0]) - tc(l)\exp(c(l))), \tag{6.11}$$

and then there exists a time $T = T(f_0)$, $T < T^* = \frac{\exp(-c(l)\mathfrak{E}_{\mathbb{B}}[f_0])}{c(l)\exp(c(l))}$ such that $\mathfrak{E}_{\mathbb{B}}[f] \leq c(f_0)$.

Now, for $\epsilon > 0$, we consider the mollifier

$$\mathcal{J}_\epsilon(x) = \frac{1}{\epsilon}\mathcal{J}\left(\frac{x}{\epsilon}\right), \tag{6.12}$$

where $\mathcal{J}$ is the heat kernel, and the regularized problem

$$\begin{cases} \partial_t f^{\epsilon,\delta} = F^{\epsilon,\delta}(f^{\epsilon,\delta}), \\ f^{\epsilon,\delta}(x,0) = \mathcal{J}_\epsilon * f_0(x), \end{cases} \tag{6.13}$$

where

$$\begin{aligned} F^{\epsilon,\delta}(f^{\epsilon,\delta}) = \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^{\epsilon,\delta}(x) - \mathcal{J}_\epsilon * \partial_x f^{\epsilon,\delta}(x-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^{\epsilon,\delta}(x) - \mathcal{J}_\epsilon * f^{\epsilon,\delta}(x-\eta)) + \delta}d\eta\right) \\ + \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^{\epsilon,\delta}(x) + \mathcal{J}_\epsilon * \partial_x f^{\epsilon,\delta}(x-\eta))\sinh(\eta)}{\cosh(\eta) + \cos(\mathcal{J}_\epsilon * f^{\epsilon,\delta}(x) + \mathcal{J}_\epsilon * f^{\epsilon,\delta}(x-\eta)) + \delta}d\eta\right). \end{aligned} \tag{6.14}$$

Step 1: applying Picard's Theorem For these regularized problems we show the existence of classical solutions $f^{\epsilon,\delta} \in C^1([0,T^\epsilon], H^3(\mathbb{R}))$. This fact follows from the proof of Theorem 4.1. Now we pass to the limit in δ, showing the existence of $f^\epsilon \in C^1([0,T^\epsilon], H^3(\mathbb{R}))$ and these functions are solutions of the following problems

$$\begin{cases} \partial_t f^{\epsilon} = F^{\epsilon}(f^{\epsilon}), \\ f^{\epsilon}(x,0) = \mathcal{J}_\epsilon * f_0(x), \end{cases}$$

where

$$\begin{aligned} F^{\epsilon}(f^{\epsilon}) = \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^{\epsilon}(x) - \mathcal{J}_\epsilon * \partial_x f^{\epsilon}(x-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(\mathcal{J}_\epsilon * f^{\epsilon}(x) - \mathcal{J}_\epsilon * f^{\epsilon}(x-\eta))}d\eta\right) \\ + \mathcal{J}_\epsilon * \left(\text{P.V.}\int_{\mathbb{R}} \frac{(\mathcal{J}_\epsilon * \partial_x f^{\epsilon}(x) + \mathcal{J}_\epsilon * \partial_x f^{\epsilon}(x-\eta))\sinh(\eta)}{\cosh(\eta) + \cos(\mathcal{J}_\epsilon * f^{\epsilon}(x) + \mathcal{J}_\epsilon * f^{\epsilon}(x-\eta))}d\eta\right). \end{aligned}$$

Step 2: obtaining uniform bounds for f^ϵ The regularized system must be integrated up to a time T, and this time must be uniform in ϵ. Due to the particular mollifier we have chosen, these solutions, f^ϵ, are analytic (see [37]). Now, we remark that

$$m^\epsilon(t) = \text{Re}\left(\frac{1}{1 + (\mathcal{J}_\epsilon * \partial_x f(\gamma_t))^2}\right) > 0,$$

at least for a small time T^ϵ. For these T^ϵ the same bounds for $\mathfrak{E}_{\mathbb{B}}$ and $E_{\mathbb{B}}$ as in the previous Lemmas are valid. Thus, with the same techniques and using the properties of mollifiers, we obtain *'a priori'* bounds in $H^3(\mathbb{B})$ for these solutions f^ϵ which hold for a time $T_1 = T_1(f_0) > 0$ uniform in ϵ. Moreover, we can bound $m^\epsilon(t)$ up to a time $T_2 = T_2(f_0)$ (uniform in ϵ) using the bound

$$m^\epsilon(t) \geq m^\epsilon(0) - \int_0^t \exp(c(l)(E_{\mathbb{B}}[f](s) + 1))ds \geq m^0(0) - \int_0^t \exp(c(l)(\mathfrak{E}_{\mathbb{B}}[f](s) + 1))ds.$$

Taking $T = \min\{T_1, T_2\}$ we conclude that for $0 < t < T$ the previous *'a priori'* bounds for $\mathfrak{E}_{\mathbb{B}}$ and $E_{\mathbb{B}}$ (see Lemmas 6.2-6.6, 6.8, and 6.10) remains valid for f^ϵ.

Following the quote[1], we give three different proofs of the analyticity of the limit f.

Step 3a: passing to the limit in ϵ Now, fixed $t < T$, the sequence $\{f^\epsilon\}$ is bounded in $H^3(\mathbb{B})$ and then, there is a weak limit $f(t) \in H^3(\mathbb{B})$. Moreover, $E_{\mathbb{B}}[f]$ is uniformly bounded, so we conclude $f \in L^\infty([0,T], H^3(\mathbb{B}))$. As $f \in H^3(\mathbb{B})$ (in particular is in the Hardy space $\mathcal{H}^2$) we have that f is holomorphic on $\mathbb{B}$. Thus, it is real analytic when restricted to the real axis. We remark that due to the uniqueness of classical solution to the equation (2.5) the limit of f^ϵ when restricted to the real axis must coincide with f obtained in Theorem 4.1.

Step 3b: We can prove the analyticity of the limit f with a second argument. This proof uses the Fourier transform (see [52]). We compute

$$\check{f}(\zeta \pm ikt) = \int_{\mathbb{R}} e^{ix\zeta} f(x) e^{\pm ktx} dx = (f e^{\pm kt\cdot})\check{}(\zeta),$$

Then we have $\check{f}(\cdot)$ and $(f(\cdot)e^{\pm kt\cdot})\check{} \in L^2(\mathbb{R})$ and (using Theorem IX.13 of [52]) we conclude the analyticity.

Step 3c: Finally, we can show the analyticity by a third argument. This reasoning is based on complex variable theory (see [17]). We have that f^ϵ are complex analytic functions on $\mathbb{B}$ and using Hadamard's Three Lines Theorem, Sobolev embedding and the uniform bound in $H^3(\mathbb{B})$ we conclude that f^ϵ are locally bounded (Lemma 2.8 in [17]). Using Montel's Theorem we have that $\{f^\epsilon\}$ is normal and then there is a subsequence converging to f which is analytic because of the fact that the set of analytic functions is closed in the set of continuous functions (Theorem 2.1 in [17]).

Finally, we are done with the proof of Theorem 6.1.

6.9 Proof of Theorem 6.2

Now we show that if we consider $\rho^2 < \rho^1$ the problem is ill-posed in Sobolev spaces, *i.e.* singularities appears for arbitrary short energies and times and arbitrary small initial energies. We remark that, if the initial data is analytic, there is local existence of solutions (see Theorem 5.1). The idea is to use the instant analyticity forward in time to conclude the result.

We prove Theorem 6.2 for the case $s = 4$, being analogous the rest of the cases. Take $g_0(x) \in H^3(\mathbb{R})$ but $g_0 \notin H^4(\mathbb{R})$ and $\bar{\rho} = 2$. We consider a fixed constant $R \geq 4$ and $0 < \lambda < 1$. Now we denote $f^\lambda(x,t)$ the solution to the problem (2.5) with initial datum $f^\lambda(x,0) = \lambda g_0(x)$. We know that f^λ exists for a positive time $T(\lambda, g_0)$ and that it is analytic in a complex strip which grows with constant $k(\lambda, g_0)$ (see Theorems 4.1, 6.1 and equation (6.9)). We can take an uniform k^* with respect to λ. Indeed, using the definition of (6.6),

$$k(\lambda, g_0) = \frac{\pi m(0)}{4} = \frac{\pi}{4} \frac{1}{1 + \lambda^2 \|\partial_x g_0\|^2_{L^\infty(\mathbb{R})}} \geq \frac{\pi}{4} \frac{1}{1 + \|\partial_x g_0\|^2_{L^\infty(\mathbb{R})}} = k^*(g_0).$$

[1] Sir Michael Atiyah said

> I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions. They are not just repetitions of each other.

Then, the condition

$$4k^*(g_0) - 2\pi m(0) = 4k^*(g_0) - \frac{2\pi}{1+\lambda^2\|\partial_x g_0\|^2_{L^\infty(\mathbb{R})}} < 0$$

is satisfied. Now, the initial energy is

$$\mathfrak{E}_{\mathbb{B}}[f^\lambda(0)] = \lambda^2\|g_0\|^2_{H^3(\mathbb{R})} + \|d^-[\lambda g_0]\|_{L^\infty(\mathbb{R})} + \|d^+[\lambda g_0]\|_{L^\infty(\mathbb{R})} + \|D[\lambda g_0]\|_{L^\infty(\mathbb{R})} + \frac{1+\lambda^2\|\partial_x g_0\|^2_{L^\infty(\mathbb{R})}}{\pi}.$$

We note that

$$\|d^-[\lambda g_0]\|_{L^\infty(\mathbb{R})} = \frac{\sinh^2(\eta/3)}{2\sinh^2(\eta/2)\left(1+\frac{\sin^2\left(\frac{\lambda}{2}(g_0(x)-g_0(x-\eta))\right)}{\sinh^2(\eta/2)}\right)} \leq \frac{2}{9},$$

$$\|d^+[\lambda g_0]\|_{L^\infty(\mathbb{R})} = \frac{\cosh^2(\eta/3)}{2\cosh^2(\eta/2)\left(1-\frac{\sin^2\left(\frac{\lambda}{2}(g_0(x)+g_0(x-\eta))\right)}{\cosh^2(\eta/2)}\right)} \leq \frac{\cosh^2(\eta/3)}{2\cosh^2(\eta/2)\left(1-\frac{\sin^2\left(\frac{1}{2}(g_0(x)+g_0(x-\eta))\right)}{\cosh^2(\eta/2)}\right)} = \|d^+[g_0]\|_{L^\infty(\mathbb{R})},$$

$$\|D[\lambda g_0]\|_{L^\infty(\mathbb{R})} = \frac{1}{\cosh(R) - 2\cosh(2\lambda\|g_0\|_{L^\infty(\mathbb{R})})} \leq \frac{1}{\cosh(4)-2\cosh(\pi)}.$$

Therefore, we obtain a uniform bound for the energy

$$\mathfrak{E}_{\mathbb{B}}[f^\lambda(0)] \leq c(g_0),$$

with $c(g_0)$ some constant depending on g_0. We define

$$\min_{\lambda\in[0,1]} T(\lambda, g_0) \geq \frac{1}{c\exp(c(g_0))} = \delta^*(g_0) > 0.$$

We consider $0 < \delta < \delta^*(g_0)$. We remark that all $f^\lambda(x,t)$ exists up to time $\delta^*(g_0)$ and, by means of the instant analyticity result (Theorem 6.1), we have

$$\mathfrak{E}_{\mathbb{B}}[f^\lambda(t)] \leq c(g_0), \quad \forall 0 < t < \delta^*(g_0). \tag{6.15}$$

Now, we define $\tilde{f}^{\lambda,\delta}(x,t) = f(x, -t+\delta)$ and we have

$$\|\tilde{f}^{\lambda,\delta}(\delta)\|_{H^4(\mathbb{R})} = \lambda\|g_0\|_{H^4(\mathbb{R})} = \infty.$$

Recall that f^λ is analytic in the common complex strip growing with constant $k^*(g_0)$ for all $0 < \lambda < 1$. Then, applying Cauchy's integral formula with the curve $\Gamma = x + k^*(g_0)\delta e^{i\theta}$ and that Hardy spaces on growing strips are a Banach scale, we get

$$\|\partial_x^4 \tilde{f}^{\lambda,\delta}(0)\|_{L^2(\mathbb{R})} = \|\partial_x^4 f^\lambda(\delta)\|_{L^2(\mathbb{R})} \leq \frac{C}{k^*(g_0)\delta}\|\partial_x^3 f^\lambda\|_{L^2(\mathbb{B}_{\delta^*})}.$$

Using the uniform energy bound (6.15), we have

$$\|\partial_x^3 f^\lambda\|_{L^2(\mathbb{B}_\delta^*)} \leq c(g_0)\lambda\|\partial_x^3 g_0\|_{L^2(\mathbb{R})}.$$

and, therefore

$$\|\partial_x^4 \tilde{f}^{\lambda,\delta}(0)\|_{L^2(\mathbb{R})} \leq \frac{c(g_0)}{\delta}\lambda.$$

Now, given $\epsilon > 0$ take $0 < \lambda = \min\left\{1, \frac{\delta\epsilon}{c(g_0)}\right\}$ to conclude $\|\partial_x^4 \tilde{f}^{\lambda,\delta}(0)\|_{L^2(\mathbb{R})} < \epsilon$. Taking $\tilde{f}^{\lambda,\delta} = \tilde{f}$ we conclude the proof of Theorem 6.2.

Remark 6.1. *Theorem 6.2 and the idea of the proof also applies to the unbounded periodic case.*

Chapter 7

Maximum principles

7.1 Foreword

In this Chapter we prove some qualitative properties of the solutions of

$$\partial_t f(x,t) = \frac{\rho^2-\rho^1}{8l}\text{P.V.}\int_{\mathbb{R}}\Big[\left(\partial_x f\left(x\right)-\partial_x f\left(x-\eta\right)\right)\Xi_1(x,\eta,f) + \left(\partial_x f\left(x\right)+\partial_x f\left(x-\eta\right)\Xi_2(x,\eta,f)\right]d\eta = \frac{\rho^2-\rho^1}{4l}A[f](x). \quad (7.1)$$

In particular, we prove

- a maximum principle for $\|f(t)\|_{L^\infty(\mathbb{R})}$ (Theorem 7.1),
- that the sign propagates (Corollary 7.1),
- a maximum principle for $\|f(t)\|_{L^2(\mathbb{R})}$ (Theorem 7.2),
- a decay estimate for $\|f(t)\|_{L^\infty(\mathbb{R})}$ (Theorem 7.3),
- a maximum principle for $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ for a special class of initial data (Theorem 7.4),
- a uniform bound for $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ for a special class of initial data (Corollary 7.3).

To obtain the maximum principles, the key point is to compare the local and the nonlocal terms who appear in the ODEs for the evolution of the $L^\infty(\mathbb{R})$ norms.

The main difference between Theorem 7.1 and the analogous result in [24] is that we have positive and negative contributions in its ODE. Thus, we have to balance them to obtain our result. We also have some local terms that we have to combine with non-local ones. We note that the local terms disappear with infinite depth.

The $L^2(\mathbb{R})$ maximum principle gives us an energy balance in terms of the total kinetic energy and differs from the logarithmic energy balance known in the infinitely deep case (see [13]).

Theorem 7.3 is intriguing. We observe that in the whole plane case (see [24]) the decay rate is faster. Theorem 7.3 will be examined numerically in Chapter 13. As a corollary of this result we obtain that there are not non-trivial, steady solutions (see Corollary 7.2).

The uniform boundedness for the slope (Theorem 7.4 and Corollary 7.3) is a very interesting result. Here three hypotheses related to the slope, the amplitude and the depth play a role. Roughly speaking, Theorem 7.4 says that if the interface is in the so-called *long wave regime* (small amplitude and large wavelenght) then the slope decays or, even if it grows, the slope is uniformly bounded.

7.2 Maximum principle for $\|f\|_{L^\infty(\mathbb{R})}$

Now we prove a maximum principle using an analysis similar to the analysis in [19] and [24]. As in the previous section, we consider $\frac{\pi}{2}=l$ and $\bar{\rho}=2$.

Theorem 7.1 (Maximum principle for $\|f\|_{L^\infty}$)**.** *Let $f(t)\in H^3_l(\mathbb{R})$ be the unique classical solution of (2.5) in the Rayleigh-Taylor stable case. Then, f satisfies that*

$$\|f(t)\|_{L^\infty(\mathbb{R})}\leq\|f_0\|_{L^\infty(\mathbb{R})}.$$

Proof. Due to the smoothness of f in space and time we have that $\|f(t)\|_{L^\infty(\mathbb{R})}=f(x_t)$ is Lipschitz. Indeed

$$\max_x|f(t_1,x)|=\max_x(|f(t_1,x)-f(t_2,x)+f(t_2,x)|)\leq\max_x(|f(t_1,x)-f(t_2,x)|)+\max_x|f(t_2,x)|,$$

thus

$$\begin{aligned}|\max_x f(t_1,x)-\max_x f(t_2,x)|\leq\max_x(|f(t_1,x)-f(t_2,x)|)&=\max_x(|\partial_t f(s,x)||t_1-t_2|)\\&\leq\max_{s\in(t_2,t_1)}\max_x(|\partial_t f(s,x)|)|t_1-t_2|.\end{aligned}$$

We take $(t_1,t_2)\subset[0,T]$ for a fixed $T>0$. We compute

$$|\max_x f(t_1,x)-\max_x f(t_2,x)|\leq\max_{s\in[0,T]}\max_x(\partial_t f(s,x))(t_1-t_2)=L(t_1-t_2).$$

Using Rademacher's Theorem, we have that $f(x_t)$ is differentiable almost everywhere. Thus, using that x_t is the point of maximum, we get that

$$\begin{aligned}\frac{d}{dt}\|f(t)\|_{L^\infty}&=\lim_{h_j\to0}\frac{\|f(t+h_j)\|_{L^\infty}-\|f(t)\|_{L^\infty}}{h_j}=\lim_{h_j\to0}\frac{f(x_{t+h_j},t+h_j)-f(x_t,t)}{h_j}\\&=\lim_{h_j\to0}\frac{f(x_{t+h_j},t+h_j)\pm f(x_t,t+h_j)-f(x_t,t)}{h_j}\\&\geq\partial_t f(x_t,t).\end{aligned}$$

In the same way we compute

$$\begin{aligned}\frac{d}{dt}\|f(t)\|_{L^\infty}&=\lim_{h_j\to0}\frac{\|f(t-h_j)\|_{L^\infty}-\|f(t)\|_{L^\infty}}{-h_j}=\lim_{h_j\to0}\frac{f(x_{t-h_j},t-h_j)-f(x_t,t)}{-h_j}\\&=\lim_{h_j\to0}\frac{f(x_{t-h_j},t-h_j)\pm f(x_t,t-h_j)-f(x_t,t)}{-h_j}\\&\leq\partial_t f(x_t,t).\end{aligned}$$

We conclude

$$\frac{d}{dt}\|f(t)\|_{L^\infty}=\partial_t f(x_t)=\text{P.V.}\int_{\mathbb{R}}\partial_\eta f(x_t-\eta)(\Xi_1(x_t,\eta,f)-\Xi_2(x_t,\eta,f))d\eta=I_1+I_2. \tag{7.2}$$

Let us introduce the following notation:

$$\theta=\frac{f(x_t)-f(x_t-\eta)}{2},\qquad\bar{\theta}=\frac{f(x_t)+f(x_t-\eta)}{2}. \tag{7.3}$$

Now, since Ξ_1 is defined as (2.6), using the classical and hyperbolic trigonometric formulas for the half-angle, we have

$$\begin{aligned} I_1 &= -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\theta\frac{2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)}{\cosh\left(\eta\right)-\frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}} = -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\theta\frac{2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)}{\cosh\left(\eta\right)-1+\frac{2\tan^2(\theta)}{1+\tan^2(\theta)}} \\ &= -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\theta\frac{(1+\tan^2(\theta))\coth\left(\frac{\eta}{2}\right)}{1+\tan^2\left(\theta\right)\coth^2\left(\frac{\eta}{2}\right)} = -2\text{P.V.}\int_{\mathbb{R}}\frac{\partial_\eta\theta}{\cos^2(\theta)}\frac{\coth\left(\frac{\eta}{2}\right)}{1+\tan^2\left(\theta\right)\coth^2\left(\frac{\eta}{2}\right)} \\ &= -\text{P.V.}\int_{\mathbb{R}}\partial_\eta\tan(\theta)\frac{2\coth\left(\frac{\eta}{2}\right)}{1+(\tan(\theta)\coth(\eta/2))^2}d\eta \\ &= -4\frac{\tan\left(\frac{f(x_t)}{2}\right)}{1+\tan^2\left(\frac{f(x_t)}{2}\right)}+\text{P.V.}\int_{\mathbb{R}}\tan(\theta)\partial_\eta\left(\frac{2\coth\left(\frac{\eta}{2}\right)}{1+(\tan(\theta)\coth(\eta/2))^2}\right)d\eta, \end{aligned}$$

where we integrate by parts. Considering

$$G(x)=\frac{-x}{1+x^2}+\arctan(x),$$

we have

$$I_1=-4\frac{\tan\left(\frac{f(x_t)}{2}\right)}{1+\tan^2\left(\frac{f(x_t)}{2}\right)}-2\int_{\mathbb{R}}\partial_\eta\left[G\left(\frac{\tan(\theta)}{\tanh(\frac{\eta}{2})}\right)\right]d\eta$$
$$-\text{P.V.}\int_{\mathbb{R}}\frac{\tan(\theta)}{\sinh^2(\frac{\eta}{2})}\frac{1}{1+(\tan(\theta)\coth(\frac{\eta}{2}))^2}d\eta.$$

Then, we get

$$I_1=-2f(x_t)-\text{P.V.}\int_{\mathbb{R}}\frac{\tan(\theta)}{\sinh^2(\eta/2)}\frac{1}{1+(\tan(\theta)\coth(\eta/2))^2}d\eta. \tag{7.4}$$

Anagously for the I_2 term, we use the classical and hyperbolic trigonometric formulas. In this case we have to write all in terms of the $\tan(\bar{\theta})$. This is possible because x_t is a maximum point. Since Ξ_2 is defined as (2.7) and using the same function G evaluated in $\tan(\bar{\theta})\tanh(\eta/2)$, we get the following expression for I_2

$$\begin{aligned} I_2 &= -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\bar{\theta}\frac{2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)}{\cosh\left(\eta\right)+\frac{1-\tan^2(\bar{\theta})}{1+\tan^2(\bar{\theta})}} = -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\bar{\theta}\frac{2\sinh\left(\frac{\eta}{2}\right)\cosh\left(\frac{\eta}{2}\right)}{\cosh\left(\eta\right)-1+\frac{2}{1+\tan^2(\bar{\theta})}} \\ &= -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\bar{\theta}\frac{(1+\cot^2(\bar{\theta}))\coth\left(\frac{\eta}{2}\right)}{1+\cot^2\left(\bar{\theta}\right)\coth^2\left(\frac{\eta}{2}\right)} = -2\text{P.V.}\int_{\mathbb{R}}\frac{\partial_\eta\bar{\theta}}{\sin^2(\bar{\theta})}\frac{\coth\left(\frac{\eta}{2}\right)}{1+\cot^2\left(\bar{\theta}\right)\coth^2\left(\frac{\eta}{2}\right)} \\ &= -2\text{P.V.}\int_{\mathbb{R}}\frac{\partial_\eta\bar{\theta}}{\cos^2(\bar{\theta})}\frac{\tanh\left(\frac{\eta}{2}\right)}{1+\tan^2\left(\bar{\theta}\right)\tanh^2\left(\frac{\eta}{2}\right)} = -2\text{P.V.}\int_{\mathbb{R}}\partial_\eta\tan(\bar{\theta})\frac{\tanh\left(\frac{\eta}{2}\right)}{1+\tan^2\left(\bar{\theta}\right)\tanh^2\left(\frac{\eta}{2}\right)} \\ &= -2f(x_t)+\text{P.V.}\int_{\mathbb{R}}\frac{\cot(\bar{\theta})}{\sinh^2(\eta/2)}\frac{1}{1+(\cot(\bar{\theta})\coth(\eta/2))^2}d\eta. \end{aligned} \tag{7.5}$$

Therefore, by (7.4) and (7.5), we have in (7.2)

$$\partial_t f(x_t)=-4f(x_t)+\int_{\mathbb{R}}\frac{\cot(\bar{\theta})}{\cosh^2(\eta/2)}\frac{1}{\tanh^2(\eta/2)+\cot^2(\bar{\theta})}d\eta$$
$$-\text{P.V.}\int_{\mathbb{R}}\frac{\tan(\theta)}{\cosh^2(\eta/2)}\frac{1}{\tanh^2(\eta/2)+\tan^2(\theta)}d\eta. \tag{7.6}$$

Due to the definition of $\bar{\theta}$

$$\cot(\bar{\theta}) = \tan\left(\frac{\pi}{2} - \bar{\theta}\right) = \tan\left(\frac{\pi}{2} - f(x_t) + \theta\right).$$

Using that

$$\arctan(\tan(f(x_t))\tanh(\eta/2))\Big|_{-\infty}^{\infty} = 2f(x_t),$$

we can write

$$4f(x_t) = \int_{\mathbb{R}} \frac{1}{\cosh^2\left(\frac{\eta}{2}\right)} \frac{\tan(\frac{\pi}{2} - f(x_t))}{\tan^2(\frac{\pi}{2} - f(x_t)) + \tanh^2\left(\frac{\eta}{2}\right)} d\eta.$$

Moreover, we use the equality

$$\tan\left(\frac{\pi}{2} - f(x_t) + \theta\right) = \frac{\tan(\frac{\pi}{2} - f(x_t)) + \tan(\theta)}{1 - \tan(\frac{\pi}{2} - f(x_t))\tan(\theta)}.$$

By notational convenience we use the notation $\sigma = \frac{\pi}{2} - f(x_t)$. We define

$$\Pi(x,\eta,t) = \frac{\tan(\sigma)}{\tan^2(\sigma) + \tanh^2\left(\frac{\eta}{2}\right)} + \frac{\tan(\theta)}{\tan^2(\theta) + \tanh^2\left(\frac{\eta}{2}\right)} - \frac{(\tan(\sigma) + \tan(\theta))(1 - \tan(\sigma)\tan(\theta))}{(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2 \tanh^2\left(\frac{\eta}{2}\right)}.$$

and, using (7.6) we have

$$\partial_t f(x_t) = -\int_{\mathbb{R}} \frac{1}{\cosh^2(\eta/2)} \Pi(x,\eta,t) d\eta. \tag{7.7}$$

So we need to prove that $\Pi \geq 0$[1] if $\|f\|_{L^\infty} = \max_x f(x)$. We have

$$\begin{aligned}\Pi = {} & \frac{\tan(\sigma)\tan(\theta)(\tan(\theta) + 2\tan(\sigma))}{[\tan^2(\sigma) + \tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\ & + \frac{\tan^2(\sigma)\tan(\theta)(\tan(\sigma)\tan(\theta) - 2)\tanh^2\left(\frac{\eta}{2}\right)}{[\tan^2(\sigma) + \tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\ & + \frac{\tan(\sigma)\tan(\theta)(\tan(\sigma) + 2\tan(\theta))}{[\tan^2(\theta) + \tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\ & + \frac{\tan(\sigma)\tan^2(\theta)(\tan(\sigma)\tan(\theta) - 2)\tanh^2\left(\frac{\eta}{2}\right)}{[\tan^2(\theta) + \tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\ & + \frac{(\tan(\sigma) + \tan(\theta))\tan(\sigma)\tan(\theta)}{(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)}.\end{aligned} \tag{7.8}$$

[1] In the case where $\|f(t)\|_{L^\infty(\mathbb{R})} = -\min_x f(x,t)$ we need to prove that $\Pi \leq 0$.

Rearranging, we get

$$
\begin{aligned}
\Pi = & \frac{\tan(\sigma)\tan^2(\theta)[1+\tan^2(\sigma)\tanh^2\left(\frac{\eta}{2}\right)]}{[\tan^2(\sigma)+\tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\
& + \frac{2\tan^2(\sigma)\tan(\theta)[1-\tanh^2\left(\frac{\eta}{2}\right)]}{[\tan^2(\sigma)+\tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\
& + \frac{\tan^2(\sigma)\tan(\theta)[1+\tan^2(\theta)\tanh^2\left(\frac{\eta}{2}\right)]}{[\tan^2(\theta)+\tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\
& + \frac{2\tan(\sigma)\tan^2(\theta)[1-\tanh^2\left(\frac{\eta}{2}\right)]}{[\tan^2(\theta)+\tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]} \\
& + \frac{(\tan(\sigma)+\tan(\theta))\tan(\sigma)\tan(\theta)}{(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)}.
\end{aligned}
$$

Now, we use that $\tanh^2\left(\frac{\eta}{2}\right)\leq 1$.

If $\|f\|_{L^\infty}=\max_x f(x)$, the definitions of θ and σ give us that $\tan(\theta),\tan(\sigma)>0$ and obtaining that $\Pi\geq 0$. This concludes the proof for this case.

For the case where the L^∞ norm is achieved in the minimum the proof is analogous. Indeed, we have that in this case $\Pi\leq 0$ because $\tan(\sigma),\tan(\theta)\leq 0$. □

Furthermore, we obtain the following corollary:

Corollary 7.1. *If $f_0(x)$ has a definite sign then this sign propagates through evolution.*

Proof. Using (7.8), the proof follows. □

Remark 7.1. *In order to obtain (7.6) we also can use the expression (2.8). In this expression we change the integration and the derivative and we exchange variables $\eta = x-\eta$. Then we differentiate the principal value integral.*

7.3 Maximum principle for $\|f\|_{L^2}$

In this section we prove the maximum principle for L^2.

Theorem 7.2 (Maximum principle for $\|f\|_{L^2}$)**.** *Let f be the unique classical solution of (2.5) in the Rayleigh-Taylor stable case. Then, f satisfies*

$$\|f(t)\|^2_{L^2(\mathbb{R})}+\frac{2}{\rho^2-\rho^1}\int_0^t\|v(s)\|^2_{L^2(S)}ds=\|f_0\|^2_{L^2(\mathbb{R})}.$$

Proof. Recall that, due to the incompressibility condition, the velocity satisfy

$$(v^+-v^-)\cdot n=0,$$

where $n=(-\partial_x f(x),1)/\sqrt{1+(\partial_x f(x))^2}$ and $v^\pm$ denote the velocity in the upper and lower subdomain respectively. In each subdomain, S^i, we have the potentials

$$\phi^+(x,y)=p^+(x,y)+\rho^1 y,\ \phi^-(x,y,t)=p^-(x,y,t)+\rho^2 y. \tag{7.9}$$

We have

$$0=\int_{S^1}\phi^+\Delta\phi^+dxdy=-\int_{S^1}|\nabla\phi^+|^2dxdy+\int_{\partial S^1}\phi^+\nabla\phi^+\cdot\frac{(\partial_x f(x),-1)}{\sqrt{1+(\partial_x f(x))^2}}ds,$$

$$0=\int_{S^2}\phi^-\Delta\phi^-dxdy=-\int_{S^2}|\nabla\phi^-|^2dxdy+\int_{\partial S^2}\phi^-\nabla\phi^-\cdot\frac{(-\partial_x f(x),1)}{\sqrt{1+(\partial_x f(x))^2}}ds.$$

We add these equalities and use (7.9) and the continuity in the normal direction. We get

$$\int_S|v|^2dxdy=\int_{\mathbb{R}}(\phi^+(x,f(x)-\phi^-(x,f(x)))(\nabla\phi^+(x,f(x))\cdot(\partial_x f(x),-1))dx.$$

Using the continuity of the pressure along the interface we obtain

$$\int_S|v|^2dxdy=(\rho^1-\rho^2)\int_{\mathbb{R}}f(x)\partial_t f(x)dx=\frac{\rho^1-\rho^2}{2}\frac{d}{dt}\|f\|^2_{L^2}=(\rho^1-\rho^2)\|f\|_{L^2}\frac{d}{dt}\|f\|_{L^2}.$$

In the Rayleigh-Taylor stable regime we have $\rho^2-\rho^1>0$, and we get that

$$\frac{d}{dt}\|f\|_{L^2}\leq 0.$$

Moreover, we have the following energy balance

$$\|f(t)\|^2_{L^2(\mathbb{R})}+\frac{2}{\rho^2-\rho^1}\int_0^t\|v(s)\|^2_{L^2(S)}ds=\|f_0\|^2_{L^2(\mathbb{R})}.$$

□

7.4 A decay estimate for $\|f\|_{L^\infty}$

Moreover, we have a decay estimate:

Theorem 7.3 (Decay for $\|f\|_{L^\infty}$)**.** *Let $f_0\in H^3_l(\mathbb{R})$ be the initial data and assume $\rho^2-\rho^1>0$. Then the solution $f(x,t)$ of equation (2.5) satisfies the inequality*

$$\frac{d}{dt}\|f(t)\|_{L^\infty}\leq -c(\|f_0\|_{L^2},\|f_0\|_{L^\infty},\rho^2,\rho^1,l)\exp\left(-\frac{2\pi}{l}\frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\left(1+\frac{\|f_0\|_{L^2}}{\|f(t)\|_{L^\infty}}\right)\right).$$

Proof. We conserve the notation and the hypothesis of the proof of Theorem 7.1, *i.e.* $\bar\rho=2$, $l=\frac{\pi}{2}$, $f(x_t)=\|f(t)\|_{L^\infty}$, $\sigma=\frac{\pi}{2}-f(x_t)$ and $\theta=\frac{f(x_t)-f(x_t-\eta)}{2}$. We have the equation (7.7) with Π defined in (7.8). Due to analysis in the proof of Theorem 7.1, we have the following bound

$$\begin{aligned}\Pi &\geq \frac{\tan^2(\sigma)\tan(\theta)[1+\tan^2(\theta)\tanh^2\left(\frac{\eta}{2}\right)]}{[\tan^2(\theta)+\tanh^2\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2\tanh^2\left(\frac{\eta}{2}\right)]}\\ &\geq \frac{\tan(\theta)}{(\tan^2(\|f_0\|_{L^\infty})+1)^2+\tan^2(\|f_0\|_{L^\infty})}\frac{1}{1+\tan^2(\|f_0\|_{L^\infty})}.\end{aligned}$$

thus

$$-\partial_t f(x_t)\geq\int_{\mathbb{R}}\frac{1}{\cosh^2(\eta/2)}\frac{\tan(\theta)}{((\tan^2(\|f_0\|_{L^\infty})+1)^2+\tan^2(\|f_0\|_{L^\infty}))(1+\tan^2(\|f_0\|_{L^\infty}))}d\eta. \quad (7.10)$$

7.4. A decay estimate for $\|f\|_{L^\infty}$

Fix the interval $[-r, r]$. We consider the sets

$$\mathcal{U}_1 = \left\{\eta : \eta \in [-r, r], \theta \geq \frac{f(x_t)}{4}\right\},$$

$$\mathcal{U}_2 = \left\{\eta : \eta \in [-r, r], \theta < \frac{f(x_t)}{4}\right\}.$$

Theorem 7.2 gives us a control for the measure of these sets. Indeed,

$$\|f_0\|_{L^2}^2 \geq \int_{\mathbb{R}} f^2(x_t - \eta) d\eta \geq \int_{\mathcal{U}_2} f^2(x_t - \eta) d\eta > \frac{f^2(x_t)}{4} |\mathcal{U}_2|.$$

Therefore, if r is big enough $\mathcal{U}_1$ is not empty. Furthermore,

$$|\mathcal{U}_1| = 2r - |\mathcal{U}_2| \geq 2\left(r - 2\frac{\|f_0\|_{L^2}^2}{f^2(x_t)}\right).$$

thus, using (7.10), we have

$$\begin{aligned}
-\partial_t f(x_t) &\geq \int_{\mathcal{U}_1} \frac{1}{\cosh^2(\eta/2)} \frac{\tan(\theta)}{((\tan^2(\|f_0\|_{L^\infty}) + 1)^2 + \tan^2(\|f_0\|_{L^\infty}))(1 + \tan^2(\|f_0\|_{L^\infty}))} d\eta \\
&\geq \frac{1}{\cosh^2(r/2)} \frac{\tan(f(x_t)/4)}{((\tan^2(\|f_0\|_{L^\infty}) + 1)^2 + \tan^2(\|f_0\|_{L^\infty}))(1 + \tan^2(\|f_0\|_{L^\infty}))} |\mathcal{U}_1| \\
&\geq \frac{2\tan(f(x_t)/4)\left(r - 2\frac{\|f_0\|_{L^2}^2}{f^2(x_t)}\right)}{2\cosh^2(r/2)((\tan^2(\|f_0\|_{L^\infty}) + 1)^2 + \tan^2(\|f_0\|_{L^\infty}))(1 + \tan^2(\|f_0\|_{L^\infty}))}.
\end{aligned}$$

Now, we take

$$r = 2\frac{\|f_0\|_{L^2}}{f(x_t)} + 2\frac{\|f_0\|_{L^2}^2}{f^2(x_t)}.$$

Then we get

$$\begin{aligned}
-\partial_t f(x_t) &\geq \frac{\operatorname{sech}^2\left(2\frac{\|f_0\|_{L^2}}{f(x_t)}\left(1 + \frac{\|f_0\|_{L^2}}{f(x_t)}\right)\right)\|f_0\|_{L^2}}{2((\tan^2(\|f_0\|_{L^\infty}) + 1)^2 + \tan^2(\|f_0\|_{L^\infty}))(1 + \tan^2(\|f_0\|_{L^\infty}))} \\
&\geq \frac{e^{-4\frac{\|f_0\|_{L^2}}{f(x_t)}\left(1 + \frac{\|f_0\|_{L^2}}{f(x_t)}\right)}\|f_0\|_{L^2}}{2((\tan^2(\|f_0\|_{L^\infty}) + 1)^2 + \tan^2(\|f_0\|_{L^\infty}))(1 + \tan^2(\|f_0\|_{L^\infty}))},
\end{aligned}$$

and we conclude the proof. □

Remark 7.2. *If the initial data is positive and integrable the decay (see [26]) is given by*

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1(\mathbb{R})}, \|f_0\|_{L^\infty(\mathbb{R})}, \rho^2, \rho^1, l) \exp\left(-\frac{\pi\|f_0\|_{L^1(\mathbb{R})}}{l\|f(t)\|_{L^\infty(\mathbb{R})}}\right).$$

We observe that in the whole plane case (see [24]) the decay rate is given by

$$\frac{d}{dt}\|f(t)\|_{L^\infty(\mathbb{R})} \leq -c(\|f_0\|_{L^1}, \|f_0\|_{L^\infty}, \rho^1, \rho^2)\|f(t)\|_{L^\infty(\mathbb{R})}^2,$$

so in the case without boundaries the decay is faster.

Moreover, we obtain the following corollary:

Corollary 7.2. *As a corollary we conclude that there are not non-trivial, steady state solutions of* (7.1).

7.5 Maximum principle for $\|\partial_x f\|_{L^\infty}$

In this section we show the maximum principle for $\|\partial_x f\|_{L^\infty}$ for a special class of initial data:

Theorem 7.4 (Maximum Principle for $\|\partial_x f\|_{L^\infty}$). *Let f_0 be a smooth initial data under the assumptions of Theorem 4.1 such that the following conditions holds:*

$$\|\partial_x f_0\|_{L^\infty} < 1, \tag{7.11}$$

$$\tan\left(\frac{\pi\|f_0\|_{L^\infty}}{2l}\right) < \|\partial_x f_0\|_{L^\infty} \tanh\left(\frac{\pi}{4l}\right), \tag{7.12}$$

and

$$\left(\|\partial_x f_0\|_{L^\infty} + |2(\cos\left(\frac{\pi}{2l}\right) - 2)\sec^4\left(\frac{\pi}{4l}\right)|\|\partial_x f_0\|^3_{L^\infty}\right)\frac{\pi^3}{8l^3} \times \frac{\left(1 + \|\partial_x f_0\|_{L^\infty}\left(\|\partial_x f_0\|_{L^\infty} + \frac{\tan\left(\frac{\pi}{2l}\frac{\|\partial_x f_0\|_{L^\infty}}{2}\right)}{\tanh\left(\frac{\pi}{4l}\right)}\right)\right)}{6\tanh\left(\frac{\pi}{4l}\right)}\frac{\pi^2}{4l^2} + 4\tan\left(\frac{\pi}{2l}\|f_0\|_{L^\infty}\right) - 4\|\partial_x f_0\|_{L^\infty}\cos\left(\frac{\pi}{l}\|f_0\|_{L^\infty}\right) < 0 \tag{7.13}$$

Then,

$$\|\partial_x f(t)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0\|_{L^\infty(\mathbb{R})}. \tag{7.14}$$

Proof. We take $\bar{\rho} = 2$ and $\pi = 2l$ without loss of generality. First, we suppose that $\partial_x f(x_t) = \max \partial_x f(x,t)$ in order to clarify the exposition. We divide the proof in some steps:

Step 1: obtaining the equation Using the same method as in Theorem 7.1 and the smoothness of f we have that the evolution of $\partial_x f(x_t) = \|\partial_x f(t)\|_{L^\infty}$ is given by

$$\frac{d}{dt}\|\partial_x f\|_{L^\infty} = \partial_t\partial_x f(x_t).$$

Since (2.5) is equivalent to (2.8), so we have to take a derivative in space in this equivalent formulation. The boundaries in the principal value integrals contributes with $-8\partial_x f(x_t)$. Thus, we get

$$\partial_t\partial_x f(x_t) = -8\partial_x f(x_t) + I_1(t) + I_2(t),$$

with

$$I_1(t) = 2\text{P.V.}\int_{\mathbb{R}} \partial_x^2\left(\arctan\left(\frac{\tan\left(\frac{f(x_t)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_t-\eta}{2}\right)}\right)\right)d\eta,$$

$$I_2(t) = 2\text{P.V.}\int_{\mathbb{R}} \partial_x^2\left(\arctan\left(\tan\left(\frac{f(x_t)+f(\eta)}{2}\right)\tanh\left(\frac{x_t-\eta}{2}\right)\right)\right)d\eta.$$

We define

$$\mu_1(t) = \frac{\tan\left(\frac{f(x_t)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_t-\eta}{2}\right)}, \qquad \mu_2(t) = \tan\left(\frac{f(x_t)+f(\eta)}{2}\right)\tanh\left(\frac{x_t-\eta}{2}\right).$$

We compute

$$2\partial_x^2\arctan(\mu_1) = \frac{\tanh^2((x_t-\eta)/2)}{\cosh^2((x_t-\eta)/2)\cos^2(\theta)}\frac{Q_1(x_t,\eta,t)}{(\tanh^2((x_t-\eta)/2)+\tan^2(\theta))^2},$$

with

$$Q_1 = \partial_x f(x_t)\mu_1^2(t) + (1-(\partial_x f(x_t))^2)\mu_1(t) - \partial_x f(x_t),$$

and

$$2\partial_x^2 \arctan(\mu_2) = \frac{1}{\cosh^2((x_t-\eta)/2)\cos^2(\bar{\theta})}\frac{Q_2(x_t,\eta,t)}{(1+\tanh^2((x_t-\eta)/2)\tan^2(\bar{\theta}))^2},$$

with

$$Q_2 = -\partial_x f(x_t)\mu_2^2(t) + ((\partial_x f(x_t))^2-1)\mu_2(t) + \partial_x f(x_t).$$

Step 2: local decay In this part of the proof we obtain local in time decay around $t=0$. Let $0<\delta$ be a sufficiently small fixed constant and notice that, if (7.11)-(7.13) hold for $f_0(x)$, then they are satisfied by $f(x,\delta)$ if δ is small enough. The sign of the integral terms are given by the sign of Q_1 and Q_2. Q_i are polynomials in the variables μ_i, respectively.

The roots of Q_1 are $\partial_x f(x_t)$ and $-1/\partial_x f(x_t)$, so if we have

$$|\mu_1(\delta)| \leq \min\left\{\|\partial_x f(\delta)\|_{L^\infty}, \frac{1}{\|\partial_x f(\delta)\|_{L^\infty}}\right\},$$

then we can ensure that the integral involving the increments of f is negative. However we have that for $\eta = x_t$ the following equality holds

$$\lim_{\eta\to x_t}\left|\frac{\tan\left(\frac{f(x_t)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_t-\eta}{2}\right)}\right| = \|\partial_x f(t)\|_{L^\infty},$$

and so we need that $\min\{\|\partial_x f(\delta)\|_{L^\infty}, 1/\|\partial_x f(\delta)\|_{L^\infty}\} = \|\partial_x f(\delta)\|_{L^\infty}$. Thus we impose condition (7.11). Moreover, if

$$|x-\eta|\geq 1 \quad \text{then} \;\; |\mu_1(\delta)| \leq \frac{\tan(\|f(\delta)\|_{L^\infty})}{\tanh\left(\frac{1}{2}\right)} < \|\partial_x f(\delta)\|_{L^\infty},$$

under the hypothesis (7.12). Then,

$$I_1^{out}(\delta) = 2\text{P.V.}\int_{B^c(x_\delta,1)} \partial_x^2\left(\arctan\left(\frac{\tan\left(\frac{f(x_\delta)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_\delta-\eta}{2}\right)}\right)\right)d\eta < 0.$$

We have to bound the following integral

$$I_1^{in}(\delta) = \text{P.V.}\int_{B(x_\delta,1)} \frac{\partial_x f(x_\delta)\mu_1^2(\delta) + (1-(\partial_x f(x_\delta))^2)\mu_1(\delta) - \partial_x f(x_\delta)}{\sinh^2((x_\delta-\eta)/2)\cos^2(\theta)\left(1+(\mu_1(\delta))^2\right)^2}d\eta.$$

Thus, using the cancellation when $\mu_1(t) = \partial_x f(x_t)$, we get

$$I_1^{in}(\delta) = \text{P.V.}\int_{B(x_\delta,1)} \frac{\partial_x f(x_\delta)(\mu_1^2(\delta) - (\partial_x f(x_\delta))^2) + (1-(\partial_x f(x_\delta))^2)(\mu_1(\delta) - \partial_x f(x_\delta))}{\sinh^2((x_\delta-\eta)/2)\cos^2(\theta)\left(1+(\mu_1(\delta))^2\right)^2}d\eta. \tag{7.15}$$

We remark that $\mu_1(\delta) - \partial_x f(x_\delta) < \mu_1(\delta) + \partial_x f(x_\delta)$. We consider the cases given by the sign of $\mu_1(\delta) - \partial_x f(x_\delta)$ and $\mu_1(\delta) + \partial_x f(x_\delta)$.

1. Case $\mu_1(\delta) > \partial_x f(x_\delta)$. In this case we have that $\mu_1(\delta) - \partial_x f(x_\delta) > 0$ and $\mu_1(\delta) + \partial_x f(x_\delta) > 0$. Using the definition of θ in (7.3) and the fact that $|x_t - \eta| \leq 1$, we have

$$0 \leq \frac{\tan(\theta)}{\tanh((x_t-\eta)/2)} - \partial_x f(x_t) \leq \frac{(x_t-\eta)^2}{48\tanh\left(\frac{1}{2}\right)}\left(\partial_x f(x_t) + 5(\partial_x f(x_t))^3\right). \tag{7.16}$$

To obtain this bound we split as follows

$$\frac{\tan(\theta)}{\tanh((x_t-\eta)/2)} - \partial_x f(x_t) = \frac{\tan(\theta)-\theta}{\tanh((x_t-\eta)/2)} + \theta\left(\frac{1}{\tanh((x_t-\eta)/2)} - \frac{2}{x_t-\eta}\right) + \frac{2\theta}{x_t-\eta} - \partial_x f(x_t). \quad (7.17)$$

Taylor theorem and the fact that the function $(x_t-\eta)/\tanh(x_t-\eta) \leq 0.5/\tanh(0.5)$ in this region gives us the desired bound. Notice that in this case we have $\mu_1^2(\delta) - (\partial_x f(x_\delta))^2 > 0$. Thus, in the same way,

$$\begin{aligned} 0 \leq \frac{\tan^2(\theta)}{\tanh^2((x_t-\eta)/2)} - (\partial_x f(x_t))^2 &= (\mu_1(t) - \partial_x f(x_t))(\mu_1(t) + \partial_x f(x_t)) \\ &\leq \frac{(x_t-\eta)^2}{48\tanh\left(\frac{1}{2}\right)}\left(\partial_x f(x_t) + 5(\partial_x f(x_t))^3\right)\left(\partial_x f(x_t) + \frac{\tan\left(\frac{\partial_x f(x_t)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right). \end{aligned} \quad (7.18)$$

Due to (7.16) and (7.18), we obtain

$$\begin{aligned} I_1^{in}(\delta) &\leq \frac{\left(\partial_x f(x_\delta) + 5(\partial_x f(x_\delta))^3\right)\left(1 + \partial_x f(x_\delta)\left(\partial_x f(x_\delta) + \frac{\tan\left(\frac{\partial_x f(x_\delta)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{48\tanh(1/2)\cos^2(\|f(\delta)\|_{L^\infty})}\int_{-1}^{1}\frac{\eta^2}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta \\ &\leq \frac{\left(\partial_x f(x_\delta) + 5(\partial_x f(x_\delta))^3\right)\left(1 + \partial_x f(x_\delta)\left(\partial_x f(x_\delta) + \frac{\tan\left(\frac{\partial_x f(x_\delta)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{6\tanh(1/2)\cos^2(\|f(\delta)\|_{L^\infty})}. \end{aligned} \quad (7.19)$$

We have

$$I_2(\delta) \leq \left|\int_{\mathbb{R}} \frac{((\partial_x f(x_\delta))^2 - 1)\tanh((x_\delta-\eta)/2)\tan(\bar\theta) + \partial_x f(x_\delta)}{\cosh^2((x_\delta-\eta)/2)\cos^2(\bar\theta)(1+\tanh^2((x_\delta-\eta)/2)\tan^2(\bar\theta))^2} d\eta\right|.$$

Easily, we get

$$|I_2(\delta)| \leq 4\frac{\tan(\|f(\delta)\|_{L^\infty}) + \|\partial_x f(\delta)\|_{L^\infty}}{\cos^2(\|f(\delta)\|_{L^\infty})}. \quad (7.20)$$

It remains to show that

$$I_1^{in}(\delta) + |I_2(\delta)| - 8\|\partial_x f(\delta)\|_{L^\infty} \leq 0.$$

We need to use the local term $-8\partial_x f(x_t)$ in order to control the remainder terms. Using the maximum principle (Theorem 7.1), we obtain

$$\begin{aligned} &\frac{\|\partial_x f(\delta)\|_{L^\infty} + 5\|\partial_x f(\delta)\|^3_{L^\infty}}{6\tanh(1/2)}\left(1 + \|\partial_x f(\delta)\|_{L^\infty}\left(\|\partial_x f(\delta)\|_{L^\infty} + \frac{\tan\left(\frac{\|\partial_x f(\delta)\|_{L^\infty}}{2}\right)}{\tanh(1/2)}\right)\right) \\ &\quad + 4\tan(\|f(\delta)\|_{L^\infty}) + 4\|\partial_x f(\delta)\|_{L^\infty}(1 - 2\cos^2(\|f(\delta)\|_{L^\infty})) < 0, \end{aligned}$$

which is the condition (7.13). Putting all together we have shown, for every $0 < \delta$ small enough, if initially the previous conditions holds there is local in time decay for $\delta < t < t^*$.

2. Case $-\partial_x f(x_\delta) < \mu_1(\delta) \leq \partial_x f(x_\delta)$. In this case we have $\mu_1(\delta) - \partial_x f(x_\delta) \leq 0$ and $\mu_1(\delta) + \partial_x f(x_\delta) > 0$. Thus, we get $I_1^{in}(\delta) < 0$ and we can neglect it. Equation (7.20) remains valid and we have

$$|I_2(\delta)| - 8\|\partial_x f(\delta)\|_{L^\infty} \leq 0.$$

due to condition (7.13). We obtain

$$\partial_t \partial_x f(x_\delta) = I_1^{in}(\delta) + I_1^{out}(\delta) + I_2(\delta) - 8\partial_x f(x_\delta) \leq |I_2| - 8\partial_x f(x_\delta) < 0.$$

3. Case $\mu_1(\delta) \leq -\partial_x f(x_\delta)$. Now, we have $\mu_1(\delta) - \partial_x f(x_\delta) \leq 0$ and $\mu_1(\delta) + \partial_x f(x_\delta) \leq 0$. We split

$$\frac{\tan(\theta)}{\tanh((x_t-\eta)/2)} + \partial_x f(x_t) = \frac{\tan(\theta)-\theta}{\tanh((x_t-\eta)/2)} + \theta\left(\frac{1}{\tanh((x_t-\eta)/2)} - \frac{2}{x_t-\eta}\right) + \frac{2\theta}{x_t-\eta} + \partial_x f(x_t). \quad (7.21)$$

The last term is now positive due to the definition of $\partial_x f(x_t)$. Then, in this case, we have

$$\partial_x f(x_\delta)(\mu_1(\delta) - \partial_x f(x_\delta))\left(\frac{2\theta}{x_t-\eta} + \partial_x f(x_t)\right) \leq 0,$$

and we can neglect its contribution. Using Taylor theorem in (7.21), we obtain the bound (7.18) and (7.19). Then we conclude using (7.19) and (7.20) as in the case 1.

We have proved that for every $0 < \delta$ small enough, if initially the previous conditions (7.11)-(7.13) holds, there is local in time decay. As δ is positive and arbitrary, we have

$$\|\partial_x f(t)\|_{L^\infty} \leq \|\partial_x f(0)\|_{L^\infty}, \text{ for } 0 \leq t < t^*.$$

Step 3: from local decay to an uniform bound First, notice that if $\|\partial_x f(t)\|_{L^\infty}$ decays faster enough then condition (7.13) is not global and t^* could be finite. If $t^* = \infty$, then we are done. If $t^* < \infty$ we have $\|\partial_x f(t^*)\|_{L^\infty} = \|\partial_x f(0)\|_{L^\infty}$. We have to show that conditions (7.11)-(7.13) are satisfied at $t = t^*$ and, as a consequence, we conclude that the maximum of the slope decays again. At time $0 < t^*$ we have $\|\partial_x f(t^*)\|_{L^\infty} = \|\partial_x f(0)\|_{L^\infty}$ and, due to Theorem 7.1, $\|f(t^*)\|_{L^\infty} \leq \|f(0)\|_{L^\infty}$. Thus the condition (7.13) again holds. Easily we get that the same is valid for the condition (7.11). We have

$$|\mu_1(t^*)| \leq \frac{\tan\left(\|f(t^*)\|_{L^\infty}\right)}{\tanh\left(\frac{1}{2}\right)} \leq \frac{\tan\left(\|f(0)\|_{L^\infty}\right)}{\tanh\left(\frac{1}{2}\right)} < \|\partial_x f_0\|_{L^\infty} = \|\partial_x f(t^*)\|_{L^\infty},$$

and we conclude that condition (7.12) also holds at time t^*. Thus (7.14) is achieved for all time $0 < t$.

Step 4: the case where $\|\partial_x f(t)\|_{L^\infty} = -\min_x \partial_x f(x,t)$ Now we address the case where $\partial_x f(x_t) = -\|\partial_x f(t)\|_{L^\infty}$, so $\frac{d}{dt}\|\partial_x f(t)\|_{L^\infty} = -\partial_t\partial_x f(x_t)$. We need to show that

$$\partial_t\partial_x f(x_t) = -8\partial_x f(x_t) + I_1^{in} + I_1^{out} + I_2 > 0.$$

Using the hypotheses (7.11) and (7.12) we obtain that $I_1^{out} > 0$. Recall that the sign in I_1^{in} is given by

$$Q_1 = \partial_x f(x_t)(\mu_1^2(t) - (\partial_x f(x_t))^2) + (1 - (\partial_x f(x_t))^2)(\mu_1(t) - \partial_x f(x_t)).$$

Notice that in this case we have $\mu_1(t) + \partial_x f(x_t) < \mu_1(t) - \partial_x f(x_t)$. As in the step 2 we consider different cases by looking to the sign of $\mu_1(t) - \partial_x f(x_t)$ and $\mu_1(t) + \partial_x f(x_t)$.

1. Case $\partial_x f(x_t) < \mu_1(t) \leq -\partial_x f(x_t)$. Since in this case we have $\mu_1(t) - \partial_x f(x_t) > 0$ and $\mu_1(t) + \partial_x f(x_t) \leq 0$, we get $I_1^{in} \geq 0$. The bound (7.20) remain valid and we get

$$\partial_t\partial_x f(x_t) > -8\partial_x f(x_t) + I_2 \geq 8\|\partial_x f(t)\|_{L^\infty} - |I_2| > 0,$$

where the last inequality is a consequence of condition (7.13).

2. Case $\mu_1(t) \leq \partial_x f(x_t)$. Now, we have $\mu_1(t) - \partial_x f(x_t) \leq 0$ and $\mu_1(t) + \partial_x f(x_t) \leq 0$. Thus, the last term in (7.17) is positive. We get

$$\partial_x f(x_t)\left(\frac{2\theta}{x_t-\eta} - \partial_x f(x_t)\right)(\mu_1(t) + \partial_x f(x_t)) \geq 0,$$

and we can neglect these terms. We obtain a lower bound for I_1^{in} following the same ideas as in the step 2. Indeed, we obtain

$$I_1^{in}(t) \geq -\frac{\left(\|\partial_x f(t)\|_{L^\infty} + 5\|\partial_x f(t)\|^3_{L^\infty}\right)\left(1 + \|\partial_x f(t)\|_{L^\infty}\left(\|\partial_x f(t)\|_{L^\infty} + \frac{\tan\left(\frac{\|\partial_x f(t)\|_{L^\infty}}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{6\tanh(1/2)\cos^2(\|f(t)\|_{L^\infty})}.$$

We use (7.20) and (7.13). We are done with this case.

3. Case $\mu_1(t) > -\partial_x f(x_t)$*:* In this case we have $\mu_1(t) - \partial_x f(x_t) > 0$ and $\mu_1(t) + \partial_x f(x_t) > 0$. Now, the last term in (7.21) is negative. We get

$$\partial_x f(x_t)(\mu_1(t) - \partial_x f(x_t))\left(\frac{2\theta}{x_t-\eta} + \partial_x f(x_t)\right) \geq 0,$$

and we can neglect it because its positiveness. We conclude following the same ideas as in step 2. We are done with this step. □

Remark 7.3. *We observe that in the whole plane case the condition is only on* $\|\partial_x f_0\|_{L^\infty}$*. In our case this appears to be impossible because of two facts: Firstly, the term with* $\bar{\theta}$ *in* (7.3) *gives us a condition on* $\|f_0\|_{L^\infty}$*, also we notice that the condition on*

$$\mu_1(t) = \frac{\tan\left(\frac{f(x_t)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_t-\eta}{2}\right)}$$

gives us implicitly a condition on $\|f(t)\|_{L^\infty}$*. Indeed, we have*

$$\left|\lim_{|\eta|\to\infty}\frac{\tan\left(\frac{f(x_t)-f(\eta)}{2}\right)}{\tanh\left(\frac{x_t-\eta}{2}\right)}\right| = |\tan(f(x_t/2))| \leq \|\partial_x f(t)\|_{L^\infty}.$$

The second fact is that the term μ_1 *can be bounded* below *by the incremential quotients, but if we want to bound it* above *we have to use* $\|\partial_x f(t)\|_{L^\infty}$ *and* $\|f(t)\|_{L^\infty}$*.*

A new region appears due to the boundaries. In this region the slope can initially grow but it remains bounded uniformly in time. We have

Corollary 7.3. *Let us define* $(x(l), y(l))$ *as the solution of the system*

$$\begin{cases} \tan\left(\frac{\pi x}{2l}\right) - y\tanh\left(\frac{\pi}{4l}\right) = 0 \\ \left(y + |2(\cos\left(\frac{\pi}{2l}\right) - 2)\sec^4\left(\frac{\pi}{4l}\right)|y^3\right)\frac{\left(1+y\left(y+\frac{\tan\left(\frac{\pi}{2l}\frac{y}{2}\right)}{\tanh\left(\frac{\pi}{4l}\right)}\right)\right)}{6\tanh\left(\frac{\pi}{4l}\right)}\left(\frac{\pi}{2l}\right)^5 \\ \qquad +4\tan\left(\frac{\pi}{2l}x\right) - 4y\cos\left(\frac{\pi}{l}x\right) = 0. \end{cases} \tag{7.22}$$

Then, if we have $\|\partial_x f_0\|_{L^\infty} < y(l)$ *and* $\|f_0\|_{L^\infty} < x(l)$*, we get*

$$\|\partial_x f(t)\|_{L^\infty(\mathbb{R})} \leq 1.$$

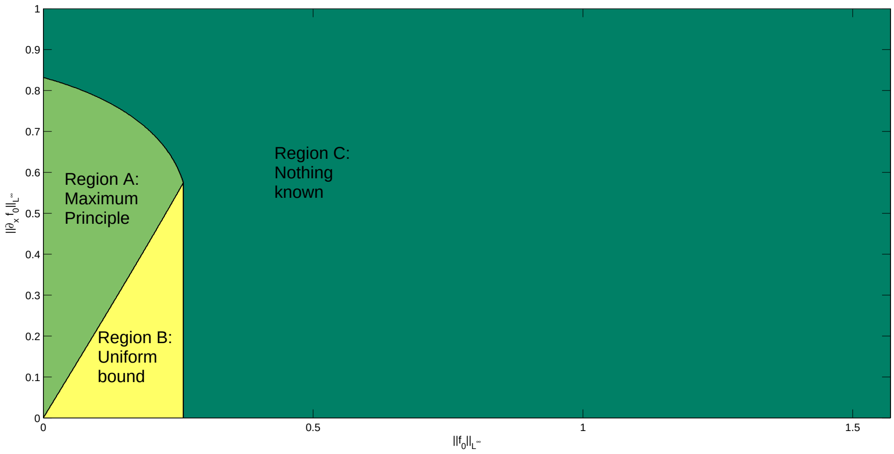

Figure 7.1: Different regions in $(\|f_0\|_{L^\infty}, \|\partial_x f_0\|_{L^\infty})$ for the behaviour of $\|\partial_x f\|_{L^\infty}$ when $\pi = 2l$.

Proof. Notice that this region is below the region with maximum principle for $\|\partial_x f(t)\|_{L^\infty}$ (see Figure 7.1). The proof is straighforward using the maximum principle for $\|f(t)\|_{L^\infty}$ (see Theorem 7.1). □

Remark 7.4. *The region without decay but with an uniform bound (see Figure 7.1) appears due to the boundaries. This region does not appear in the case with infinite depth* $\mathcal{A} = 0$*. We notice that if we now take the* $l \to \infty$ *limit we recover the well-known result (contained in [24]) for the whole plane case. Indeed, if* $l \to \infty$*, the conditions* (7.12) *and* (7.13) *are automatically achieved and we only have* (7.11) *as in [24].*

We have the following corollary:

Corollary 7.4. *Let* $f_0 \in H^3_l(\mathbb{R})$ *be an admissible initial datum and let* $0 < l$ *be the depth. If* $\frac{l}{2} \leq \|f_0\|_{L^\infty}$ *then hypothesis* (7.13) *can not hold.*

Proof. Notice that $\cos(\pi\|f_0\|/l)$ must be positive. □

Chapter 8

Turning waves

8.1 Foreword

In this Chapter we show the existence of finite time singularities for (2.5) in the Rayleigh-Taylor stable case, *i.e.* $\rho^2 > \rho^1$. In particular, we start with an interface which is a graph, and then after a finite time there is a point with vertical tangent. After this time the Rayleigh-Taylor unstable regime is reached and the curve is no longer a graph. The idea is that some singularities for (2.5) are equivalent to solutions of (2.9) that initially can be parametrized as graphs and then they can not.[1] The precise statement is the following:

Theorem 8.1 (Turning waves)**.** *Take* $\rho^2 - \rho^1 > 0$*. Then, there exist analytic initial data* $z_0 = z(\alpha, 0)$*, that can be parametrized as a graph, such that the solution of* (2.9)*, at finite time* $t = t_1$*, is no longer a graph.*

An analogous result was proven in [10] in the case of infinitely deep medium.

We also show that the amplitude plays no role on the turning

Theorem 8.2 (Turning waves with small amplitude)**.** *Take* $\rho^2 - \rho^1 > 0$*. Given any* $0 < \gamma << 1$*,*

[1]Notice that (2.9) is the equation for the interface when the curve is not necessarily parametrized as a graph.

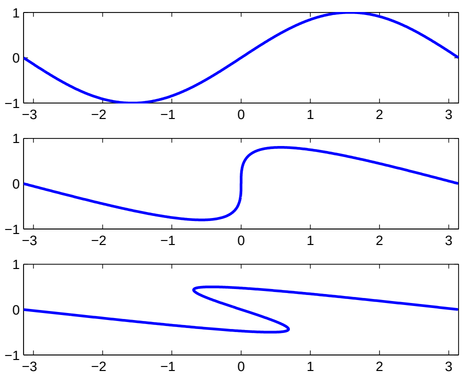

Figure 8.1: The evolution at times $t = -t_1, 0, t_1$ respectively.

there exist analytic initial data f_0 with $\|f_0\|_{L^\infty(\mathbb{R})} \leq \gamma$, such that the solution of (2.5), at finite time $t = t_1$, has a singularity.

In order to prove these results we show (see Theorem 8.3) forward and backward solvability for the interface parametrized as a curve (2.9). Then the proof is achieved by constructing a family of curves which turn.

8.2 Existence of solution without assuming RT stability

In this section we show that there is an unique local smooth solution even when the Rayleigh-Taylor condition is not satisfied but our initial data is analytic. We prove this result by a Cauchy-Kowalevski Theorem (see [10], [46], [49] and [50]). We consider curves z satisfying the arc-chord condition and such that

$$\lim_{|\alpha|\to\infty} |z(\alpha) - (\alpha, 0)| = 0.$$

As in Chapter 5, we define the complex strip $\mathbb{B}_r = \{x + i\xi, |\xi| < r\}$, and the following Banach spaces

$$X_r = \{z = (z_1, z_2) \text{ analytic curves satisfying the arc-chord condition on } \mathbb{B}_r\}, \tag{8.1}$$

with norm

$$\|z\|_r^2 = \|z(\gamma) - (\gamma, 0)\|^2_{H^3(\mathbb{B}_r)}.$$

These spaces for a Banach scale[2]. This can be easily proved (see [2] for different but equivalent definitions of the Hardy spaces on the strip). For notational convenience we write $\gamma = \alpha \pm ir$, $\gamma' = \alpha \pm ir'$ and we take $\bar{\rho} = 2$ and $l = \pi/2$. We claim that, for $0 < r' < r$,

$$\|\partial_\alpha \cdot\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C}{r - r'}\|\cdot\|_{L^2(\mathbb{B}_r)}. \tag{8.2}$$

Indeed, we apply Cauchy's integral formula with $\Gamma = \gamma' + (r - r')e^{i\theta}$ to conclude the claim.

The complex extension of the equation can be written as

$$\begin{aligned}\partial_t z = \text{P.V.} \int_{\mathbb{R}} & \frac{(\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \eta)) \sinh(z_1(\gamma) - z_1(\gamma - \eta))}{\cosh(z_1(\gamma) - z_1(\gamma - \eta)) - \cos(z_2(\gamma) - z_2(\gamma - \eta))} \\ & + \frac{(\partial_\alpha z_1(\gamma) - \partial_\alpha z_1(\gamma - \eta), \partial_\alpha z_2(\gamma) + \partial_\alpha z_2(\gamma - \eta)) \sinh(z_1(\gamma) - z_1(\gamma - \eta))}{\cosh(z_1(\gamma) - z_1(\gamma - \eta)) + \cos(z_2(\gamma) + z_2(\gamma - \eta))} d\eta. \end{aligned} \tag{8.3}$$

We define

$$d^+[z](\gamma, \eta) = \frac{\cosh^2(\eta/2)}{\cosh(z_1(\gamma) - z_1(\gamma - \eta)) + \cos(z_2(\gamma) + z_2(\gamma - \eta))},$$

and

$$d^-[z](\gamma, \eta) = \frac{\sinh^2(\eta/2)}{\cosh(z_1(\gamma) - z_1(\gamma - \eta)) - \cos(z_2(\gamma) - z_2(\gamma - \eta))}.$$

Then we have the following result:

Proposition 8.1. *Consider $0 \leq r' < r$ and the set*

$$O_R = \{z \in X_r \text{ such that } \|z\|_r < R, \|d^-[z]\|_{L^\infty(\mathbb{B}_r)} < R, \|d^+[z]\|_{L^\infty(\mathbb{B}_r)} < R\},$$

where $d^-[z]$ and $d^+[z]$ are defined above. Then, for $z, w \in O_R$, the spatial operator in (8.3), $F : O_R \to X_{r'}$ is continuous. Moreover, the following inequalities holds:

[2] *i.e.* $X_r \subset X_{r'}$ and $\|\cdot\|_{r'} \leq \|\cdot\|_r$ for $r' \leq r$.

1. $\|F[z]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z\|_r$,
2. $\|F[z]-F[w]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}$,
3. $\sup_{\gamma\in\mathbb{B}_r,\beta\in\mathbb{R}} |F[z](\gamma)-F[z](\gamma-\beta)| \leq C_R|\beta|$.

Proof. Proof of 3. We need to bound $\|\partial_t\partial_\alpha z\|_{L^\infty(\mathbb{B}_r)}$. Due to the Hadamard's Three Lines Theorem it is enough to bound the quantities on the boundary of the strip. For notational simplicity, we write

$$
\begin{aligned}
z_1^-(\gamma) &= z_1(\gamma)-z_1(\gamma-\eta),\\
z_2^\pm(\gamma) &= z_2(\gamma)\pm z_2(\gamma-\eta),\\
\sinh(z_1) &= \sinh(z_1^-(\gamma)),\\
\cosh(z_1) &= \cosh(z_1^-(\gamma)),\\
\sin^\pm(z_2) &= \sin(z_2^\pm(\gamma)),\\
\cos^\pm(z_2) &= \cos(z_2^\pm(\gamma)).
\end{aligned}
$$

We have

$$
\begin{aligned}
\partial_t\partial_\alpha z &= \text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\frac{(\partial_\alpha^2 z(\gamma)-\partial_\alpha^2 z(\gamma-\eta))\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}\\
&\quad+\frac{(\partial_\alpha^2 z_1(\gamma)-\partial_\alpha^2 z_1(\gamma-\eta),\partial_\alpha^2 z_2(\gamma)+\partial_\alpha^2 z_2(\gamma-\eta))\sinh(z_1)}{\cosh(z_1)+\cos^+(z_2)}\\
&\quad+\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\eta))\cosh(z_1)(\partial_\alpha z_1(\gamma)-\partial_\alpha z_1(\gamma-\eta))}{\cosh(z_1)-\cos^-(z_2)}\\
&\quad+\frac{(\partial_\alpha z_1^-(\gamma),\partial_\alpha z_2^+(\gamma))\cosh(z_1)\partial_\alpha z_1^-(\gamma)}{\cosh(z_1)+\cos^+(z_2)}\\
&\quad-\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\eta))\sinh^2(z_1)}{(\cosh(z_1)-\cos^-(z_2))^2}\partial_\alpha z_1^-(\gamma)\\
&\quad-\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\eta))\sinh(z_1)}{(\cosh(z_1)-\cos^-(z_2))^2}\sin^-(z_2)\partial_\alpha z_2^-(\gamma)\\
&\quad-\frac{(\partial_\alpha z_1^-(\gamma),\partial_\alpha z_2^+(\gamma))\sinh(z_1)}{(\cosh(z_1)+\cos^+(z_2))^2}\sin^+(z_2)\partial_\alpha z_2^+(\gamma)\\
&\quad-\frac{(\partial_\alpha z_1^-(\gamma),\partial_\alpha z_2^+)\sinh^2(z_1)}{(\cosh(z_1)+\cos^+(z_2))^2}\partial_\alpha z_1^-(\gamma)d\eta\\
&= A_1+A_2.
\end{aligned}
$$

The inner part can be easily bounded as in Chapters 5 and 6 using Sobolev embedding and the definitions of $d^\pm[z]$ and O_R as

$$A_1 \leq C_R.$$

In order to bound the outer part we have to integrate by parts in η. We have $A_2 = B_1+B_2+B_3$, where

$$
\begin{aligned}
B_1 &= \partial_\alpha^2 z(\gamma)\text{P.V.}\int_{B^c(0,1)}\frac{\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}+\frac{\sinh(z_1)}{\cosh(z_1)+\cos^+(z_2)}d\eta,\\
B_2 &= \text{P.V.}\int_{B^c(0,1)}\frac{-\partial_\alpha^2 z(\gamma-\eta)\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}+\frac{(-\partial_\alpha^2 z_1(\gamma-\eta),\partial_\alpha^2 z_2(\gamma-\eta))\sinh(z_1)}{\cosh(z_1)+\cos^+(z_2)}d\eta,
\end{aligned}
$$

$$\begin{aligned} B_3 &= \text{P.V.}\int_{B^c(0,1)} \frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\eta))\partial_\alpha z_1^-(\gamma)}{(\cosh(z_1)-\cos^-(z_2))^2}(1-\cosh(z_1)\cos^-(z_2)) \\ &+\frac{(\partial_\alpha z_1^-(\gamma),\partial_\alpha z_2^+(\gamma))\partial_\alpha z_1^-(\gamma)}{(\cosh(z_1)+\cos^+(z_2))^2}(1+\cos^+(z_2)\cosh(z_1)) \\ &-\frac{(\partial_\alpha z_1^-(\gamma),\partial_\alpha z_2^+(\gamma))\sinh(z_1)}{(\cosh(z_1)+\cos^+(z_2))^2}\sin^+(z_2)\partial_\alpha z_2^+(\gamma) \\ &-\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\eta))\sinh(z_1)}{(\cosh(z_1)-\cos^-(z_2))^2}\sin^-(z_2)\partial_\alpha z_2^-(\gamma)d\eta. \end{aligned}$$

All the terms in B_3 are easily bounded using the definition of $d^\pm$ and O_R. We obtain $B_3\leq C_R$. In B_1 we integrate by parts and use the cancelation coming from the Principal Value integration. The boundary terms in ±1 are harmless and can be easily bounded. The boundary terms at infinity are

$$\lim_{M\to\infty} M\Bigg(\frac{\sinh(z_1(\gamma)-z_1(\gamma-M))}{\cosh(z_1(\gamma)-z_1(\gamma-M))\pm\cos(z_2(\gamma)\pm z_2(\gamma-M))} + \frac{\sinh(z_1(\gamma)-z_1(\gamma+M))}{\cosh(z_1(\gamma)-z_1(\gamma+M))\pm\cos(z_2(\gamma)\pm z_2(\gamma+M))}\Bigg)=0.$$

The integral now is

$$\tilde{B}_1=-\partial_\alpha^2 z(\gamma)\text{P.V.}\int_{B^c(0,1)}\eta\partial_\eta\left(\frac{\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}+\frac{\sinh(z_1)}{\cosh(z_1)+\cos^+(z_2)}\right)d\eta,$$

and this integral can be easily bounded as $\tilde{B}_1\leq C_R$ using the definition of $d^\pm, O_R$ and the Sobolev embedding. For B_2 we integrate by parts again. The boundary terms are harmless. We obtain

$$\tilde{B}_2=-\text{P.V.}\int_{B^c(0,1)}\partial_\alpha z(\gamma-\eta)\partial_\eta\left(\frac{\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}\right) + (-\partial_\alpha z_1(\gamma-\eta),\partial_\alpha z_2(\gamma-\eta))\partial\eta\left(\frac{\sinh(z_1)}{\cosh(z_1)+\cos^+(z_2)}\right)d\eta.$$

Using the decay coming from the derivatives we conclude $\tilde{B}_2\leq C_R$, and then $\|\partial_t\partial_\alpha z\|_{L^\infty}\leq C_R$. This concludes with the point 3.

Proof of 2. We show now the second point. For the sake of simplicity we bound only some terms. We consider first the low order terms; we have that the terms corresponding to the first kernel Ξ_1 are

$$\begin{aligned} F_1[z]-F_1[w] &= \text{P.V.}\int_{\mathbb{R}}\frac{(\partial_\alpha z(\gamma')-\partial_\alpha z(\gamma'-\eta)-(\partial_\alpha w(\gamma')-\partial_\alpha w(\gamma'-\eta)))\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}d\eta \\ &+\text{P.V.}\int_{\mathbb{R}}(\partial_\alpha w(\gamma')-\partial_\alpha w(\gamma'-\eta))\left(\frac{\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}-\frac{\sinh(w_1)}{\cosh(w_1)-\cos^-(w_2)}\right)d\eta \\ &= A_3+A_4. \end{aligned}$$

Now we split the integral:

$$\begin{aligned} A_3 &= \text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\frac{(\partial_\alpha z(\gamma')-\partial_\alpha z(\gamma'-\eta)-(\partial_\alpha w(\gamma')-\partial_\alpha w(\gamma'-\eta)))\sinh(z_1)}{\cosh(z_1)-\cos^-(z_2)}d\eta \\ &= B_4+B_5. \end{aligned}$$

The inner term can be written as

$$B_4(\gamma') = \int_0^1 \text{P.V.} \int_{B(0,1)} \frac{(\partial_\alpha^2(z_2 - w_2)(\gamma' + (s-1)\eta))\eta \sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta ds,$$

and then, using Cauchy–Schwarz inequality as in Chapter 5 we can obtain

$$\|B_4\|_{L^2(\mathbb{B}_{r'})} \leq C_R \|\partial_\alpha^2(z_2 - w_2)\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z - w\|_r.$$

where in the last step we used (8.2), *i.e.* we can recover one derivative using the Cauchy integral formula. Now we split $B_5 = C_1 + C_2$, where

$$C_1 = (\partial_\alpha z(\gamma') - \partial_\alpha w(\gamma'))\text{P.V.} \int_{B^c(0,1)} \frac{\sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta,$$

$$C_2 = \text{P.V.} \int_{B^c(0,1)} \frac{(\partial_\eta z(\gamma' - \eta) - \partial_\eta w(\gamma' - \eta)) \sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta.$$

Integrating by parts in the previous integrals we can bound it. Thus, appliying Cauchy–Schwarz inequality, we obtain

$$\|B_5\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z - w\|_r.$$

The term A_4 can be bounded with the same techniques, thus we conclude

$$\|A_4\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z - w\|_{H^3(\mathbb{B}_r)}.$$

Putting all the inequalities together we obtain

$$\|F_1[z] - F_1[w]\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z - w\|_{H^3(\mathbb{B}_r)}.$$

The third derivative of the first kernel, Ξ_1, has the more singular integral:

$$\begin{aligned} I_1 = \text{P.V.} &\int_{\mathbb{R}} \frac{(\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 z(\gamma' - \eta) - (\partial_\alpha^4 w(\gamma') - \partial_\alpha^4 w(\gamma' - \eta))) \sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta \\ &+ \text{P.V.} \int_{\mathbb{R}} (\partial_\alpha^4 w(\gamma') - \partial_\alpha^4 w(\gamma' - \eta)) \left(\frac{\sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} - \frac{\sinh(w_1)}{\cosh(w_1) - \cos^-(w_2)} \right) d\eta \\ &= A_5 + A_6. \end{aligned}$$

First consider the term $A_5 = B_6 + B_7$, where

$$\begin{aligned} B_6 = (\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 w(\gamma'))\text{P.V.} &\int_{B(0,1)} \frac{\sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta \\ &- \text{P.V.} \int_{B(0,1)} \frac{(\partial_\alpha^4 z(\gamma' - \eta) - \partial_\alpha^4 w(\gamma' - \eta)) \sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta = C_3 + C_4, \end{aligned}$$

$$\begin{aligned} B_7 = (\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 w(\gamma'))\text{P.V.} &\int_{B^c(0,1)} \frac{\sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta \\ &- \text{P.V.} \int_{B^c(0,1)} \frac{(\partial_\alpha^4 z(\gamma' - \eta) - \partial_\alpha^4 w(\gamma' - \eta)) \sinh(z_1)}{\cosh(z_1) - \cos^-(z_2)} d\eta. \end{aligned}$$

B_7 can be easily bounded integrating by parts as before obtaining

$$\|B_7\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

We need handle the singularity arising in B_6 without using more derivatives. In order to do this we split the integrals:

$$\begin{aligned} C_3 &= (\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 w(\gamma'))\text{P.V.}\int_{B(0,1)} \frac{\sinh(z_1) - (z_1(\gamma') - z_1(\gamma'-\eta))}{\cosh(z_1) - \cos^-(z_2)} d\eta \\ &+ (\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 w(\gamma'))\text{P.V.}\int_{B(0,1)} \frac{z_1(\gamma') - z_1(\gamma'-\eta) - \eta\partial_\alpha z_1(\gamma')}{\cosh(z_1) - \cos^-(z_2)} d\eta \\ &+ (\partial_\alpha^4 z(\gamma') - \partial_\alpha^4 w(\gamma'))\partial_\alpha z_1(\gamma')\text{P.V.}\int_{B(0,1)} \frac{\eta}{\sinh^2(\eta/2)}\left(d^-[z](\gamma',\eta) - d^-[z](\gamma',0)\right), \end{aligned}$$

where

$$d^-[z](\gamma',0) = \frac{1}{2}\frac{1}{|\partial_\alpha z(\gamma')|^2}.$$

The integrals in C_3 are not singular and then

$$\|C_3\|_{L^2(\mathbb{B}_{r'})} \leq C_R\|\partial_\alpha^4(z-w)\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

In the term C_4 we have a Hilbert transform:

$$\begin{aligned} C_4 &= -\text{P.V.}\int_{B(0,1)} \frac{(\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta))(\sinh(z_1) - (z_1(\gamma') - z_1(\gamma'-\eta)))}{\cosh(z_1) - \cos^-(z_2)} \\ &\quad -\text{P.V.}\int_{B(0,1)} \frac{(\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta))(z_1(\gamma') - z_1(\gamma'-\eta) - \partial_\alpha z_1(\gamma')\eta)}{\cosh(z_1) - \cos^-(z_2)} \\ &\quad -\partial_\alpha z_1(\gamma')\text{P.V.}\int_{B(0,1)} (\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta))\eta\frac{d^-[z](\gamma',\eta) - d^-[z](\gamma',0)}{\sinh^2(\eta/2)} \\ &\quad -\partial_\alpha z_1(\gamma')d^-[z](\gamma',0)\text{P.V.}\int_{B(0,1)} (\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta))\eta\left(\frac{1}{\sinh^2(\eta/2)} - \frac{4}{\eta^2}\right) \\ &\quad +4\partial_\alpha z_1(\gamma')d^-[z](\gamma',0)\text{P.V.}\int_{B^c(0,1)} \frac{\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta)}{\eta} \\ &\quad -4\partial_\alpha z_1(\gamma')d^-[z](\gamma',0)H(\partial_\alpha^4 z(\gamma'-\eta) - \partial_\alpha^4 w(\gamma'-\eta)) \\ &= D_1 + D_2 + D_3 + D_4 + D_5 + D_6. \end{aligned}$$

The integrals D_1, D_2, D_3 and D_4 are not singular. Then, as before, it can be easily bounded

$$\|D_i\|_{L^2(\mathbb{B}(r'))} \leq C_R\|\partial_\alpha^4(z-w)\|_{L^2(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}, \quad i = 1, ..., 4.$$

The term D_5 can be bounded by means of an integration by parts to ensure the decay at infinity. The term D_6 can be bounded using the boundedness of the Hilbert transform in L^2, so we conclude

$$\|C_4\|_{L^2(\mathbb{B}(r'))} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

The term A_6 can be bounded as before. The terms corresponding to the second kernel are not singular and can be easily bounded with the same techniques and the definition of $d^+[\cdot]$.

Proof of 1. This proof follows with the same techniques. □

The previous proposition gives us the following result:

Theorem 8.3 (Existence in the unstable case)**.** *Let $z_0 \in X_{r_0}$, for some $r_0 > 0$ (see definition (8.1)), be the initial datum and assume that this initial data does not reach the boundaries. Then, there exists an analytic solution of the Muskat problem (2.9) for $t \in [-T, T]$ for a small enough $T > 0$.*

Proof. The proof follows the same argument as in [10, 46, 49, 50]. As $z_0 \in X_{r_0}$, satisfies the arc-chord condition and does not reach the boundaries, there exists R_0 such that $z_0 \in O_{R_0}$. We take $r < r_0$ and $R > R_0$ in order to define O_R and we consider the iterates

$$z_{n+1} = z_0 + \int_0^t F[z_n]ds,$$

and assume by induction that $z_k \in O_R$ for $k \leq n$. Recall that $F = (F^1, F^2)$ is the spatial operator in (8.3). Then, we obtain a time $T_{CK} > 0$ such that $\|z_n\|_r < R$ for all n. It remains to show that

$$\|d^-[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)}, \|d^+[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R,$$

for some times $T_A, T_B > 0$ respectively (see Theorem 5.1). Using the classical trigonometric formulas we obtain

$$\begin{aligned}
(d^-[z_{n+1}])^{-1} &= \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta) + \int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&\quad - \frac{\cos\left(z_{0_2}(\gamma) - z_{0_2}(\gamma-\eta) + \int_0^t F^2[z](\gamma) - F^2[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&= \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\cosh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&\quad + \frac{\sinh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\sinh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&\quad - \frac{\cos\left(z_{0_2}(\gamma) - z_{0_2}(\gamma-\eta)\right)\cos\left(\int_0^t F^2[z](\gamma) - F^2[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&\quad + \frac{\sin\left(z_{0_2}(\gamma) - z_{0_2}(\gamma-\eta)\right)\sin\left(\int_0^t F^2[z](\gamma) - F^2[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(d^-[z_{n+1}])^{-1} &= (d^-[z_0])^{-1} + \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)2\sinh^2\left(\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)/2\right)}{\sinh^2(\eta/2)}\\
&\quad + \frac{\sinh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\sinh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}\\
&\quad - \frac{\cos\left(z_{0_2}(\gamma) - z_{0_2}(\gamma-\eta)\right)2\sin^2\left(\left(\int_0^t F^2[z](\gamma) - F^2[z](\gamma-\eta)ds\right)/2\right)}{\sinh^2(\eta/2)}\\
&\quad + \frac{\sin\left(z_{0_2}(\gamma) - z_{0_2}(\gamma-\eta)\right)\sin\left(\int_0^t F^2[z](\gamma) - F^2[z](\gamma-\eta)ds\right)}{\sinh^2(\eta/2)}.
\end{aligned}$$

Take $t \leq 1$ and assume that $\eta \in B(0,1)$, then, using the inequality (see Proposition 8.1)

$$\sup_{\gamma\in\mathbb{B}_r,\beta\in\mathbb{R}} |F[f](\gamma) - F[f](\gamma-\beta)| \leq C_R|\beta|,$$

we have

$$(d^-[z_{n+1}])^{-1} > \frac{1}{R_0} - C_R^1(t^2+t).$$

In the case where $\eta \in B^c(0,1)$, to ensure the decay at infinity, we use the inequalities (obtained from Proposition 8.1)

$$\|F[z_n]\|_{L^\infty(\mathbb{B}_r)} \leq C_R,$$
$$\|z_0(\gamma) - (\gamma,0)\|_{L^\infty(\mathbb{B}_r)} \leq C_R,$$

to conclude

$$(d^-[z_{n+1}])^{-1} > \frac{1}{R_0} - C_R^2(t^2+t).$$

Thus we can take

$$0 < T_A < \min\left\{1, \sqrt{\left(\frac{1}{R_0} - \frac{1}{R}\right)\frac{1}{4\max\{C_R^1, C_R^2\}}}\right\},$$

and then $\|d^-[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R$. We do in the same way for $d^+[z_{n+1}]$:

$$\begin{aligned}(d^+[z_{n+1}])^{-1} &= \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta) + \int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &+ \frac{\cos\left(z_{0_2}(\gamma) + z_{0_2}(\gamma-\eta) + \int_0^t F^2[z](\gamma) + F^2[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)}.\end{aligned}$$

Using the classical trigonometric formulas and the previous inequalities we get

$$\begin{aligned}(d^+[z_{n+1}])^{-1} &= \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\cosh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &+ \frac{\sinh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\sinh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &- \frac{\cos\left(z_{0_2}(\gamma) + z_{0_2}(\gamma-\eta)\right)\cos\left(\int_0^t F^2[z](\gamma) + F^2[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &+ \frac{\sin\left(z_{0_2}(\gamma) + z_{0_2}(\gamma-\eta)\right)\sin\left(\int_0^t F^2[z](\gamma) + F^2[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &= (d^+[z_0])^{-1} + \frac{\cosh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)2\sinh^2\left((\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds)/2\right)}{\cosh^2(\eta/2)} \\ &+ \frac{\sinh\left(z_{0_1}(\gamma) - z_{0_1}(\gamma-\eta)\right)\sinh\left(\int_0^t F^1[z](\gamma) - F^1[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)} \\ &- \frac{\cos\left(z_{0_2}(\gamma) + z_{0_2}(\gamma-\eta)\right)2\sin^2\left((\int_0^t F^2[z](\gamma) + F^2[z](\gamma-\eta)ds)/2\right)}{\cosh^2(\eta/2)} \\ &+ \frac{\sin\left(z_{0_2}(\gamma) + z_{0_2}(\gamma-\eta)\right)\sin\left(\int_0^t F^2[z](\gamma) + F^2[z](\gamma-\eta)ds\right)}{\cosh^2(\eta/2)}.\end{aligned}$$

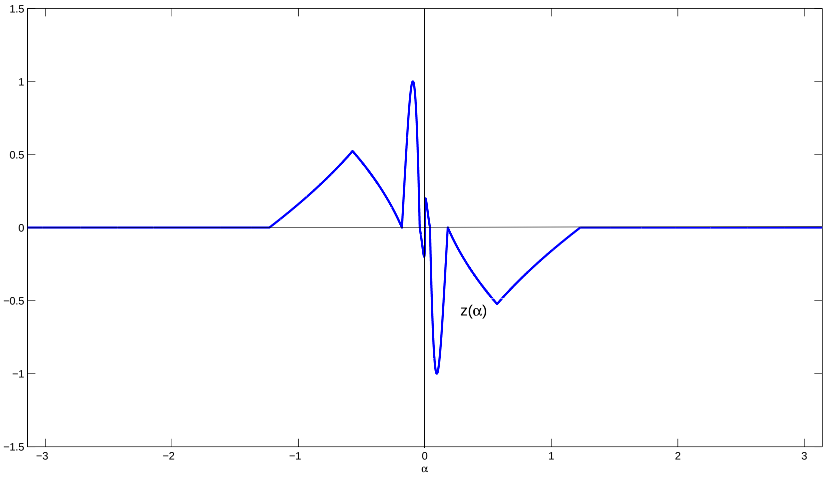

Figure 8.2: The curve in the case $a = 5, b = 3$.

Thus, we can take

$$0 < T_B < \min\left\{1, \sqrt{\left(\frac{1}{R_0} - \frac{1}{R}\right)\frac{1}{2C_R^3}}\right\},$$

and then $\|d^+[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R$. Taking $0 < T < \min\{T_{CK}, T_A, T_B\}$, we conclude the proof. □

8.3 Singularity formation

Now, we can prove the existence of turning waves in the stable Rayleigh-Taylor regime:

Proof of Theorem 8.1. We take $\bar{\rho} = 2$ and $l = \pi/2$. First, we show that there exists curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ such that:

C1. z_i are analytic, odd functions.

C2. $\partial_\alpha z_1(\alpha) > 0, \forall \alpha \neq 0$, $\partial_\alpha z_1(0) = 0$, and $\partial_\alpha z_2(0) > 0$.

C3. $\partial_\alpha v_1(0) = \partial_\alpha \partial_t z_1(0) < 0$.

By integration by parts in expression (2.9) and using the definition of z_i, we obtain

$$\partial_\alpha v_1(0) = 2\partial z_2(0) \int_0^\infty \partial_\alpha z_1(\eta) \sinh(z_1(\eta)) \sin(z_2(\eta)) \left(\frac{1}{(\cosh(z_1(\eta)) - \cos(z_2(\eta)))^2} + \frac{1}{(\cosh(z_1(\eta)) + \cos(z_2(\eta)))^2} \right) d\eta. \quad (8.4)$$

Now, we define piecewise smooth and odd functions (see Figure 8.2)

$$z_1(\alpha) = \alpha - \alpha \exp\left(-\alpha^2\right)$$

and, fixed $2 < b \leq a$ positive constants,

$$z_2(\alpha) = \begin{cases} \dfrac{1}{a}\sin(a\alpha) & \text{if } 0 \leq \alpha \leq \dfrac{\pi}{a}, \\ \sin\left(\pi\dfrac{\alpha - (\pi/a)}{(\pi/a) - (\pi/b)}\right) & \text{if } \dfrac{\pi}{a} < \alpha < \dfrac{\pi}{b}, \\ -\alpha + \dfrac{\pi}{b} & \text{if } \dfrac{\pi}{b} \leq \alpha < \dfrac{\pi}{2}, \\ \alpha - \pi + \dfrac{\pi}{b} & \text{if } \dfrac{\pi}{2} \leq \alpha < \pi(1 - \dfrac{1}{b}), \\ 0 & \text{if } \pi(1 - \dfrac{1}{b}) \leq \alpha. \end{cases} \tag{8.5}$$

We observe that C2 is achieved for this (z_1, z_2). Moreover, these curves satisfy the arc-chord condition in the whole domain. Using the definition of z_2 we have that

$$\partial_\alpha v_1(0) \leq 2\partial z_2(0)(I_a + I_b),$$

where I_a, I_b are the integrals (8.4) on the intervals $(0, \pi/a)$ and $(\pi/b, \pi)$, respectively. Easily, we show $I_b < 0$ and this is independent of the choice of a. The integral I_a is well defined and positive, but goes to zero as a grows. Therefore, by approximating, there exists curves (z_1, z_2) that satisfies the conditions C1–C3.

Now, we consider (z_1, z_2) as the analytic initial datum for the equation (2.9). By a Cauchy-Kowalevski Theorem (see Theorem 8.3), there exists a curve, $w(\alpha, t)$, solution of (2.9) for any $t \in [-T, T]$.

We denote $m(t) = \min_\alpha \partial_\alpha z_1(\alpha, t)$. We notice that, if $\partial_\alpha v_1(0,0) = \partial_\alpha \partial_t z_1(0,0) > 0$ then we have $m(0) = \partial_\alpha z_1(0,0) = 0$ and $\frac{d}{dt}m(t) > 0$ for $0 < t$ small enough. This implies

$$m(\delta) = m(0) + \int_0^\delta m'(s)ds > 0$$

for a small enough $\delta > 0$ and the curve can be parametrized as a graph. If $\partial_\alpha v_1(0,0) = \partial_\alpha \partial_t z_1(0,0) < 0$ then $m(t) < 0$ if t is small enough and the curve can not be parametrized as a graph.

Due to C3, we get the following

1. for $-T < t < 0$, we have $\min_\alpha \partial_\alpha w_1(\alpha, t) > \min_\alpha \partial_\alpha w_1(\alpha, 0) = 0$ and $s_0(\alpha) = s(\alpha, 0) = w(\alpha, -T/2)$ can be parametrized as a graph.

2. At $t = 0$, $w(\alpha, 0) = s(\alpha, T/2) = z(\alpha)$ has a vertical tangent.

3. For $0 < t < T$ we get $\min_\alpha \partial_\alpha w_1(\alpha, t) < \min_\alpha \partial_\alpha w_1(\alpha, 0) = 0$. Thus, for $0 < t < T$, the curve is no longer a graph and the Rayleigh-Taylor condition is not satisfied in a neighbourhood of $\alpha = 0$.

□

Remark 8.1. *This theorem implies that there exist initial data f_0, parametrized as graphs, such that the solution of (2.5) develops a blow up for $\|\partial_x f(t)\|_{L^\infty}$ at finite time t_1.*

We have the following result

Proof of Theorem 8.2. We take

$$z_1(\alpha) = \alpha - \alpha\exp\left(-\alpha^2\right)$$

and, fixed $2 < b \leq a$ positive constants,

$$z_2(\alpha) = \begin{cases} \frac{1}{a}\sin(a\alpha) & \text{if } 0 \leq \alpha \leq \frac{\pi}{a}, \\ \frac{\sin\left(\pi\frac{\alpha-(\pi/a)}{(\pi/a)-(\pi/b)}\right)}{a} & \text{if } \frac{\pi}{a} < \alpha < \frac{\pi}{b}, \\ -\alpha+\frac{\pi}{b} & \text{if } \frac{\pi}{b} \leq \alpha < \frac{\pi}{2}, \\ \alpha-\pi+\frac{\pi}{b} & \text{if } \frac{\pi}{2} \leq \alpha < \pi(1-\frac{1}{b}), \\ 0 & \text{if } \pi(1-\frac{1}{b}) \leq \alpha. \end{cases} \tag{8.6}$$

We take b close enough to 2 such that

$$\frac{\pi}{2}-\frac{\pi}{b} < \gamma.$$

Now we take a large enough such that $\partial_\alpha v_1(0) < 0$ and $1/a < \gamma$. This curve turns, so we can solve backward in time to obtain a graph such that the amplitude is smaller than γ. We conclude the result. □

Remark 8.2. *Notice that this family of functions can be used to prove the existence of singularities for the infinitely deep case $l = \infty$.*

Chapter 9

Global solvability for the confined Muskat problem

9.1 Foreword

In this Chapter we prove global existence of Lipschitz continuous solution for the stable, confined Muskat problem

$$\partial_t f = A[f],$$

with

$$A[f](x) = \frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}} \partial_x \left(\arctan \left(\frac{\tan\left(\frac{\pi}{2l}\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)} \right) \right) d\eta$$
$$+ \frac{2l}{\pi}\text{P.V.}\int_{\mathbb{R}} \partial_x \left(\arctan\left(\tan\left(\frac{\pi}{2l}\frac{f(x)+f(x-\eta)}{2} \right) \tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)\right)\right) d\eta. \quad (9.1)$$

and initial data satisfying some smallness conditions relating the amplitude, the slope and the depth. The cornerstone of the argument is that, for these *small* initial data, both the amplitude and the slope remain uniformly bounded for all positive times. We notice that, for some of these solutions, the slope can grow but it remains bounded. This is very different from the infinite deep case, where the slope of the solutions satisfy a maximum principle.

For equation (9.1) (see Theorem 7.4 in Chapter 7), we obtain the maximum principle for $\|f(t)\|_{L^\infty(\mathbb{R})}$ and the maximum principle for $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ for initial data satisfying the following hypotheses:

$$\|\partial_x f_0\|_{L^\infty(\mathbb{R})} < 1, \quad (9.2)$$

$$\tan\left(\frac{\pi\|f_0\|_{L^\infty(\mathbb{R})}}{2l}\right) < \|\partial_x f_0\|_{L^\infty(\mathbb{R})} \tanh\left(\frac{\pi}{4l}\right), \quad (9.3)$$

and

$$\left(\|\partial_x f_0\|_{L^\infty(\mathbb{R})} + |2(\cos\left(\frac{\pi}{2l}\right) - 2)\sec^4\left(\frac{\pi}{4l}\right)|\|\partial_x f_0\|^3_{L^\infty(\mathbb{R})}\right)\frac{\pi^3}{8l^3}$$
$$\times \frac{\left(1 + \|\partial_x f_0\|_{L^\infty(\mathbb{R})}\left(\|\partial_x f_0\|_{L^\infty(\mathbb{R})} + \frac{\tan\left(\frac{\pi}{2l}\frac{\|\partial_x f_0\|_{L^\infty(\mathbb{R})}}{2}\right)}{\tanh\left(\frac{\pi}{4l}\right)}\right)\right)}{6\tanh\left(\frac{\pi}{4l}\right)}\frac{\pi^2}{4l^2}$$
$$+ 4\tan\left(\frac{\pi}{2l}\|f_0\|_{L^\infty}\right) - 4\|\partial_x f_0\|_{L^\infty(\mathbb{R})}\cos\left(\frac{\pi}{l}\|f_0\|_{L^\infty(\mathbb{R})}\right) < 0 \quad (9.4)$$

These hypotheses are smallness conditions relating $\|\partial_x f_0\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^\infty(\mathbb{R})}$ and the depth. We define $(x(l), y(l))$ as the solution of the system

$$\begin{cases} \tan\left(\frac{\pi x}{2l}\right) - y\tanh\left(\frac{\pi}{4l}\right) = 0 \\ \left(y + |2(\cos\left(\frac{\pi}{2l}\right) - 2)\sec^4\left(\frac{\pi}{4l}\right)|y^3\right)\frac{\left(1+y\left(y+\frac{\tan\left(\frac{\pi}{2l}\frac{y}{2}\right)}{\tanh\left(\frac{\pi}{4l}\right)}\right)\right)}{6\tanh\left(\frac{\pi}{4l}\right)}\left(\frac{\pi}{2l}\right)^5 \\ \qquad +4\tan\left(\frac{\pi}{2l}x\right) - 4y\cos\left(\frac{\pi}{l}x\right) = 0. \end{cases} \quad (9.5)$$

Then, for initial data satisfying

$$\|\partial_x f_0\|_{L^\infty(\mathbb{R})} < y(l) \text{ and } \|f_0\|_{L^\infty(\mathbb{R})} < x(l), \quad (9.6)$$

we show (see Corollary 7.3 in Chapter 7) that

$$\|\partial_x f\|_{L^\infty(\mathbb{R})} \leq 1.$$

These inequalities define a region where the slope of the solution can grow but it is bounded uniformly in time. This region only appears in the finite depth case (see Figure 7.1).

In this Chapter the question of global existence of weak solution (in the sense of Definition 9.1) for (9.1) in the stable regime is adressed. In particular we show the following Theorem:

Theorem 9.1. *Let $f_0(x) \in W^{1,\infty}(\mathbb{R})$ be the initial datum satisfying hypotheses (9.2), (9.3) and (9.4) or (9.6) in the Rayleigh-Taylor stable regime. Then there exists a global solution*

$$f(x,t) \in C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty), W^{1,\infty}(\mathbb{R})).$$

Moreover, if the initial datum satisfy (9.2), (9.3) and (9.4) the solution fulfills the following bounds:

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})} \text{ and } \|\partial_x f(t)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0\|_{L^\infty(\mathbb{R})},$$

while, if the initial datum satisfy (9.6), the solution satisfies the following bounds:

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})} \text{ and } \|\partial_x f(t)\|_{L^\infty(\mathbb{R})} \leq 1.$$

This result excludes the formation of cusps (blow up of the first and second derivatives) and turning waves for these initial data. Notice that in the limit $l \to \infty$, the previous conditions (9.2)-(9.4) reduces to

$$\|\partial_x f_0\|_{L^\infty(\mathbb{R})} < 1,$$

and we recover the result contained in [13].

9.2 The regularized system

In this Section we define the regularized system and obtain some useful *'a priori'* bounds for the amplitude and the slope. To clarify the exposition we write $f^\epsilon(x,t)$ for the solution of the regularized system.

9.2.1 Motivation and methodology

We remark that the term

$$\Xi_1(x,\eta) = \partial_x \arctan\left(\frac{\tan\left(\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\eta}{2}\right)}\right) d\eta,$$

in (9.1) is a singular integral operator, while

$$\Xi_2(x,\eta) = \partial_x \arctan\left(\tan\left(\frac{f(x)+f(x-\eta)}{2}\right)\tanh\left(\frac{\eta}{2}\right)\right) d\eta$$

is not if the curve does not reach the boundaries. In order to remove the singularity while preserving the inner structure, we put a term $|\tanh\left(\frac{\eta}{2}\right)|^\epsilon$ for $0<\epsilon<1/10$ in both kernels. We define

$$\Xi_1^\epsilon(x,\eta) = \partial_x \arctan\left(\frac{\tan\left(\frac{f^\epsilon(x)-f^\epsilon(x-\eta)}{2}\right)|\tanh(\frac{\eta}{2})|^\epsilon}{\tanh\left(\frac{\eta}{2}\right)}\right) d\eta, \tag{9.7}$$

$$\Xi_2^\epsilon(x,\eta) = \partial_x \arctan\left(\frac{\tan\left(\frac{f^\epsilon(x)+f^\epsilon(x-\eta)}{2}\right)}{|\tanh(\frac{\eta}{2})|^\epsilon}\tanh\left(\frac{\eta}{2}\right)\right) d\eta, \tag{9.8}$$

To pass to the limit we use compactness coming from an uniform bound in $L^\infty([0,T],W^{1,\infty}(\mathbb{R}))$. Thus, we need to obtain *'a priori'* bounds for the amplitude and the slope. Taking derivatives in Ξ_i^ϵ, we obtain some terms with positive contribution. So, we attach some diffusive operators to the regularized system. Given a smooth function ϕ, we define

$$\Lambda_l^{1-\epsilon}\phi(x) = \text{PV}\int_{\mathbb{R}} \frac{(\phi(x)-\phi(x-\eta))|\tanh\left(\frac{\eta}{2}\right)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)} d\eta. \tag{9.9}$$

We notice that, if the depth is not $l=\pi/2$, the previous operators should be rescaled and we write the subscript l to keep this dependence in mind. These operators are finite depth versions of the classical $\Lambda^\alpha = (-\Delta)^{\alpha/2}$.

Roughly speaking, there are three different types of *extra* terms appearing in the derivatives of (9.7) and (9.8) that we need to control to obtain the *'a priori'* bound for the slope:

1. There are terms which have an integrable singularity and they appear multiplied by ϵ. In order to handle these terms we add $-\epsilon\alpha_2\Lambda_l^{1-\epsilon}f^\epsilon(x)$ and $-\epsilon\alpha_3\Lambda_l^{1-3\epsilon}f^\epsilon(x)$, with α_2,α_3 positive constants that will be fixed below depending only on the initial datum. These two scales $1-\epsilon$, $1-3\epsilon$, appear naturally due to the nonlinearity present in (9.1).

2. There are terms which are nonlinear versions of $\Lambda_l - \Lambda_l^{1-\epsilon}$ and $\Lambda_l - \Lambda_l^{1-3\epsilon}$. These terms go to zero due to the convergence of the operators but they are not multiplied by ϵ. In order to handle these terms, we add $-(\Lambda_l - \Lambda_l^{1-\epsilon})f^\epsilon(x)$ and $-\alpha_4(\Lambda_l-\Lambda_l^{1-3\epsilon})f^\epsilon(x)$, with α_4 a positive constant taht will be fixed later.

3. To absorb the nonsingular terms we add $-\sqrt{\epsilon}\alpha_1 f^\epsilon(x)$, with α_1 a positive constant that will be chosen below. We notice that, as $\epsilon < 1/10$, the square root converges to zero less than linearly. This factor will be used because the contribution of some terms is $O(\epsilon^a)$ with $1/2 < a < 1$.

Once that the *'a priori'* bounds are achieved, we should prove global solvability in H^3 for the regularized system. To get this bound, we add $\epsilon\partial_x^2 f^\epsilon(x)$. We also regularize the initial datum. We take $\mathcal{J} \in C_c^\infty(\mathbb{R})$, $\mathcal{J} \geq 0$ and $\|\mathcal{J}\|_{L^1} = 1$, a symmetric mollifier and define $\mathcal{J}_\epsilon(x) = \mathcal{J}(x/\epsilon)/\epsilon$. Given $f_0 \in W^{1,\infty}(\mathbb{R})$ we define the initial data for the regularized system as

$$f^\epsilon(x,0) = \frac{\mathcal{J}_\epsilon * f_0}{1+\epsilon^2 x^2}. \tag{9.10}$$

Putting all together, we define the regularized system

$$\begin{aligned}\partial_t f^\epsilon(x) = &-\sqrt{\epsilon}\alpha_1 f^\epsilon(x) + \epsilon\partial_x^2 f^\epsilon(x) - \epsilon\alpha_2\Lambda_l^{1-\epsilon} f^\epsilon(x) \\ &- \epsilon\alpha_3\Lambda_l^{1-3\epsilon} f^\epsilon(x) - (\Lambda_l - \Lambda_l^{1-\epsilon})f^\epsilon(x) - \alpha_4(\Lambda_l - \Lambda_l^{1-3\epsilon})f^\epsilon(x) \\ &+ 2\text{P.V.}\int_{\mathbb{R}} \Xi_1^\epsilon(x,\eta)d\eta + 2\text{P.V.}\int_{\mathbb{R}} \Xi_2^\epsilon(x,\eta)d\eta,\end{aligned} \tag{9.11}$$

where α_i are constants that will be fixed below depending only on the initial datum f_0. We remark that $f_0^\epsilon \in H^k(\mathbb{R})$ for all $k > 0$. We need to obtain that $\|f_0^\epsilon\|_{L^\infty(\mathbb{R})} \to \|f_0\|_{L^\infty(\mathbb{R})}$ and $\|\partial_x f_0^\epsilon\|_{L^\infty(\mathbb{R})} \to \|\partial_x f_0\|_{L^\infty(\mathbb{R})}$. Notice that, due to the continuity of f_0,

$$f_0^\epsilon = \mathcal{J}_\epsilon * f_0 - \epsilon^2 x^2 f_0^\epsilon \to f_0$$

uniformly on any compact set in $\mathbb{R}$. Since $\partial_x f_0 \in L^\infty(\mathbb{R})$, we get $\partial_x f_0 \in L^1_{loc}(\mathbb{R})$ and then, as $\epsilon \to 0$, we have $\mathcal{J}_\epsilon * \partial_x f_0 \to \partial_x f_0$ a.e. Thus, we have $\|\mathcal{J}_\epsilon * \partial_x f_0\|_{L^\infty(\mathbb{R})} \to \|\partial_x f_0\|_{L^\infty(\mathbb{R})}$. Moreover

$$\partial_x f_0^\epsilon = \mathcal{J}_\epsilon * \partial_x f_0 - \frac{\epsilon^2 x^2 \mathcal{J}_\epsilon * \partial_x f_0}{1+\epsilon^2 x^2} - \mathcal{J}_\epsilon * f_0 \frac{2\epsilon^2 x}{(1+\epsilon^2 x^2)^2} \to \partial_x f_0 \text{ a.e. on every compact,}$$

and we get $\|\partial_x f_0^\epsilon\|_{L^\infty(\mathbb{R})} \to \|\partial_x f_0\|_{L^\infty(\mathbb{R})}$. Therefore, we have that if f_0 satisfies the hypotheses (9.2),(9.3) and (9.4), f_0^ϵ also satisfy these hypotheses if ϵ is small enough. Moreover, if $f_0, \partial_x f_0$ satisfy (9.6) the same remains valid for f^ϵ and $\partial_x f^\epsilon$ if ϵ is small enough.

We use some properties of the operators $\Lambda_l^{1-\epsilon}$. For the reader's convenience, we collect them in the following lemma:

Lemma 9.1. *For the operators $\Lambda_l^{1-\epsilon}$ (see (9.9)), the following properties hold:*

1. *$\Lambda_l^{1-\epsilon}$ is L^2-symmetric.*
2. *$\Lambda_l^{1-\epsilon}$ is positive definite.*
3. *Let ϕ be a Schwartz function. Then, they converge acting on ϕ as ϵ goes to zero:*

$$\|(\Lambda_l - \Lambda_l^{1-\epsilon})\phi\|_{L^1(\mathbb{R})} \leq c\|\phi\|_{W^{2,1}(\mathbb{R})}\epsilon.$$

4. *Let ϕ be a Schwartz function. Then, the derivative can be written in two different forms as*

$$\begin{aligned}\Lambda_l^{1-\epsilon}\partial_x\phi(x) = &(1-\epsilon)P.V.\int_{\mathbb{R}} \frac{\left(\partial_x\phi(x) - \frac{\phi(x)-\phi(\eta)}{\sinh(x-\eta)}\right)|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta \\ &+ P.V.\int_{\mathbb{R}} \frac{\left(\partial_x\phi(x) - \frac{\phi(x)-\phi(\eta)}{\tanh(x-\eta)}\right)|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta + 4\partial_x\phi(x) \\ = &P.V.\int_{\mathbb{R}} \frac{\partial_x(\phi(x)-\phi(x-\eta))|\tanh(\eta/2)|^\epsilon}{\sinh^2(\frac{\eta}{2})}d\eta.\end{aligned}$$

Proof. Proof of 1: Changing variables appropriately, we have

$$\begin{aligned}\int_{\mathbb{R}}\psi(x)\Lambda_l^{1-\epsilon}\phi(x)dx &= \int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\psi(x)\frac{(\phi(x)-\phi(x-\eta))|\tanh(\eta/2)|^\epsilon}{\sinh^2(\frac{\eta}{2})}d\eta dx\\ &= \int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\psi(x)\frac{(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2(\frac{(x-\eta)}{2})}d\eta dx\\ &= -\int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\psi(\eta)\frac{(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2(\frac{(x-\eta)}{2})}d\eta dx\\ &= \frac{1}{2}\int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\frac{(\psi(x)-\psi(\eta))(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2(\frac{(x-\eta)}{2})}d\eta dx\\ &= \int_{\mathbb{R}}\Lambda_l^{1-\epsilon}\psi(x)\phi(x)dx.\end{aligned}$$

Proof of 2: In the same way, we compute

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)\Lambda_l^{1-\epsilon}\phi(x)dx &= \int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\phi(x)\frac{(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2(\frac{(x-\eta)}{2})}d\eta dx\\ &= \frac{1}{2}\int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\frac{(\phi(x)-\phi(\eta))^2|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2(\frac{(x-\eta)}{2})}d\eta dx>0.\end{aligned}$$

Proof of 3: We recall some useful facts: if $|y|\geq\delta>0$, due to the Mean Value Theorem, we get

$$||\tanh(y)|^\epsilon-1|=\left|\frac{d}{d\gamma}|\tanh(y)|^\gamma\Big|_{\gamma=\xi}\epsilon\right|\leq\epsilon|\log\left(|\tanh(y)|\right)|, \tag{9.12}$$

and

$$\int_0^\infty|\log\left(|\tanh(y)|\right)|dy\leq c<\infty. \tag{9.13}$$

We have

$$\begin{aligned}\|(\Lambda_l-\Lambda_l^{1-\epsilon})\phi\|_{L^1(\mathbb{R})}&=\int_{\mathbb{R}}\left|\text{P.V.}\int_{\mathbb{R}}\frac{(\phi(x)-\phi(x-\eta)-\partial_x\phi(x)\eta)\left(1-|\tanh(\eta/2)|^\epsilon\right)}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\right|dx\\ &=\int_{\mathbb{R}}\left|\text{P.V.}\int_{\mathbb{R}}\int_0^1\int_0^1\frac{\eta^2((s-1)\partial_x^2\phi(x+r(s-1)\eta))\left(1-|\tanh(\eta/2)|^\epsilon\right)}{\sinh^2\left(\frac{\eta}{2}\right)}dsdrd\eta\right|dx.\end{aligned}$$

We split the integral in η

$$\begin{aligned}\|(\Lambda_l-\Lambda_l^{1-\epsilon})\phi\|_{L^1(\mathbb{R})}&\leq\|\partial_x^2\phi\|_{L^1(\mathbb{R})}\int_{B(0,\epsilon)}\frac{\eta^2}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\\ &\quad+\|\partial_x^2\phi\|_{L^1(\mathbb{R})}\int_{B^c(0,\epsilon)}\frac{\epsilon|\log\left(|\tanh(\eta/2)|\right)|\eta^2}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\\ &\leq\|\partial_x^2\phi\|_{L^1(\mathbb{R})}4\epsilon+\|\partial_x^2\phi\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}\frac{\epsilon|\log\left(|\tanh(\eta/2)|\right)|\eta^2}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\leq c\|\phi\|_{W^{2,1}(\mathbb{R})}\epsilon.\end{aligned}$$

Proof of 4: We only prove the first expression, being the second one straightforward. We use the cancellation coming from the principal value to write

$$\Lambda_l^{1-\epsilon}\phi(x)=\text{PV}\int_{\mathbb{R}}\frac{(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2\left(\frac{x-\eta}{2}\right)}-2\partial_x\phi(x)\frac{|\tanh((x-\eta)/2)|^\epsilon}{\tanh((x-\eta)/2)}d\eta. \tag{9.14}$$

Notice that, in this formulation,

$$\text{P.V.}\int = \lim_{R\to\infty}\int_{B^c(x,1/R)\cap B(x,R)}.$$

We take the derivative of (9.14). Due to the principal value the boundaries contribute with some terms:

$$\begin{aligned}
&\frac{(\phi(x)-\phi(x+R))|\tanh(-R/2)|^{\epsilon}}{\sinh^2\left(\frac{-R}{2}\right)}-2\partial_x\phi(x)\frac{|\tanh(-R/2)|^{\epsilon}}{\tanh(-R/2)}\\
&\quad-\left(\frac{(\phi(x)-\phi(x+1/R))|\tanh((-1/R)/2)|^{\epsilon}}{\sinh^2\left(\frac{-1/R}{2}\right)}-2\partial_x\phi(x)\frac{|\tanh((-1/R)/2)|^{\epsilon}}{\tanh((-1/R)/2)}\right)\\
&\quad+\frac{(\phi(x)-\phi(x-1/R))|\tanh((1/R)/2)|^{\epsilon}}{\sinh^2\left(\frac{1/R}{2}\right)}-2\partial_x\phi(x)\frac{|\tanh((1/R)/2)|^{\epsilon}}{\tanh((1/R)/2)}\\
&\quad-\left(\frac{(\phi(x)-\phi(x-R))|\tanh(R/2)|^{\epsilon}}{\sinh^2\left(\frac{R}{2}\right)}-2\partial_x\phi(x)\frac{|\tanh(R/2)|^{\epsilon}}{\tanh(R/2)}\right)\to 4\partial_x\phi(x),
\end{aligned}$$

when $\epsilon\to 0$. Now we take the derivative under the integral sign in (9.14), and obtain

$$\begin{aligned}
\Lambda_l^{1-\epsilon}\partial_x\phi(x)&=4\partial_x\phi(x)+\text{P.V.}\int_{\mathbb{R}}\frac{(\epsilon-2)\partial_x\phi(x)(-\sinh(x-\eta))|\tanh((x-\eta)/2)|^{\epsilon}}{2\cosh\left(\frac{x-\eta}{2}\right)\sinh^3\left(\frac{x-\eta}{2}\right)}\\
&\quad+\frac{(\phi(x)-\phi(\eta))|\tanh((x-\eta)/2)|^{\epsilon}(\epsilon-\cosh((x-\eta)/2)-1)}{2\cosh\left(\frac{x-\eta}{2}\right)\sinh^3\left(\frac{x-\eta}{2}\right)}d\eta\\
&=(1-\epsilon)\text{P.V.}\int_{\mathbb{R}}\frac{\left(\partial_x\phi(x)-\frac{\phi(x)-\phi(\eta)}{\sinh(x-\eta)}\right)|\tanh((x-\eta)/2)|^{\epsilon}}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta\\
&\quad+\text{PV}\int_{\mathbb{R}}\frac{\left(\partial_x\phi(x)-\frac{\phi(x)-\phi(\eta)}{\tanh(x-\eta)}\right)|\tanh((x-\eta)/2)|^{\epsilon}}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta+4\partial_x\phi(x).
\end{aligned}$$

□

9.2.2 Maximum principle for f^{ϵ}

In this section we prove an *a priori* bound for f^{ϵ}. To simplify notation we define

$$\theta=\frac{f^{\epsilon}(x)-f^{\epsilon}(\eta)}{2}\text{ and }\bar{\theta}=\frac{f^{\epsilon}(x)+f^{\epsilon}(\eta)}{2}. \tag{9.15}$$

Proposition 9.1. *Let $f_0\in W^{1,\infty}(\mathbb{R})$ be the initial datum in (9.1), define f_0^{ϵ} as in (9.10) and let f^{ϵ} be the classical solution of (9.11) corresponding to the initial datum f_0^{ϵ}. Then f^{ϵ} verifies*

$$\|f^{\epsilon}(t)\|_{L^{\infty}(\mathbb{R})}\leq\|f_0^{\epsilon}\|_{L^{\infty}(\mathbb{R})}\leq\|f_0\|_{L^{\infty}(\mathbb{R})}.$$

Moreover, if f_0 has a sign, then this sign is preserved during the evolution of f^{ϵ}.

Proof. Changing variables and taking the derivative we obtain that (9.11) is equivalent to

$$\begin{aligned}\partial_t f^\epsilon(x) &= -(4+\sqrt{\epsilon}\alpha_1)f^\epsilon(x)+\epsilon\partial_x^2 f^\epsilon(x)-\epsilon\alpha_2\Lambda_l^{1-\epsilon}f^\epsilon(x)\\ &\quad -\epsilon\alpha_3\Lambda_l^{1-3\epsilon}f^\epsilon(x)-(\Lambda_l-\Lambda_l^{1-\epsilon})f^\epsilon(x)-\alpha_4(\Lambda_l-\Lambda_l^{1-3\epsilon})f^\epsilon(x)\\ &\quad +\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f^\epsilon(x)\sec^2(\theta)\frac{|\tanh((x-\eta)/2)|^\epsilon}{\tanh((x-\eta)/2)}+(\epsilon-1)\tan(\theta)\frac{|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2((x-\eta)/2)}}{1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}}d\eta\\ &\quad +\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f^\epsilon(x)\sec^2(\bar\theta)\frac{\tanh((x-\eta)/2)}{|\tanh((x-\eta)/2)|^\epsilon}+\frac{(1-\epsilon)\tan(\bar\theta)}{|\tanh((x-\eta)/2)|^\epsilon\cosh^2((x-\eta)/2)}}{1+\frac{\tan^2(\bar\theta)\tanh^2((x-\eta)/2)}{|\tanh((x-\eta)/2)|^{2\epsilon}}}d\eta.\end{aligned} \tag{9.16}$$

If $\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}=\max f^\epsilon(x,t)$ we define $f^\epsilon(x_t)=\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}$. Then, we have $\partial_t f^\epsilon(x_t)=\frac{d}{dt}\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}$ (see [26] for the details). If $\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}=\min f^\epsilon(x,t)$ we write $f^\epsilon(x_t)=-\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}$ and we get $-\partial_t f^\epsilon(x_t)=\frac{d}{dt}\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}$. We compute

$$\begin{aligned}4f^\epsilon(x)&=2\int_{\mathbb{R}}\partial_\eta\arctan\left(\tan(f^\epsilon(x))\frac{\tanh(\eta/2)}{|\tanh(\eta/2)|^\epsilon}\right)d\eta\\ &=-\int_{\mathbb{R}}\frac{1}{\cosh^2(\eta/2)}\frac{(\epsilon-1)\tan(f^\epsilon(x))|\tanh(\eta/2)|^\epsilon}{|\tanh(\eta/2)|^{2\epsilon}+\tanh^2(\eta/2)\tan^2(f^\epsilon(x))}d\eta\\ &=-\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^{-\epsilon}}{\cosh^2(\eta/2)}\frac{(\epsilon-1)\cot(f^\epsilon(x))}{\cot^2(f^\epsilon(x))+\tanh^{2-2\epsilon}(\eta/2)}d\eta.\end{aligned}$$

By notational convenience we use the notation $\sigma=\frac{\pi}{2}-f^\epsilon(x_t)$ and we define

$$\Pi^\epsilon=\frac{\tan(\theta)}{\tanh^{2-2\epsilon}(\eta/2)+\tan^2(\theta)}+\frac{\tan(\sigma)}{\tan^2(\sigma)+\tanh^{2-2\epsilon}(\eta/2)}-\frac{\cot(\bar\theta)}{\tanh^{2-2\epsilon}(\eta/2)+\cot^2(\bar\theta)}.$$

Evaluating (9.16) in x_t we have

$$\begin{aligned}\partial_t f^\epsilon(x_t)&=-\sqrt{\epsilon}\alpha_1 f^\epsilon(x_t)+\epsilon\partial_x^2 f^\epsilon(x_t)-\epsilon\alpha_2\Lambda_l^{1-\epsilon}f^\epsilon(x_t)-\epsilon\alpha_3\Lambda_l^{1-3\epsilon}f^\epsilon(x_t)\\ &\quad -(\Lambda_l-\Lambda_l^{1-\epsilon})f^\epsilon(x_t)-\alpha_4(\Lambda_l-\Lambda_l^{1-3\epsilon})f^\epsilon(x_t)\\ &\quad -(1-\epsilon)\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^{-\epsilon}}{\cosh^2(\eta/2)}\Pi^\epsilon d\eta.\end{aligned}$$

Using the definition of $\bar\theta$ and classical trigonometric identities, we have

$$\cot(\bar\theta)=\tan\left(\frac{\pi}{2}-\bar\theta\right)=\tan\left(\frac{\pi}{2}-f(x_t)+\theta\right)=\frac{\tan(\frac{\pi}{2}-f(x_t))+\tan(\theta)}{1-\tan(\frac{\pi}{2}-f(x_t))\tan(\theta)}.$$

Putting together all the terms in Π^ϵ, we obtain

$$\begin{aligned}\Pi^\epsilon&=\frac{\tan(\sigma)\tan^2(\theta)[1+\tan^2(\sigma)|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}{[\tan^2(\sigma)+|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}\\ &\quad+\frac{2\tan^2(\sigma)\tan(\theta)[1-|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}{[\tan^2(\sigma)+|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}\\ &\quad+\frac{\tan^2(\sigma)\tan(\theta)[1+\tan^2(\theta)|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}{[\tan^2(\theta)+|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}\\ &\quad+\frac{2\tan(\sigma)\tan^2(\theta)[1-|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}{[\tan^2(\theta)+|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)][(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)]}\\ &\quad+\frac{(\tan(\sigma)+\tan(\theta))\tan(\sigma)\tan(\theta)}{(\tan(\sigma)+\tan(\theta))^2+(1-\tan(\sigma)\tan(\theta))^2|\tanh|^{2-2\epsilon}\left(\frac{\eta}{2}\right)}.\end{aligned}$$

Assuming that $0 < f^\epsilon(x_t) = \max_x f^\epsilon(x)$, then $0 < \tan(\theta), \tan(\sigma)$ and we obtain $\Pi^\epsilon \geq 0$ and $\partial_t f^\epsilon(x_t) \leq 0$. In the case $f^\epsilon(x_t) = \min_x f^\epsilon(x) < 0$, we have $0 > \tan(\theta), \tan(\sigma)$ and we get $\Pi^\epsilon \leq 0$ and $\partial_t f^\epsilon(x_t) \geq 0$. Integrating this in time, we get

$$\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0^\epsilon\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})},$$

where in the last step we use the definition (9.10). In order to prove that the initial sign propagates we observe that if f_0 is positive (respectively negative) the same remains valid for f_0^ϵ. Assume now that $f_0 \geq 0$ and suppose that the line $y = 0$ is reached (if this line is not reached at any time t we are done). We write $f^\epsilon(x_t) = \min_x f^\epsilon(x,t) = 0$. We have $\tan(\theta) < 0$, $\sigma = \pi/2$ and we get $\Pi^\epsilon \leq 0$ and $\partial_t f^\epsilon(x_t) \geq 0$. If $f_0 \leq 0$ we denote $f^\epsilon(x_t) = \max_x f^\epsilon(x,t) = 0$. We have $\tan(\theta) > 0$ and $\Pi^\epsilon \geq 0$. Integrating in time we conclude the result. □

9.2.3 Maximum principle for $\partial_x f^\epsilon$

In this section we prove an *a priori* bound for $\partial_x f^\epsilon$. We define

$$\mu_1(t) = \frac{\tan(\theta)}{\tanh\left(\frac{x_t-\eta}{2}\right)}, \qquad \mu_2(t) = \tan\left(\bar{\theta}\right)\tanh\left(\frac{x_t-\eta}{2}\right),$$

where θ and $\bar{\theta}$ are defined in (9.15) and x_t is a critical point for $\partial_x f^\epsilon(x)$. We will use some bounds for μ_1 and, for the reader's convenience, we collect them in the following lemma:

Lemma 9.2. *Let f_0 be an initial datum that fulfills (9.2), (9.3) and (9.4) (or (9.6)), and let f^ϵ be the solution with initial datum f_0^ϵ defined in (9.10). Then for μ_1 the following inequalities hold*

1. *If $|x_t - \eta| \geq 1$, due to (9.3), we have*

$$|\mu_1(t)| \leq \frac{\tan\left(\|f^\epsilon(t)\|_{L^\infty}\right)}{\tanh\left(\frac{1}{2}\right)} \leq \frac{\tan\left(\|f_0\|_{L^\infty}\right)}{\tanh\left(\frac{1}{2}\right)} < \|\partial_x f_0^\epsilon\|_{L^\infty} < 1. \tag{9.17}$$

2. *If $|x_t - \eta| \leq 1$, we get*

$$|\mu_1(t)| \leq c\left(\|f_0\|^2_{L^\infty(\mathbb{R})} + 1\right)\|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})}. \tag{9.18}$$

3. *If $|x_t - \eta| \leq 1$ and x_t is the point where $\partial_x f^\epsilon$ reaches its maximum,*

$$\mu_1(t) - \partial_x f^\epsilon(x_t) \leq \frac{(x_t-\eta)^2}{48\tanh\left(\frac{1}{2}\right)}\left(|\partial_x f^\epsilon(x_t)| + 5|\partial_x f^\epsilon(x_t)|^3\right). \tag{9.19}$$

4. *If $|x_t - \eta| \leq 1$ and $\mu_1(t) - \partial_x f^\epsilon(x_t) \geq 0$*

$$\begin{aligned} 0 &\leq \mu_1^2(t) - (\partial_x f^\epsilon(x_t))^2 \\ &\leq \frac{(x_t-\eta)^2}{48\tanh\left(\frac{1}{2}\right)}\left(|\partial_x f^\epsilon(x_t)| + 5|\partial_x f^\epsilon(x_t)|^3\right)\left(|\partial_x f^\epsilon(x_t)| + \frac{\tan\left(\frac{|\partial_x f^\epsilon(x_t)|}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right). \end{aligned} \tag{9.20}$$

Proof. To prove this lemma we use the following splitting

$$\frac{\tan(\theta)}{\tanh((x_t-\eta)/2)} = \frac{\tan(\theta)-\theta}{\tanh((x_t-\eta)/2)} + \frac{\theta}{\tanh((x_t-\eta)/2)},$$

Taylor's theorem and the appropriate bounds using Proposition 9.1. □

First, we assume $\partial_x f^\epsilon(x_t) = \max_x \partial_x f^\epsilon(x,t)$. Notice that we can take $0<\epsilon<1/10$ small enough to ensure that $f^\epsilon(x,0)$ defined in (9.10) also fulfills the hypotheses (9.2)-(9.4). From (9.16), taking one derivative and using Lemma 9.1, we get

$$\begin{aligned}\partial_t\partial_x f^\epsilon(x_t) &= -8\partial_x f^\epsilon(x_t) - \sqrt{\epsilon}\alpha_1\partial_x f^\epsilon(x_t) + \epsilon\partial_x^3 f^\epsilon(x_t) - \epsilon\alpha_2\Lambda_l^{1-\epsilon}f^\epsilon(x_t)\\ &\quad -\epsilon\alpha_3\Lambda_l^{1-3\epsilon}f^\epsilon(x_t) - \alpha_4(\Lambda_l - \Lambda_l^{1-3\epsilon})f^\epsilon(x_t) - (\Lambda_l - \Lambda_l^{1-\epsilon})f^\epsilon(x_t)\\ &\quad +\text{P.V.}\int_{\mathbb{R}}\mathcal{I}_1 d\eta + \text{P.V.}\int_{\mathbb{R}}\mathcal{I}_2 d\eta + \text{P.V.}\int_{\mathbb{R}}\mathcal{I}_3 d\eta\end{aligned} \tag{9.21}$$

where $\mathcal{I}_1$ is the integral corresponding to Ξ_1^ϵ, $\mathcal{I}_2$ is the integral corresponding to Ξ_2^ϵ and

$$\mathcal{I}_3 = +\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\partial_x\theta\left(1-|\tanh(\eta/2)|^{2\epsilon}\mu_1^2(t)\right)d\eta}{\sinh^2(\eta/2)\cos^2(\theta)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} \tag{9.22}$$

$$-\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^{-\epsilon}\partial_x\bar\theta d\eta}{\cosh^2(\eta/2)\cos^2(\bar\theta)\left(1+\frac{\tan^2(\bar\theta)\tanh^2(\eta/2)|}{|\tanh(\eta/2)|^{2\epsilon}}\right)} \tag{9.23}$$

$$+\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{\mu_2^2(t)|\tanh((x_t-\eta)/2)|^{-3\epsilon}2\partial_x\bar\theta d\eta}{\cosh^2((x_t-\eta)/2)\cos^2(\bar\theta)\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x-\eta)/2)|}{|\tanh((x-\eta)/2)|^{2\epsilon}}\right)^2}. \tag{9.24}$$

We have

$$\mathcal{I}_1 = \Gamma_1 + \epsilon\Gamma_2$$

where

$$\begin{aligned}\Gamma_1 = \frac{\left(|\tanh((x_t-\eta)/2)|^{3\epsilon}-1\right)\Gamma_1^1}{\left(1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}\right)^2} + \frac{\left(|\tanh((x_t-\eta)/2)|^{\epsilon}-1\right)\Gamma_1^2}{\left(1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}\right)^2}\\ + \frac{\Gamma_1^1+\Gamma_1^2}{\left(1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}\right)^2},\end{aligned} \tag{9.25}$$

with

$$\Gamma_1^1 = \frac{-(\partial_x f^\epsilon(x_t))^2\mu_1}{\cos^2(\theta)\tanh^2((x_t-\eta)/2)} + \frac{\mu_1^3}{\cosh^2((x_t-\eta)/2)} + \frac{\partial_x f^\epsilon(x_t)\mu_1^2}{\sinh^2((x_t-\eta)/2)\cos^2(\theta)},$$

$$\Gamma_1^2 = \frac{\mu_1 - \partial_x f^\epsilon(x_t)}{\sinh^2((x_t-\eta)/2)} - \frac{\partial_x f^\epsilon(x_t)\mu_1^2}{\cosh^2((x_t-\eta)/2)} + \frac{(\partial_x f^\epsilon(x_t))^2\mu_1}{\cos^2(\theta)}.$$

The second term is given by

$$\Gamma_2 = \frac{|\tanh((x_t-\eta)/2)|^{3\epsilon}\Gamma_2^1}{2\left(1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}\right)^2} + \frac{|\tanh((x_t-\eta)/2)|^{\epsilon}\Gamma_2^2}{2\left(1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}\right)^2}, \tag{9.26}$$

where

$$\Gamma_2^1 = \mu_1^2(t)\frac{\frac{\partial_x f^\epsilon(x_t)}{\cos^2(\theta)} + \frac{-\mu_1(t)}{\cosh^2((x_t-\eta)/2)}}{\sinh^2((x_t-\eta)/2)}, \text{ and } \Gamma_2^2 = \frac{\frac{\partial_x f^\epsilon(x_t)}{\cos^2(\theta)} + \frac{-\mu_1(t)}{\cosh^2((x_t-\eta)/2)}}{\sinh^2((x_t-\eta)/2)}.$$

We compute

$$\mathcal{I}_2 = \Omega_1 + \epsilon\Omega_2,$$

with

$$\Omega_1=\frac{\left(|\tanh((x_t-\eta)/2)|^{-3\epsilon}-1\right)\Omega_1^1}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2}+\frac{\left(|\tanh((x_t-\eta)/2)|^{-\epsilon}-1\right)\Omega_1^2}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2}+\frac{\Omega_1^1+\Omega_1^2}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2},\qquad(9.27)$$

where

$$\Omega_1^1=-\frac{\partial_x f^\epsilon(x_t)\mu_2^2(t)\sec^2(\bar\theta)}{\cosh^2((x_t-\eta)/2)}+(\partial_x f^\epsilon(x_t))^2\mu_2^3(t)\sec^2(\bar\theta)-\frac{\mu_2^3(t)}{\cosh^2((x_t-\eta)/2)}-(\partial_x f^\epsilon(x_t))^2\mu_2(t)\tanh^2((x_t-\eta)/2)\sec^4(\bar\theta)-\frac{\tan^2(\bar\theta)\mu_2(t)}{\cosh^4((x_t-\eta)/2)},$$

$$\Omega_1^2=\frac{\partial_x f^\epsilon(x_t)\sec^2(\bar\theta)-\mu_2(t)}{\cosh^2((x_t-\eta)/2)}+(\partial_x f^\epsilon(x_t))^2\mu_2(t)\sec^2(\bar\theta).$$

The second term is given by

$$\Omega_2=\frac{|\tanh((x_t-\eta)/2)|^{-3\epsilon}\left(\frac{\partial_x f^\epsilon(x_t)\mu_2^2(t)\sec^2(\bar\theta)}{2\cosh^2((x_t-\eta)/2)}+\frac{\tan^2(\bar\theta)\mu_2(t)}{2\cosh^4((x_t-\eta)/2)}\right)}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2}+\frac{|\tanh((x_t-\eta)/2)|^{-\epsilon}\left(\frac{-\partial_x f^\epsilon(x_t)\sec^2(\bar\theta)}{2\cosh^2((x_t-\eta)/2)}-\frac{\tan(\bar\theta)}{2\cosh^4((x_t-\eta)/2)\tanh((x_t-\eta)/2)}\right)}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2}.\qquad(9.28)$$

We need to obtain the local decay $\|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})}\leq\|\partial_x f^\epsilon(0)\|_{L^\infty(\mathbb{R})}$ for $0\leq t<t^*$. Assuming the classical solvability for (9.11) with an initial datum f_0 fulfilling the hypotheses (9.2)-(9.4) we have that $f^\epsilon(x,\delta)$ also fulfills (9.2),(9.3) and (9.4) if $0\leq\delta<<1$ is small enough. Recall that $\partial_x f^\epsilon(x_\delta)=\|\partial_x f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})}$ and $\partial_x\theta>0$. The linear terms in (9.21) have the appropriate sign and they will be used to control the the positive contributions of the nonlinear terms. We need to prove that $\partial_t\partial_x f^\epsilon(x_\delta)<0$. For the sake of simplicity, we split the proof of this inequality in different lemmas.

Lemma 9.3. *If $\alpha_2>2\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})$, we have*

$$\mathcal{I}_3\leq\epsilon c\tan^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta).$$

Proof. Using the linear term $\Lambda_l^{1-\epsilon}$ to control (9.22), we have

$$A_1=\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\partial_x\theta\left(1-|\tanh(\eta/2)|^{2\epsilon}\mu_1^2(\delta)\right)d\eta}{\sinh^2(\eta/2)\cos^2(\theta)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}-\epsilon\frac{\alpha_2}{2}\Lambda_l^{1-\epsilon}\partial_x f^\epsilon(x_\delta)$$

$$=\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\partial_x\theta\left(\frac{1}{\cos^2(\theta)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}-\frac{\alpha_2}{2}\right)d\eta}{\sinh^2(\eta/2)}-\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\partial_x\theta|\tanh(\eta/2)|^{2\epsilon}\mu_1^2(\delta)d\eta}{\sinh^2(\eta/2)\cos^2(\theta)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}<0,$$

if $\alpha_2/2 > \sec^2(\|f_0\|_{L^\infty(\mathbb{R})})$. Due to $\partial_x f^\epsilon(x_\delta) = \|\partial_x f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})}$, we have $\partial_x\bar{\theta} > 0$. Then, the term (9.23) is

$$A_2 = -\epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{-\epsilon}\partial_x\bar{\theta}d\eta}{\cosh^2(\eta/2)\cos^2(\bar{\theta})\left(1+\frac{\tan^2(\bar{\theta})\tanh^2(\eta/2)|}{|\tanh(\eta/2)|^{2\epsilon}}\right)} < 0.$$

The term (9.24) is

$$A_3 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{\mu_2^2(\delta)|\tanh((x_\delta-\eta)/2)|^{-3\epsilon}2\partial_x\bar{\theta}d\eta}{\cosh^2((x_\delta-\eta)/2)\cos^2(\bar{\theta})\left(1+\frac{\tan^2(\bar{\theta})\tanh^2((x_\delta-\eta)/2)|}{|\tanh((x_\delta-\eta)/2)|^{2\epsilon}}\right)^2}$$
$$\leq \epsilon c\tan^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta).$$

□

This kind of terms will be absorbed by α_1. We have to deal with $\mathcal{I}_1$. We start with the term corresponding to Γ_2^2 in (9.26). We write

$$A_4 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh((x_\delta-\eta)/2)|^\epsilon\left(\frac{\partial_x f^\epsilon(x_\delta)}{\cos^2(\theta)}+\frac{-\mu_1(\delta)}{\cosh^2((x_\delta-\eta)/2)}\right)}{2\sinh^2((x_\delta-\eta)/2)\left(1+\frac{\tan^2(\theta)|\tanh((x_\delta-\eta)/2)|^{2\epsilon}}{\tanh^2((x_\delta-\eta)/2)}\right)^2}d\eta.$$

Lemma 9.4. *If $\alpha_2 > 2\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})$, we have*

$$A_4 \leq c\epsilon\partial_x f(x_\delta)\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)+c\epsilon\partial_x f(x_\delta)\frac{\alpha_2}{2}.$$

Proof. We split

$$A_4 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh((x_\delta-\eta)/2)|^\epsilon\left(\frac{\partial_x f^\epsilon(x_\delta)}{\cos^2(\theta)}-\partial_x f^\epsilon(x_\delta)+\frac{-\mu_1(\delta)}{\cosh^2((x_\delta-\eta)/2)}+\mu_1(t)\right)}{2\sinh^2((x_\delta-\eta)/2)\left(1+\frac{\tan^2(\theta)|\tanh((x_\delta-\eta)/2)|^{2\epsilon}}{\tanh^2((x_\delta-\eta)/2)}\right)^2}d\eta$$
$$+\epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh((x_\delta-\eta)/2)|^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\mu_1(\delta)\right)}{2\sinh^2((x_\delta-\eta)/2)\left(1+\frac{\tan^2(\theta)|\tanh((x_\delta-\eta)/2)|^{2\epsilon}}{\tanh^2((x_\delta-\eta)/2)}\right)^2}d\eta = B_1 + B_2.$$

Since $0<\delta<<1$ is small enough to ensure that the hypotheses (9.2)-(9.4) hold at time δ, we have that, if $|\eta|>1$,

$$|\mu_1(\delta)| \leq \frac{\tan(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})})}{\tanh(1/2)} < \partial_x f^\epsilon(x_\delta). \tag{9.29}$$

The term B_1 is not singular and can be bounded using (9.18) and (9.29):

$$|B_1| \leq \epsilon\text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\frac{\partial_x f^\epsilon(x_\delta)\tan^2(\theta)+|\mu_1(\delta)|\tanh^2(\eta/2)}{2\sinh^2(\eta/2)}d\eta$$
$$\leq c\epsilon\partial_x f(x_\delta)\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right).$$

We compute

$$B_2 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\frac{\tan(\theta)-\theta}{\tanh(\eta/2)}-\frac{\theta}{\tanh(\eta/2)}+\frac{2\theta}{\eta}-\frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = C_1 + C_2,$$

with

$$C_1=\epsilon\text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\frac{|\tanh(\eta/2)|^\epsilon\left(-\frac{\tan(\theta)-\theta}{\tanh(\eta/2)}-\frac{\theta}{\tanh(\eta/2)}+\frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta=D_1+D_2,$$

$$C_2=\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta.$$

Using the Mean Value Theorem, we bound the inner term D_1 as

$$|D_1|\leq c\epsilon\partial_x f^\epsilon(x_\delta).$$

Due to (9.29), the outer term is

$$|D_2|\leq\epsilon\text{P.V.}\int_{B^c(0,1)}\frac{|\mu_1(\delta)|+\partial_x f^\epsilon(x_\delta)}{2\sinh^2(\eta/2)}d\eta\leq\epsilon c\partial_x f^\epsilon(x_\delta).$$

Putting all together, we obtain

$$|C_1|\leq\epsilon c\partial_x f^\epsilon(x_\delta).$$

Then, using the diffusion given by $\Lambda_l^{1-\epsilon}$ to control C_2, we get

$$\begin{aligned}C_2-\epsilon\frac{\alpha_2}{2}\Lambda_l^{1-\epsilon}\partial_x f^\epsilon(x_\delta)=&\,\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta\\&-\epsilon\frac{\alpha_2}{2}\left((1-\epsilon)\text{PV}\int_{\mathbb{R}}\frac{\left(\partial_x f^\epsilon(x_\delta)-\frac{f^\epsilon(x_\delta)-f^\epsilon(x_\delta-\eta)}{\sinh(\eta)}\right)|\tanh(\eta/2)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\right.\\&\left.+\text{P.V.}\int_{\mathbb{R}}\frac{\left(\partial_x f^\epsilon(x_\delta)-\frac{f^\epsilon(x_\delta)-f^\epsilon(x_\delta-\eta)}{\tanh(\eta)}\right)|\tanh(\eta/2)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta+4\partial_x f^\epsilon(x_\delta)\right).\end{aligned}$$

Due to $|\eta/\sinh(\eta)|<1$ and $0<\epsilon<1/10$, some terms have the appropriate sign:

$$\epsilon\frac{\alpha_2}{2}\left((1-\epsilon)\text{PV}\int_{\mathbb{R}}\frac{\left(\partial_x f^\epsilon(x_\delta)-\frac{f^\epsilon(x_\delta)-f^\epsilon(x_\delta-\eta)}{\sinh(\eta)}\right)|\tanh(\eta/2)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta+4\partial_x f^\epsilon(x_\delta)\right)\geq 0,$$

thus we can neglect their contribution. Furthermore, we have

$$\begin{aligned}C_2-\epsilon\frac{\alpha_2}{2}\Lambda_l^{1-\epsilon}\partial_x f^\epsilon(x_\delta)<&\,\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)|^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta\\&-\epsilon\frac{\alpha_2}{2}\text{PV}\int_{\mathbb{R}}\frac{\left(\partial_x f^\epsilon(x_\delta)-\frac{2\theta}{\eta}+\frac{2\theta}{\eta}-\frac{2\theta}{\tanh(\eta)}\right)|\tanh(\eta/2)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta\\\leq&\,\epsilon\text{P.V.}\int_{\mathbb{R}}\frac{|\tanh(\eta/2)^\epsilon\left(\partial_x f^\epsilon(x_\delta)-\frac{2\theta}{\eta}\right)}{\sinh^2(\eta/2)}\left(\frac{1}{2\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}-\frac{\alpha_2}{L^2}\right)d\eta\\&-\epsilon\frac{\alpha_2}{2}\text{PV}\int_{\mathbb{R}}\frac{\left(\frac{2\theta}{\eta}-\frac{2\theta}{\tanh(\eta)}\right)|\tanh(\eta/2)|^\epsilon}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta.\end{aligned}$$

Taking $\alpha_2/2 > 1$ and using the Mean Value Theorem, we get

$$C_2 - \epsilon\frac{\alpha_2}{2}\Lambda_l^{1-\epsilon}\partial_x f^\epsilon(x_\delta) < \epsilon\frac{\alpha_2}{2}\partial_x f^\epsilon(x_\delta)c.$$

Combining these terms we conclude this result. □

The term corresponding to Γ_2^1 in (9.26) is

$$A_5 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\mu_1^2(\delta)\left(\frac{\partial_x f^\epsilon(x_\delta)}{\cos^2(\theta)} + \frac{-\mu_1(\delta)}{\cosh^2(\eta/2)}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta.$$

Lemma 9.5. *If $\alpha_3 > 1$, we have*

$$A_5 \leq c\epsilon\partial_x f(x_\delta)\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right) + c\epsilon\partial_x f(x_\delta)\alpha_3.$$

Proof.

$$A_5 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\mu_1^2(\delta)\left(\frac{\partial_x f^\epsilon(x_\delta)}{\cos^2(\theta)} - \partial_x f^\epsilon(x_\delta) + \frac{-\mu_1(\delta)}{\cosh^2(\eta/2)} + \mu_1(t)\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta$$
$$+ \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\mu_1^2(\delta)\left(\partial_x f^\epsilon(x_\delta) - \mu_1(\delta)\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = B_3 + B_4.$$

The term B_3 is not singular and can be bounded as B_1. We obtain

$$|B_3| \leq c\epsilon\partial_x f(x_\delta)\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right).$$

The term B_4 is similar to the term B_2, so we use the same splitting:

$$B_4 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\mu_1^2(\delta)\left(\partial_x f^\epsilon(x_\delta) - \frac{\tan(\theta)-\theta}{\tanh(\eta/2)} - \frac{\theta}{\tanh(\eta/2)} + \frac{2\theta}{\eta} - \frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = C_3 + C_4,$$

with

$$|C_3| \leq \epsilon c\partial_x f^\epsilon(x_\delta),$$

and

$$C_4 = \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\left(\mu_1^2(\delta)-1\right)\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta$$
$$+ \epsilon\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{3\epsilon}\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta}\right)}{2\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = D_3 + D_4.$$

Due to (9.17) and the Mean Value Theorem, we have

$$D_3 < 0.$$

The term D_4 can be controlled as C_2 by using the diffusion $\Lambda_l^{1-3\epsilon}$. Taking $\alpha_3 > 1$ and using the Mean Value Theorem, we get

$$D_4 - \epsilon\alpha_3\Lambda_l^{1-3\epsilon}\partial_x f^\epsilon(x_\delta) < \epsilon\alpha_3\partial_x f^\epsilon(x_\delta)c.$$

Putting it all together we obtain the result. □

We are done with Γ_2^1, thus, using the previous bound for Γ_2^2, we are done with Γ_2 in (9.26). The terms in Γ_1 are not multiplied by ϵ and we have to obtain this decay from the integral. We write

$$A_6 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\Gamma_1^2}{\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}.$$

Lemma 9.6. *We have*

$$A_6 \leq c\epsilon\partial_x f(x_\delta)\left(\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right).$$

Proof. We have

$$A_6 = B_5 + B_6 + B_7,$$

with

$$B_5 = \text{P.V.}\left(\int_{B(0,\epsilon)} + \int_{B^c(0,\epsilon)\cap B(0,1)} + \int_{B^c(0,1)}\right)\frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\left(-\partial_x f^\epsilon(x_\delta)\mu_1^2(\delta)\right)}{\cosh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta,$$

$$B_6 = \text{P.V.}\left(\int_{B(0,\epsilon)} + \int_{B^c(0,\epsilon)\cap B(0,1)} + \int_{B^c(0,1)}\right)\frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\left(\partial_x f^\epsilon(x_\delta)\right)^2\mu_1(\delta)}{\cos^2(\theta)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta,$$

$$B_7 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\left(\mu_1(\delta) - \partial_x f^\epsilon(x_\delta)\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta.$$

The term B_5 is not singular and can be bounded using (9.12) and (9.13) as follows:

$$|B_5| \leq 4\epsilon\partial_x f^\epsilon(x_\delta) + \epsilon\int_{B(0,1)} |\log\left(|\tanh(\eta/2)|\right)|\partial_x f^\epsilon(x_\delta)d\eta$$
$$+\epsilon\int_{B^c(0,1)}\frac{|\log\left(|\tanh(\eta/2)|\right)|\partial_x f^\epsilon(x_\delta)}{\cosh^2(\eta/2)}d\eta \leq c\epsilon\partial_x f^\epsilon(x_\delta).$$

We can bound B_6 in the same way,

$$|B_6| \leq 4\epsilon\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta) + \epsilon\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\int_{\mathbb{R}} |\log\left(|\tanh(\eta/2)|\right)|\partial_x f^\epsilon(x_\delta)d\eta$$
$$\leq c\epsilon\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta).$$

We split the term B_7 as follows

$$B_7 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\left(\frac{\tan(\theta)-\theta}{\tanh(\eta/2)} + \frac{\theta}{\tanh(\eta/2)} - \frac{2\theta}{\eta} + \frac{2\theta}{\eta} - \partial_x f^\epsilon(x_\delta)\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = C_5 + C_6,$$

where

$$C_5 = \text{P.V.}\left(\int_{B(0,\epsilon)} + \int_{B^c(0,\epsilon)}\right)\frac{\left(|\tanh(\eta/2)|^\epsilon - 1\right)\left(\frac{\tan(\theta)-\theta}{\tanh(\eta/2)} + \frac{\theta}{\tanh(\eta/2)} - \frac{2\theta}{\eta}\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta \leq c\epsilon\partial_x f^\epsilon(x_\delta),$$

$$C_6 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(1 - |\tanh(\eta/2)|^\epsilon\right)\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta}\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta.$$

To bound C_6 we need to use the diffusion coming from $\Lambda_l - \Lambda_l^{1-\epsilon}$. Notice that, according to Lemma 9.1, we have

$$\left(\Lambda_l - \Lambda_l^{1-\epsilon}\right)\partial_x\phi(x) = (1-\epsilon)\text{PV}\int_{\mathbb{R}} \frac{\left(\partial_x\phi(x) - \frac{\phi(x)-\phi(\eta)}{\sinh(x-\eta)}\right)(1-|\tanh((x-\eta)/2)|^\epsilon)}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta$$
$$+\epsilon\text{PV}\int_{\mathbb{R}} \frac{\left(\partial_x\phi(x) - \frac{\phi(x)-\phi(\eta)}{\sinh(x-\eta)}\right)}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta$$
$$+\text{PV}\int_{\mathbb{R}} \frac{\left(\partial_x\phi(x) - \frac{\phi(x)-\phi(\eta)}{\tanh(x-\eta)}\right)(1-|\tanh((x-\eta)/2)|^\epsilon)}{\sinh^2\left(\frac{x-\eta}{2}\right)}d\eta,$$

and, when evaluating in the point where $\partial_x\phi(x)$ reaches its maximum, the first two terms are positive and they can be neglected. We get

$$C_6 - \left(\Lambda_l - \Lambda_l^{1-\epsilon}\right)\partial_x f^\epsilon(x_\delta) < \text{P.V.}\int_{\mathbb{R}} \frac{(1-|\tanh(\eta/2)|^\epsilon)\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta}\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}$$
$$-\text{PV}\int_{\mathbb{R}} \frac{\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta} + \frac{2\theta}{\eta} - \frac{f^\epsilon(x_\delta)-f^\epsilon(\eta)}{\tanh(\eta)}\right)(1-|\tanh(\eta/2)|^\epsilon)}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta$$
$$< \text{PV}\int_{\mathbb{R}} \frac{\left(\partial_x f^\epsilon(x_\delta) - \frac{2\theta}{\eta}\right)(1-|\tanh(\eta/2)|^\epsilon)}{\sinh^2\left(\frac{\eta}{2}\right)}\left(\frac{1}{\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} - 1\right)d\eta$$
$$-\text{PV}\int_{\mathbb{R}} \frac{\left(\frac{2\theta}{\eta} - \frac{f^\epsilon(x_\delta)-f^\epsilon(\eta)}{\tanh(\eta)}\right)(1-|\tanh(\eta/2)|^\epsilon)}{\sinh^2\left(\frac{\eta}{2}\right)}d\eta \leq c\epsilon\partial_x f^\epsilon(x_\delta),$$

where in the last step we have used the previous splitting in $B(0,\epsilon)$ and $\mathbb{R} - B(0,\epsilon)$, (9.12) and (9.13). This concludes the result. □

Now that we have finished with Γ_1^2, the term with Γ_1^1 is

$$A_7 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{3\epsilon} - 1\right)\Gamma_1^1}{\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta.$$

We have

Lemma 9.7. *If $\alpha_4 > \sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)$, we have*

$$A_7 \leq c\epsilon\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)^7\partial_x f^\epsilon(x_\delta) + \alpha_4 c\epsilon\partial_x f^\epsilon(x_\delta).$$

Proof. We decompose

$$A_7 = B_8 + B_9 + B_{10},$$

with

$$B_8 = \text{P.V.}\int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{3\epsilon} - 1\right)\mu_1^3(\delta)}{\cosh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta,$$

$$B_9 = \text{P.V.} \int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{3\epsilon} - 1\right)\left(-(\partial_x f^\epsilon(x_t))^2 \mu_1(\delta)\right)}{\cos^2(\theta)\left(1 + \frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} \left(\frac{1}{\tanh^2(\eta/2)} - \frac{1}{\sinh^2(\eta/2)}\right) d\eta,$$

$$B_{10} = \text{P.V.} \int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{3\epsilon} - 1\right)\left(\partial_x f^\epsilon(x_t)\mu_1^2(\delta) - (\partial_x f^\epsilon(x_t))^2 \mu_1(\delta)\right)}{\sinh^2(\eta/2)\cos^2(\theta)\left(1 + \frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} d\eta.$$

The terms B_8 and B_9 are not singular and can be bounded in a way similar to the previous proof. We get

$$|B_8| \leq c\epsilon \left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right) + 1\right)^3 \partial_x f^\epsilon(x_\delta),$$

and, using (9.13),

$$|B_9| \leq c\epsilon \left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right) + 1\right)^3 \partial_x f^\epsilon(x_\delta).$$

The term B_{10} is

$$B_{10} = \text{P.V.} \int_{\mathbb{R}} \frac{\partial_x f^\epsilon(x_\delta)\mu_1(\delta)\left(\frac{\tan(\theta)-\theta}{\tanh(\eta/2)} + \frac{\theta}{\tanh(\eta/2)} - \frac{2\theta}{\eta} + \frac{2\theta}{\eta} - \partial_x f^\epsilon(x_t)\right)}{\left(|\tanh(\eta/2)|^{3\epsilon} - 1\right)^{-1}\sinh^2(\eta/2)\cos^2(\theta)\left(1 + \frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} d\eta$$
$$= C_7 + C_8 + C_9.$$

Since the terms C_7 and C_8 are not singular, they can be bounded as

$$|C_7| + |C_8| \leq c\left(\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right) + 1\right)^7 \epsilon \partial_x f^\epsilon(x_\delta).$$

We use the diffusive term $\Lambda_l - \Lambda_l^{1-3\epsilon}$ to control the term C_9 because of its singularity.

$$C_9 - \alpha_4\left(\Lambda_l - \Lambda_l^{1-3\epsilon}\right)\partial_x f^\epsilon(x_\delta) < \text{P.V.} \int_{\mathbb{R}} \frac{\left(1 - |\tanh(\eta/2)|^{3\epsilon}\right)\left(\partial_x f^\epsilon(x_t) - \frac{2\theta}{\eta}\right)}{\sinh^2(\eta/2)}$$
$$\cdot \left(\frac{\partial_x f^\epsilon(x_t)\mu_1(\delta)}{\cos^2(\theta)\left(1 + \frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} - \alpha_4\right) d\eta$$
$$- \alpha_4 \text{PV} \int_{\mathbb{R}} \frac{\left(\frac{2\theta}{\eta} - \frac{f^\epsilon(x_\delta) - f^\epsilon(\eta)}{\tanh(\eta)}\right)\left(1 - |\tanh(\eta/2)|^{3\epsilon}\right)}{\sinh^2\left(\frac{\eta}{2}\right)} d\eta \leq \alpha_4 c\epsilon \partial_x f^\epsilon(x_\delta),$$

where we use $\alpha_4 > \sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)$. This concludes the result. □

In order to finish with Γ_1, we have to bound the term

$$A_8 = \text{P.V.} \int_{\mathbb{R}} \frac{\Gamma_1^1 + \Gamma_1^2}{\left(1 + \frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2} d\eta,$$

with Γ_1^i defined in (9.25). This term, akin to the singular term in Chapter 7, is bounded using the hypotheses (9.2) and (9.3).

Lemma 9.8. *Using (9.2)-(9.4), we obtain*

$$A_8 \leq \frac{\left(\partial_x f^\epsilon(x_\delta) + 5(\partial_x f^\epsilon(x_\delta))^3\right)\left(1 + \partial_x f^\epsilon(x_\delta)\left(\partial_x f^\epsilon(x_\delta) + \frac{\tan\left(\frac{\partial_x f^\epsilon(x_\delta)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{6\tanh(1/2)\cos^2(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})})}.$$

Proof. Using classical trigonometric identities we can write

$$\Gamma_1^1 = \frac{\partial_x f^\epsilon(x_\delta)\sec^2(\theta)\mu_1^2(\delta)}{\sinh^2(\eta/2)} + (\partial_x f^\epsilon(x_\delta))^2\mu_1^3(\delta)\sec^2(\theta) + \frac{\mu_1^3(\delta)}{\sinh^2(\eta/2)} - \frac{(\partial_x f^\epsilon(x_\delta))^2\sec^4(\theta)\mu_1(\delta)}{\tanh^2(\eta/2)} - \frac{\mu_1(\delta)\tan^2(\theta)}{\sinh^4(\eta/2)},$$

$$\Gamma_1^2 = -\frac{\partial_x f^\epsilon(x_\delta)\sec^2(\theta)}{\sinh^2(\eta/2)} + (\partial_x f^\epsilon(x_\delta))^2\mu_1(\delta)\sec^2(\theta) + \frac{\mu_1(\delta)}{\sinh^2(\eta/2)},$$

$$A_8 = \text{P.V.}\left(\int_{B(0,1)} + \int_{B^c(0,1)}\right)\frac{(\partial_x f^\epsilon(x_\delta)\mu_1^2(\delta) + \mu_1(\delta)(1-(\partial_x f^\epsilon(x_\delta))^2) - \partial_x f^\epsilon(x_\delta)))}{\cos^2(\theta)\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta = B_{11}+B_{12}.$$

Therefore, as in Chapter 7, the sign of A_8 is the sign of

$$Q_1(\mu_1(\delta)) = \partial_x f^\epsilon(x_\delta)\mu_1^2(\delta) + \mu_1(\delta)(1-(\partial_x f^\epsilon(x_\delta))^2) - \partial_x f^\epsilon(x_\delta).$$

The roots of Q_1 are $\partial_x f^\epsilon(x_\delta)$ and $-1/\partial_x f^\epsilon(x_\delta)$. So, if we have

$$|\mu_1(\delta)| \leq \min\left\{\|\partial_x f^\epsilon(\delta)\|_{L^\infty}, \frac{1}{\|\partial_x f^\epsilon(\delta)\|_{L^\infty}}\right\},$$

then we can ensure that this contribution is negative. Since (9.29), we get

$$B_{12} = \text{P.V.}\int_{B^c(0,1)}\frac{\sec^2(\theta)\left(\partial_x f^\epsilon(x_\delta)\mu_1^2(\delta) + \mu_1(\delta)(1-(\partial_x f^\epsilon(x_\delta))^2) - \partial_x f^\epsilon(x_\delta))\right)}{\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta < 0.$$

Using the cancellation when $\mu_1(\delta) = \partial_x f(x_\delta)$, we obtain

$$B_{11} = \text{P.V.}\int_{B(0,1)}\frac{Q_1(\mu_1(\delta))}{\cos^2(\theta)\sinh^2(\eta/2)\left(1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}\right)^2}d\eta, \tag{9.30}$$

where

$$Q_1(\mu_1(\delta)) = \partial_x f^\epsilon(x_\delta)(\mu_1^2(\delta) - (\partial_x f^\epsilon(x_\delta))^2) + (1-(\partial_x f^\epsilon(x_\delta))^2)(\mu_1(\delta) - \partial_x f^\epsilon(x_\delta)).$$

We remark that $\mu_1(\delta) - \partial_x f^\epsilon(x_\delta) < \mu_1(\delta) + \partial_x f^\epsilon(x_\delta)$. We consider the cases given by the sign and the size of $\mu_1(\delta)$.

1. Case $\mu_1(\delta) > \partial_x f(x_\delta)$: In this case, we have $\mu_1(\delta) - \partial_x f(x_\delta) > 0$ and $\mu_1(\delta) + \partial_x f(x_\delta) > 0$. Using the definition of θ in (9.15) and the fact that $|\eta| \leq 1$, we have (9.19) (see Lemma 9.2). Notice that, in this case, we have $\mu_1^2(\delta) - (\partial_x f(x_\delta))^2 > 0$ and we get (9.20). Due to (9.19) and (9.20) we obtain

$$B_{11} \leq \frac{(\partial_x f^\epsilon(x_\delta) + 5(\partial_x f^\epsilon(x_\delta))^3)\left(1+\partial_x f^\epsilon(x_\delta)\left(\partial_x f^\epsilon(x_\delta) + \frac{\tan\left(\frac{\partial_x f^\epsilon(x_\delta)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{48\tanh(1/2)\cos^2(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})})}\int_{B(0,1)}\frac{\eta^2 d\eta}{\sinh^2\left(\frac{\eta}{2}\right)}$$
$$\leq \frac{(\partial_x f^\epsilon(x_\delta) + 5(\partial_x f^\epsilon(x_\delta))^3)\left(1+\partial_x f^\epsilon(x_\delta)\left(\partial_x f^\epsilon(x_\delta) + \frac{\tan\left(\frac{\partial_x f^\epsilon(x_\delta)}{2}\right)}{\tanh\left(\frac{1}{2}\right)}\right)\right)}{6\tanh(1/2)\cos^2(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})})}. \tag{9.31}$$

2. Case $-\partial_x f^\epsilon(x_\delta) < \mu_1(\delta) < \partial_x f^\epsilon(x_\delta) > 0$*:* In this case we have $\mu_1(\delta) - \partial_x f^\epsilon(x_\delta) \leq 0$ and $\mu_1(\delta) + \partial_x f^\epsilon(x_\delta) > 0$. Therefore, we get $B_{11} < 0$ and we can neglect it.

3. Case $\mu_1(\delta) < -\partial_x f^\epsilon(x_\delta)$*:* We remark that in this case we have $\mu_1(\delta) - \partial_x f^\epsilon(x_\delta) \leq 0$ and $\mu_1(\delta) + \partial_x f^\epsilon(x_\delta) \leq 0$. We split

$$\mu_1(\delta) + \partial_x f^\epsilon(x_\delta) = \frac{\tan(\theta) - \theta}{\tanh(\eta/2)} + \theta\left(\frac{1}{\tanh(\eta/2)} - \frac{2}{\eta}\right) + \frac{2\theta}{\eta} + \partial_x f^\epsilon(x_\delta). \tag{9.32}$$

The last term is now positive due to the definition of $\partial_x f^\epsilon(x_\delta)$. Then, in this case, we have

$$\partial_x f^\epsilon(x_\delta)(\mu_1(\delta) - \partial_x f(x_\delta))\left(\frac{2\theta}{x_t - \eta} + \partial_x f^\epsilon(x_t)\right) \leq 0,$$

and we can neglect its contribution. Using Taylor's theorem in (9.32), we obtain the bound (9.20) and (9.31). ☐

We are done with $\mathcal{I}_1$ in (9.21) and now we move on to $\mathcal{I}_2$. These terms are easier because the integrals are not singular. With the same ideas as before we can bound the term involving Ω_2:

Lemma 9.9. *The contribution of* Ω_2 *is bounded by*

$$\epsilon\left|\int_{\mathbb{R}} \Omega_2 d\eta\right| \leq \epsilon c \sec^4\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta).$$

Proof. Using (9.28), we get

$$\begin{aligned}\left|\int_{\mathbb{R}} \Omega_2 d\eta\right| \leq\ & 6\partial_x f^\epsilon(x_\delta)\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(1 + \sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right) \\ & + 4\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\tan\left(\|f(\delta)\|_{L^\infty(\mathbb{R})}\right) \\ & + \left|\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{-\epsilon}\frac{\tan(\bar{\theta})}{2\cosh^4(\eta/2)\tanh(\eta/2)}}{\left(1 + \frac{\tan^2(\bar{\theta})\tanh^2(\eta/2)}{|\tanh(\eta/2)|^{2\epsilon}}\right)^2} d\eta\right|.\end{aligned}$$

We compute

$$\begin{aligned}&\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{-\epsilon}\tan(\bar{\theta})}{2\cosh^4(\eta/2)\tanh(\eta/2)}\left(\frac{1}{\left(1 + \frac{\tan^2(\bar{\theta})\tanh^2(\eta/2)}{|\tanh(\eta/2)|^{2\epsilon}}\right)^2} - 1\right) d\eta \\ &\quad + \int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^{-\epsilon}\left(\tan(\bar{\theta}) - \tan(f^\epsilon(x_\delta))\right)}{2\cosh^4(\eta/2)\tanh(\eta/2)} d\eta \\ &\leq 2\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\partial_x f^\epsilon(x_\delta) + 2\sec^4\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\tan\left(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})}\right).\end{aligned}$$

Putting everything together, and using (9.3), we obtain the result. ☐

We are left with Ω_1 in (9.27). First, we consider

$$A_9 = \int_{\mathbb{R}} \frac{\Omega_1^1 + \Omega_1^2}{\left(1 + \frac{\tan^2(\bar{\theta})\tanh^2((x_t - \eta)/2)}{|\tanh((x_t - \eta)/2)|^{2\epsilon}}\right)^2} d\eta.$$

Lemma 9.10. *The term A_9 is bounded as*

$$|A_9| \leq 4\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(\tan\left(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})}\right)+\partial_x f^\epsilon(x_\delta)\right).$$

Proof. Using classical trigonometric identities, we compute

$$A_9 = \int_{\mathbb{R}} \frac{-\partial_x f^\epsilon(x_\delta)\mu_2^2(\delta)+\left((\partial_x f^\epsilon(x_\delta))^2-1\right)\mu_2(\delta)+\partial_x f^\epsilon(x_\delta)}{\cosh^2(\eta/2)\cos^2(\bar\theta)\left(1+\frac{\tan^2(\bar\theta)\tanh^2((x_t-\eta)/2)}{|\tanh((x_t-\eta)/2)|^{2\epsilon}}\right)^2}d\eta$$
$$\leq 4\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(\tan\left(\|f^\epsilon(\delta)\|_{L^\infty(\mathbb{R})}\right)+\partial_x f^\epsilon(x_\delta)\right). \quad (9.33)$$

□

We have to bound the terms containing Ω_1^i. These terms are

$$A_{10} = \int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{-3\epsilon}-1\right)\Omega_1^1}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2(\eta/2)}{|\tanh(\eta/2)|^{2\epsilon}}\right)^2}d\eta \text{ and } A_{11} = \int_{\mathbb{R}} \frac{\left(|\tanh(\eta/2)|^{-\epsilon}-1\right)\Omega_1^2}{\left(1+\frac{\tan^2(\bar\theta)\tanh^2(\eta/2)}{|\tanh(\eta/2)|^{2\epsilon}}\right)^2}d\eta.$$

To obtain the decay with ϵ we split the integral in the regions $B(0,\epsilon)$ and $B^c(0,\epsilon)$ as before.

Lemma 9.11. *The terms A_{10} and A_{11} are bounded by*

$$|A_{10}|+|A_{11}| \leq c\partial_x f^\epsilon(x_\delta)\sec^4\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(1+\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right)\left(\epsilon^{7/10}+\epsilon\right).$$

Proof. Using this splitting, $0<\epsilon<1/10$, (9.12), (9.13) and (9.3), we get

$$A_{10} \leq c\partial_x f^\epsilon(x_\delta)\sec^4\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(1+\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right)\left(\int_0^\epsilon \frac{d\eta}{|\tanh(\eta/2)|^{3/10}}+\epsilon\right).$$

With the same ideas and using (9.2), we have

$$A_{11} \leq c\partial_x f^\epsilon(x_\delta)\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\left(1+\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right)\left(\int_0^\epsilon \frac{d\eta}{|\tanh(\eta/2)|^{1/10}}+\epsilon\right).$$

In order to estimate the decay with ϵ of these integrals, we compute

$$\int_0^\epsilon \frac{1}{|\tanh(\eta/2)|^{3/10}}-\frac{1}{|\eta/2|^{3/10}}d\eta+\int_0^\epsilon \frac{d\eta}{|\eta/2|^{3/10}} \leq \epsilon+2\epsilon^{7/10},$$

$$\int_0^\epsilon \frac{1}{|\tanh(\eta/2)|^{/10}}-\frac{1}{|\eta/2|^{1/10}}d\eta+\int_0^\epsilon \frac{d\eta}{|\eta/2|^{1/10}} \leq \epsilon+2\epsilon^{9/10}.$$

□

We have the following result concerning the evolution of the slope:

Proposition 9.2. *Let $f_0 \in W^{1,\infty}(\mathbb{R})$ be the initial datum in (9.1) satisfying (9.2)-(9.4), define f_0^ϵ as in (9.10) and let f^ϵ be the classical solution of (9.11) corresponding to the initial datum f_0^ϵ. Then, f^ϵ verifies*

$$\|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0^\epsilon\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0\|_{L^\infty(\mathbb{R})} < 1.$$

Proof. For the sake of simplicity we split the proof in different steps.

Step 1: local decay Combining B_{11} in (9.30) and A_9 in Lemma 9.10, and using the bounds (9.31) and (9.33) and the hypothesis (9.4) we obtain

$$B_{11} + |A_9| < 0.$$

We take $\alpha_4 = 2\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)$, $\alpha_3 = 2$, $\alpha_2 = 3\left(1+\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right)$. Since we have a term $\sqrt{\epsilon}$ and $0 < \epsilon < 1/10$, we can compare the bounds in Lemmas 9.3- 9.11 with $-\sqrt{\epsilon}\alpha_1\partial_x f^\epsilon(x_\delta)$ if $\alpha_1 = \alpha_1\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)$ is chosen big enough. The universal constant c in all these bounds can be $c = 1000$. We have shown that for every $0 < \delta << 1$ small enough, there is local in time decay. As δ is positive and arbitrary, we have

$$\|\partial_x f^\epsilon(t)\|_{L^\infty} \leq \|\partial_x f^\epsilon(0)\|_{L^\infty}, \text{ for } 0 \leq t < t^*.$$

Step 2: from local decay to an uniform bound Then, in the worst case, we have

$$\|\partial_x f^\epsilon(t^*)\|_{L^\infty(\mathbb{R})} = \|\partial_x f_0^\epsilon\|_{L^\infty(\mathbb{R})} \text{ and } \|f^\epsilon(t^*)\|_{L^\infty(\mathbb{R})} \leq \|f_0^\epsilon\|_{L^\infty(\mathbb{R})}.$$

These inequalities ensure that the hypotheses (9.2)-(9.4) hold at time $t = t^*$ and $\|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})}$ decays again.

Step 3: the case where $f^\epsilon(x_t) = \min_x \partial_x f^\epsilon(x,t)$ This case follows the same ideas, and we conclude, thus, the proof of this proposition. □

Proposition 9.3. *Let $f_0 \in W^{1,\infty}(\mathbb{R})$ be the initial datum in (9.1) satisfying (9.6) and define f_0^ϵ as in (9.10). Let f^ϵ be the classical solution of (9.11) corresponding to the initial datum f_0^ϵ. Then, f^ϵ verifies*

$$\|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})} < 1 \quad \forall t > 0.$$

Proof. The region delimited by $(x(l), y(l))$ is below the region with maximum principle (see [26]). Then, in the worst case, at some $t^* > 0$ we have that $(\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})}, \|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})})$ fulfills the hypotheses (9.2)-(9.4). From them the result follows. □

9.3 Global existence for f^ϵ

In this section we obtain *'a priori'* estimates in $H^3(\mathbb{R})$ that ensure the global existence for the regularized systems (9.11) for initial data satisfying hypotheses (9.2)-(9.4) or (9.6). First, notice that if the initial datum satisfies hypotheses (9.2)-(9.4), by Propositions 9.1 and 9.2, the solution satisfies

$$\|f^\epsilon(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty(\mathbb{R})} \text{ and } \|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq 1. \tag{9.34}$$

If the initial datum satisfies (9.6), by Propositions 9.1 and 9.3, the solution to the regularized system again satisfies the bounds (9.34). Then we have the following Proposition:

Proposition 9.4. *Let $f_0 \in W^{1,\infty}(\mathbb{R}) \cap L^2(\mathbb{R})$ be the initial datum in (9.1) satisfying (9.2)-(9.4) or (9.6) and define f_0^ϵ as in (9.10). Then for every $\epsilon > 0$ and $T > 0$ there exists a solution $f^\epsilon(x,t) \in C([0,T], H^3(\mathbb{R}))$.*

Proof. We have to bound the L^2 norm of the function and its third derivative. We split the proof in different steps.

Step 1: L^2 norm of the function We have

$$\frac{1}{2}\frac{d}{dt}\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})} = -\sqrt{\epsilon}\alpha_1\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})} - \epsilon\|\partial_x f^\epsilon(t)\|^2_{L^2(\mathbb{R})} - I_1 + I_2 + I_3$$

Using Lemma 9.1, we get

$$I_1 = \alpha_2\epsilon\int_{\mathbb{R}} f^\epsilon(x)\Lambda_l^{1-\epsilon}f^\epsilon(x)dx + \alpha_3\epsilon\int_{\mathbb{R}} f^\epsilon(x)\Lambda_l^{1-3\epsilon}f^\epsilon(x)dx \\ + \int_{\mathbb{R}} f^\epsilon(x)\left(\Lambda_l - \Lambda_l^{1-\epsilon}\right)f^\epsilon(x)dx + \alpha_4\int_{\mathbb{R}} f^\epsilon(x)\left(\Lambda_l - \Lambda_l^{1-3\epsilon}\right)f^\epsilon(x)dx \geq 0$$

and we obtain that the contribution of the linear terms is negative. The nonlinear term Ξ_1^ϵ defined in (9.7) is

$$I_2 = \int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} \frac{f^\epsilon(x)\partial_x f^\epsilon(x)\sec^2(\theta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx \\ + \int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} \frac{-f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\sec^2(\theta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx = A_1 + A_2.$$

Using the cancellation coming from the principal value, we have

$$A_1 = \int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x)\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\left(\frac{\sec^2(\theta)}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}} - 1\right)d\eta dx \\ = \int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x)\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\frac{\frac{-\tan^2(\theta)}{\sinh^2(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx \\ + \int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x)\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\frac{\mu_1^2(t)(1-|\tanh(\eta/2)|^{2\epsilon})}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx.$$

Inserting (9.17) and (9.18) in the expression for A_1, we obtain

$$|A_1| \leq c(\epsilon)\|f^\epsilon(t)\|_{L^2(\mathbb{R})}\|\partial_x f^\epsilon(t)\|_{L^2(\mathbb{R})}\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)^4.$$

The second term in I_2 is

$$A_2 = -\int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\left(\frac{\sec^2(\theta)}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}} - 1 + 1\right)d\eta dx \\ = -\int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\frac{\frac{-\tan^2(\theta)}{\sinh^2(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx \\ + \int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\frac{\mu_1^2(t)(1-|\tanh(\eta/2)|^{2\epsilon})}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta dx \\ + \int_{\mathbb{R}} \text{P.V.}\int_{\mathbb{R}} f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}d\eta dx.$$

Using the Cauchy–Schwarz inequality, the equality $\partial_x f^\epsilon(x-\eta) = -\partial_\eta f^\epsilon(x-\eta)$ and integrating by parts we get

$$|A_2| \leq c(\epsilon)\|f^\epsilon(t)\|_{L^2(\mathbb{R})}\|\partial_x f^\epsilon(t)\|_{L^2(\mathbb{R})}\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)^4 + c(\epsilon)\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})}.$$

To finish with the L^2 norm we have to deal with I_3. We have

$$I_3 = \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \frac{f^\epsilon(x)\partial_x f^\epsilon(x)\sec^2(\bar{\theta})\frac{\tanh(\eta/2)}{|\tanh(\eta/2)|^\epsilon}}{1+\frac{\mu_2^2(t)}{|\tanh(\eta/2)|^{2\epsilon}}} d\eta dx$$
$$+ \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \frac{f^\epsilon(x)\partial_x f^\epsilon(x-\eta)\sec^2(\bar{\theta})\frac{\tanh(\eta/2)}{|\tanh(\eta/2)|^\epsilon}}{1+\frac{\mu_2^2(t)}{|\tanh(\eta/2)|^{2\epsilon}}} d\eta dx,$$

where $\bar{\theta}$ is defined in (9.15). Using the same ideas as in I_2 and

$$|\mu_2(t)| \leq \tan\left(\|f_0\|_{L^\infty}\right),$$

we conclude the bound

$$|I_3| \leq c(\epsilon)\|f^\epsilon(t)\|_{L^2(\mathbb{R})}\|\partial_x f^\epsilon(t)\|_{L^2(\mathbb{R})}\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)^4 + c(\epsilon)\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})}.$$

Putting all these bounds together, we get

$$\frac{d}{dt}\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})} \leq c(\epsilon)\|f^\epsilon(t)\|_{L^2(\mathbb{R})}\|\partial_x f^\epsilon(t)\|_{L^2(\mathbb{R})}\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right)^4 + c(\epsilon)\|f^\epsilon(t)\|^2_{L^2(\mathbb{R})}. \quad (9.35)$$

Step 2: L^2 norm of the third derivative To study the L^2 norm of the third derivative, we compute

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^3 f^\epsilon(t)\|^2_{L^2(\mathbb{R})} = -\sqrt{\epsilon}\alpha_1\|\partial_x^3 f^\epsilon(t)\|^2_{L^2(\mathbb{R})} - \epsilon\|\partial_x^4 f^\epsilon(t)\|^2_{L^2(\mathbb{R})} - I_4 + I_5 + I_6.$$

The term I_4 is positive due to Lemma 9.1:

$$I_4 = \alpha_2\epsilon\int_{\mathbb{R}} \partial_x^3 f^\epsilon(x)\Lambda_l^{1-\epsilon}\partial_x^3 f^\epsilon(x)dx + \alpha_3\epsilon\int_{\mathbb{R}} \partial_x^3 f^\epsilon(x)\Lambda_l^{1-3\epsilon}\partial_x^3 f^\epsilon(x)dx$$
$$\int_{\mathbb{R}} \partial_x^3 f^\epsilon(x)\left(\Lambda_l - \Lambda_l^{1-\epsilon}\right)\partial_x^3 f^\epsilon(x)dx + \alpha_4\int_{\mathbb{R}} \partial_x^3 f^\epsilon(x)\left(\Lambda_l - \Lambda_l^{1-3\epsilon}\right)\partial_x^3 f^\epsilon(x)dx \geq 0.$$

The nonlinear terms related to θ are

$$I_5 = -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2\left[\text{P.V.}\left(\int_{B(0,1)} + \int_{B^c(0,1)}\right)\frac{2\partial_x\theta\sec^2(\theta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}}d\eta\right]dx = A_3 + A_4.$$

The term A_3 is not singular if $\epsilon > 0$ and can be bounded using Hölder and Nirenberg interpolation inequalities. For the sake of brevity, we write some terms detailedly, being the rest analogous to them. We have

$$A_3 = B_1 + B_2 + \text{ lower order terms }.$$

Using

$$f^\epsilon(x) - f^\epsilon(x-\eta) = \eta\int_0^1 \partial_x^2 f^\epsilon(x+(s-1)\eta)ds,$$

we obtain

$$\begin{aligned} B_1 &= \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\text{P.V.}\int_{B(0,1)} \frac{16(\partial_x\theta)^3\sec^6(\theta)\frac{|\tanh(\eta/2)|^{5\epsilon}}{\tanh^5(\eta/2)}\tan^2(\theta)}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)^3}d\eta dx \\ &= \int_0^1\int_0^1\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\text{P.V.}\int_{B(0,1)} \frac{4\partial_x^2 f^\epsilon(x+(s-1)\eta)\partial_x^2 f^\epsilon(x+(r-1)\eta)\partial_x\theta\eta^2 d\eta dx dr ds}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)^3\cos^6(\theta)\frac{\tanh^5(\eta/2)}{|\tanh(\eta/2)|^{5\epsilon}}\cot^2(\theta)} \\ &\leq \|\partial_x^4 f^\epsilon(t)\|_{L^2}\|\partial_x^2 f^\epsilon(t)\|^2_{L^4(\mathbb{R})}c\sec^6\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\int_{B(0,1)} |\tanh(\eta/2)|^{5\epsilon-5}\eta^2\tan^2(\eta/2)d\eta. \end{aligned}$$

The second term is

$$\begin{aligned} B_2 &= -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x) \text{P.V.} \int_{B(0,1)} \frac{4(\partial_x\theta)^3 \sec^6(\theta)\frac{|\tanh(\eta/2)|^{3\epsilon}}{\tanh^3(\eta/2)}}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)^2} d\eta dx \\ &= \int_0^1\int_0^1\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\text{P.V.}\int_{B(0,1)} \frac{\partial_x^2 f^\epsilon(x+(s-1)\eta)\partial_x^2 f^\epsilon(x+(r-1)\eta)\partial_x\theta\eta^2 d\eta dx dr ds}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)^3\cos^6(\theta)\frac{\tanh^5(\eta/2)}{|\tanh(\eta/2)|^{5\epsilon}}} \\ &\leq \|\partial_x^4 f^\epsilon(t)\|_{L^2}\|\partial_x^2 f^\epsilon(t)\|^2_{L^4(\mathbb{R})} c\sec^6\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\int_{B(0,1)}|\tanh(\eta/2)|^{3\epsilon-3}\eta^2 d\eta, \end{aligned}$$

and using the classical interpolation inequality

$$\|\partial_x^2 f\|^2_{L^4(\mathbb{R})} \leq c\|\partial_x f\|_{L^\infty(\mathbb{R})}\|\partial_x^3 f\|_{L^2(\mathbb{R})},$$

we get

$$|A_3| \leq c(\epsilon)\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}\left(1+\sec\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\right)^6.$$

We split the term A_4 as follows

$$\begin{aligned} A_4 = &\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\theta\sec^2(\theta)}{\tanh(\eta/2)}\left(\frac{1}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}} - \frac{1}{1+\mu_1^2(t)}\right) d\eta dx \\ &+ \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\theta\sec^2(\theta)\frac{|\tanh(\eta/2)|^{\epsilon}-1}{\tanh(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}} d\eta dx \\ &+ \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\theta\sec^2(\theta)}{\tanh(\eta/2)}\frac{1}{1+\mu_1^2(t)} d\eta dx = B_3+B_4+B_5. \end{aligned}$$

These terms are not singular because of the domain of integration. We have to deal with the integrability at infinity in η. We compute

$$B_3 = \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\theta\sec^2(\theta)}{\tanh(\eta/2)}\frac{\mu_1^2(t)\left(1-|\tanh(\eta/2)|^{2\epsilon}\right)}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)\left(1+\mu_1^2(t)\right)} d\eta dx.$$

The integrability at infinity is obtained using (9.12) and (9.13). We only bound the more singular terms in B_3 and B_4. The most singular term in B_3 is

$$C_1 = \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x^3\theta\sec^2(\theta)}{\tanh(\eta/2)}\frac{\mu_1^2(t)\left(1-|\tanh(\eta/2)|^{2\epsilon}\right)}{\left(1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}\right)\left(1+\mu_1^2(t)\right)} d\eta dx.$$

Using (9.12), (9.13) and (9.17), we obtain

$$|C_1| \leq c\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}\tan\left(\|f_0\|_{L^\infty}\right)\sec^2\left(\|f_0\|_{L^\infty}\right).$$

Analogously, the more singular term in B_4 is

$$C_2 = \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x^3\theta\sec^2(\theta)\frac{|\tanh(\eta/2)|^{\epsilon}-1}{\tanh(\eta/2)}}{1+\mu_1^2(t)|\tanh(\eta/2)|^{2\epsilon}} d\eta dx.$$

Using the same bounds as in C_1, we get

$$|C_2| \leq c\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}\sec^2\left(\|f_0\|_{L^\infty}\right).$$

Using classical trigonometric identities, we obtain

$$B_5 = \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\theta\sinh(\eta)}{\cosh(\eta)-\cos(2\theta)} d\eta.$$

The most singular term in B_5 is

$$C_3 = \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x) \text{P.V.} \int_{B^c(0,1)} \frac{\partial_x^3 f^\epsilon(x)\sinh(\eta)}{2\sinh^2(\eta/2)\left(1+\frac{\sin^2(\theta)}{\sinh^2(\eta/2)}\right)} d\eta - \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x) \text{P.V.} \int_{B^c(0,1)} \frac{\partial_x^3 f^\epsilon(x-\eta)\sinh(\eta)}{\cosh(\eta)-\cos(2\theta)} d\eta = D_1 + D_2.$$

Using the cancellation of the principal value integral, we obtain

$$D_1 = -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^3 f^\epsilon(x) \text{P.V.} \int_{B^c(0,1)} \frac{\sin^2(\theta)\sinh(\eta)}{2\sinh^4(\eta/2)\left(1+\frac{\sin^2(\theta)}{\sinh^2(\eta/2)}\right)} d\eta,$$

thus,

$$|D_1| \leq c\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}.$$

Integrating by parts in D_2, we obtain the required decay at infinity and we conclude

$$|D_2| \leq c\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}.$$

Putting all together, we get

$$|I_5| \leq c(\epsilon)\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}\left(1+\sec\left(\|f_0\|_{L^\infty}\right)\right)^6.$$

The nonlinear terms related to $\bar\theta$ are

$$I_6 = -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\left(\int_{B(0,1)} + \int_{B^c(0,1)}\right) \frac{2\partial_x\bar\theta\sec^2(\bar\theta)\frac{\tanh(\eta/2)}{|\tanh(\eta/2)|^\epsilon}}{1+\mu_2^2(t)|\tanh(\eta/2)|^{-2\epsilon}} d\eta dx = A_5 + A_6.$$

We observe that, due to $1/10 > \epsilon > 0$ and $\|f_0\|_{L^\infty(\mathbb{R})} < \pi/2$, this integral is not singular. Thus the inner part A_5 can be bounded following the same ideas as for A_3. The integrability at infinity is obtained with the following splitting

$$\begin{aligned} A_6 = &-\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \left(\frac{2\partial_x\bar\theta\sec^2(\bar\theta)\tanh(\eta/2)}{1+\mu_2^2(t)|\tanh(\eta/2)|^{-2\epsilon}} - \frac{2\partial_x\bar\theta\sec^2(\bar\theta)\tanh(\eta/2)}{1+\mu_2^2(t)}\right) d\eta dx \\ &- \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\bar\theta\sec^2(\bar\theta)\tanh(\eta/2)\left(\frac{1}{|\tanh(\eta/2)|^\epsilon}-1\right)}{1+\mu_2^2(t)|\tanh(\eta/2)|^{-2\epsilon}} d\eta dx \\ &- \int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{2\partial_x\bar\theta\sec^2(\bar\theta)\tanh(\eta/2)}{1+\mu_2^2(t)} d\eta dx = B_6 + B_7 + B_8. \end{aligned}$$

The term B_8 is

$$\begin{aligned} B_8 &= -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{\partial_x\bar\theta\sinh(\eta)}{\cosh(\eta)+\cos(2\bar\theta)} d\eta dx \\ &= -\int_{\mathbb{R}} \partial_x^4 f^\epsilon(x)\partial_x^2 \text{P.V.}\int_{B^c(0,1)} \frac{\partial_x\bar\theta\sinh(\eta)}{2\sinh^2(\eta/2)\left(1+\frac{\cos^2(\bar\theta)}{\sinh^2(\eta/2)}\right)} d\eta dx, \end{aligned}$$

and it can be handled as B_5. The terms B_6 and B_7 have a term $|\tanh(\eta/2)|^{k\epsilon}-1|$ and they can be bounded following the steps in B_3 and B_4 and using (9.12) and (9.13). Putting all the estimates together we obtain

$$|I_6| \leq c(\epsilon)\|\partial_x^4 f^\epsilon(t)\|_{L^2(\mathbb{R})}\|f^\epsilon(t)\|_{H^3(\mathbb{R})}\left(1+\sec\left(\|f_0\|_{L^\infty}\right)\right)^6.$$

Using (9.35), Young's inequality and the dissipation given by the Laplacian, we get the *'a priori'* estimate

$$\frac{d}{dt}\|f^\epsilon(t)\|^2_{H^3(\mathbb{R})} \leq c(\epsilon)\|f^\epsilon(t)\|^2_{H^3(\mathbb{R})} C\left(\|f_0\|_{L^\infty(\mathbb{R})}\right). \tag{9.36}$$

The existence follows from the *'a priori'* estimate (9.36) by classical energy methods (see [44]). □

9.4 Convergence of f^ϵ

In this section we study the limit of f^ϵ as $\epsilon \to 0$.

Lemma 9.12. *The regularized solutions f^ϵ corresponding to an initial datum satisfying the hypotheses (9.2)-(9.4), or (9.6), converge (up to a subsequence) weakly-* to $f \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$. Moreover, up to a subsequence, $f^\epsilon \to f$ in $L^\infty(K)$ for all compact set $K \subset \mathbb{R}\times\mathbb{R}^+$.*

Proof. First, notice that, due to Propositions 9.1-9.3 and hypotheses (9.2)-(9.4), the regularized solutions satisfy

$$\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})} < \frac{\pi}{4}, \quad \|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0\|_{L^\infty(\mathbb{R})},$$

while, if the initial datum, instead of hypotheses (9.2)-(9.4), satisfies (9.6) then

$$\|f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|f_0\|_{L^\infty(\mathbb{R})}, \quad \|\partial_x f^\epsilon(t)\|_{L^\infty(\mathbb{R})} \leq 1.$$

Due to the Banach-Alaoglu Theorem, these bounds imply that there exists a subsequence such that

$$\int_0^T\int_{\mathbb{R}} f^\epsilon(x,t)g(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} f(x,t)g(x,t)dxdt,$$
$$\int_0^T\int_{\mathbb{R}} \partial_x f^\epsilon(x,t)g(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} \partial_x f(x,t)g(x,t)dxdt,$$

with $f \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$, any $g \in L^1([0,T]\times\mathbb{R})$ and every $T > 0$. Fixing t, due to the uniform bound in $W^{1,\infty}(\mathbb{R})$ and the Ascoli-Arzela Theorem we have that $f^\epsilon(t) \to f(t)$ uniformly on any bounded interval $I \subset \mathbb{R}$. Moreover, for all N, we have

$$\|f^\epsilon - f\|_{L^\infty(B(0,N)\times[0,T])} \to 0.$$

In order to prove this uniform convergence on compact sets we use the spaces and results contained in [13]. For $v \in L^\infty(B(0,N))$, we define the norm

$$\|v\|_{W^{-2,\infty}_*(B(0,N))} = \sup_{\substack{\phi\in W^{2,1}_0(B(0,N)),\\ \|\phi\|_{W^{2,1}}\leq 1}} \left|\int_{B(0,N)} \phi(x)v(x)dx\right|. \tag{9.37}$$

We define the Banach space $W^{-2,\infty}_*(B(0,N))$ as the completion of $L^\infty(B(0,N))$ with respect to the norm (9.37). We have

$$W^{1,\infty}(B(0,N)) \subset L^\infty(B(0,N)) \subset W^{-2,\infty}_*(B(0,N)).$$

The embedding $L^\infty(B(0,N)) \subset W^{-2,\infty}_*(B(0,N))$ is continuous and, due to the Ascoli-Arzela Theorem, the embedding $W^{1,\infty}(B(0,N)) \subset L^\infty(B(0,N))$ is compact. We use the following Lemma proved in [13]

Lemma 9.13 ([13]). *Consider a sequence $\{u_m\} \in C([0,T]\times B(0,N))$ that is uniformly bounded in the space $L^\infty([0,T], W^{1,\infty}(B(0,N)))$. Assume further that the weak derivative du_m/dt is in $L^\infty([0,T], L^\infty(B(0,N)))$ (not necessarily uniform) and is uniformly bounded in $L^\infty([0,T], W_*^{-2,\infty}(B(0,N)))$. Finally, suppose that $\partial_x u_m \in C([0,T]\times B(0,N))$. Then, there exists a subsequence of u_m that converges strongly in $L^\infty([0,T]\times B(0,N))$.*

Due to this Lemma, we only need to bound $\partial_t f^\epsilon$ not uniformly in $L^\infty([0,T]\times B(0,N))$ and in uniformly in $L^\infty([0,T], W_*^{-2,\infty}(B(0,N)))$. Using that $f^\epsilon \in C([0,T], H^3(\mathbb{R}))$, the linear terms in (9.16) can be bounded easily with a bound depending on ϵ. To bound the nonlinear terms, we split the integral

$$\text{P.V.}\int_{\mathbb{R}} = \text{P.V.}\int_{B(0,1)} + \text{P.V.}\int_{B^c(0,1)},$$

and we compute

$$\begin{aligned}&\left|\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f^\epsilon(x)\sec^2(\theta)\frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}}{1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}}d\eta\right| \leq c(\epsilon)\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})\\
&\quad+\left|\text{P.V.}\int_{B^c(0,1)} \frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}\frac{-\frac{\tan^2(\theta)}{\sinh^2(\eta/2)}+\mu_1^2(t)(1-|\tanh(\eta/2)|^{2\epsilon})}{1+\frac{\tan^2(\theta)|\tanh(\eta/2)|^{2\epsilon}}{\tanh^2(\eta/2)}}d\eta\right|\\
&\qquad\leq c(\epsilon)\left(\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})+\tan^2(\|f_0\|_{L^\infty(\mathbb{R})})\right),\end{aligned}$$

where we have used $\sec^2(\theta)-1=\tan^2(\theta)$, (9.13), (9.17) and

$$\text{P.V.}\int_{\mathbb{R}} \frac{|\tanh(\eta/2)|^\epsilon}{\tanh(\eta/2)}d\eta = 0.$$

The second term with the kernel involving θ is

$$\left|\text{P.V.}\int_{\mathbb{R}} \frac{(\epsilon-1)\tan(\theta)\frac{|\tanh((x-\eta)/2)|^\epsilon}{\sinh^2((x-\eta)/2)}}{1+\frac{\tan^2(\theta)|\tanh((x-\eta)/2)|^{2\epsilon}}{\tanh^2((x-\eta)/2)}}d\eta\right| \leq c(\epsilon)\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right).$$

The terms with the kernel involving $\bar\theta$ are not singular and can be bounded following the same ideas

$$\begin{aligned}&\left|\text{P.V.}\int_{\mathbb{R}} \frac{\sec^2(\bar\theta)\frac{\tanh((x-\eta)/2)}{|\tanh((x-\eta)/2)|^\epsilon}}{1+\frac{\tan^2(\bar\theta)\tanh^2((x-\eta)/2)}{|\tanh((x-\eta)/2)|^{2\epsilon}}}d\eta\right| \leq c(\epsilon)\sec^2\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)\\
&\quad+\left|\text{P.V.}\int_{B^c(0,1)} \frac{\tanh(\eta/2)}{|\tanh(\eta/2)|^\epsilon}\frac{-\frac{\tan^2(\bar\theta)}{\sinh^2(\eta/2)}+\frac{\mu_2^2(t)}{|\tanh(\eta/2)|^\epsilon}(|\tanh(\eta/2)|^{2\epsilon}-1)}{1+\frac{\tan^2(\bar\theta)\tanh^2((x-\eta)/2)}{|\tanh((x-\eta)/2)|^{2\epsilon}}}d\eta\right|\\
&\qquad\leq c(\epsilon)\left(\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})+\tan^2(\|f_0\|_{L^\infty(\mathbb{R})})\right)\end{aligned}$$

and

$$\left|\text{P.V.}\int_{\mathbb{R}} \frac{\frac{(1-\epsilon)\tan(\bar\theta)}{|\tanh((x-\eta)/2)|^\epsilon\cosh^2((x-\eta)/2)}}{1+\frac{\tan^2(\bar\theta)\tanh^2((x-\eta)/2)}{|\tanh((x-\eta)/2)|^{2\epsilon}}}d\eta\right| \leq c(\epsilon)\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right).$$

Putting together all these estimates, we get

$$|\partial_t f^\epsilon(x,t)| \leq c(\epsilon)\left(\|f_0\|_{L^2(\mathbb{R})}+\sec^2(\|f_0\|_{L^\infty(\mathbb{R})})+\tan^2(\|f_0\|_{L^\infty(\mathbb{R})})\right),$$

and we conclude with the bound in $L^\infty([0,T]\times B(0,N))$.

To obtain the bound in $L^\infty([0,T], W_*^{-2,\infty}(B(0,N)))$ we extend $\phi\in W_0^{2,1}(B(0,N))$ by zero outside of this ball of radius N. Then, using Lemma 9.1, we *integrate by parts* and obtain

$$\int_{\mathbb{R}}\phi(x)\Lambda_l^{1-\epsilon}f^\epsilon(x)dx=\int_{\mathbb{R}}\Lambda_l^{1-\epsilon}\phi(x)f^\epsilon(x)dx\leq\|\Lambda_l^{1-\epsilon}\phi\|_{L^1(\mathbb{R})}\|f_0\|_{L^\infty(\mathbb{R})},$$

$$\int_{\mathbb{R}}\phi(x)\Lambda_l^{1-3\epsilon}f^\epsilon(x)dx=\int_{\mathbb{R}}\Lambda_l^{1-3\epsilon}\phi(x)f^\epsilon(x)dx\leq\|\Lambda_l^{1-3\epsilon}\phi\|_{L^1(\mathbb{R})}\|f_0\|_{L^\infty(\mathbb{R})},$$

$$\int_{\mathbb{R}}\phi(x)\left(\Lambda_l-\Lambda_l^{1-\epsilon}\right)f^\epsilon(x)dx=\int_{\mathbb{R}}\left(\Lambda_l-\Lambda_l^{1-\epsilon}\right)\phi(x)f^\epsilon(x)dx\leq\|\left(\Lambda_l-\Lambda_l^{1-\epsilon}\right)\phi\|_{L^1(\mathbb{R})}\|f_0\|_{L^\infty(\mathbb{R})},$$

$$\int_{\mathbb{R}}\phi(x)\left(\Lambda_l-\Lambda_l^{1-3\epsilon}\right)f^\epsilon(x)dx=\int_{\mathbb{R}}\left(\Lambda_l-\Lambda_l^{1-3\epsilon}\right)\phi(x)f^\epsilon(x)dx\leq\|\left(\Lambda_l-\Lambda_l^{1-3\epsilon}\right)\phi\|_{L^1(\mathbb{R})}\|f_0\|_{L^\infty(\mathbb{R})}.$$

Using

$$\phi(x)-\phi(x-\eta)-\eta\partial_x\phi(x)=\eta^2\int_0^1\int_0^1(s-1)\partial_x^2\phi(x+r(s-1)\eta)drds,$$

we bound the linear terms in (9.6) as

$$\begin{aligned}\|\left(\Lambda_l-\Lambda_l^{1-\epsilon}\right)f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}&+\|\left(\Lambda_l-\Lambda_l^{1-3\epsilon}\right)f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}\\&+\|\Lambda_l^{1-3\epsilon}f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}+\|\Lambda_l^{1-\epsilon}f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}\\&+\|\partial_x^2f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}+\|f^\epsilon\|_{W_*^{-2,\infty}(B(0,N))}\leq c\|f_0\|_{L^\infty(\mathbb{R})},\end{aligned}$$

being c a universal constant. The nonlinear terms are

$$I_1=\int_{\mathbb{R}}\phi(x)\partial_x\text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\arctan\left(\mu_1(t)\left|\tanh\left(\frac{\eta}{2}\right)\right|^\epsilon\right)d\eta dx=J_1+J_2,$$

$$I_2=\int_{\mathbb{R}}\phi(x)\partial_x\text{P.V.}\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\arctan\left(\frac{\mu_2(t)}{\left|\tanh\left(\frac{\eta}{2}\right)\right|^\epsilon}\right)d\eta dx=J_3+J_4.$$

Using the boundedness of arctan, we get

$$|J_i|\leq\pi\|\partial_x\phi\|_{L^1(\mathbb{R})},\text{ for }i=1,3.$$

The outer part is not singular and can be bounded (as it was done in the previous sections) applying $\epsilon<1/10$. We get

$$|J_i|\leq c\|\phi\|_{L^1(\mathbb{R})}\left(\tan\left(\|f_0\|_{L^\infty(\mathbb{R})}\right)+1\right),\text{ for }i=2,4.$$

Putting together all these bounds we obtain

$$\sup_{t\in[0,T]}\|\partial_tf(t)\|_{W_*^{-2,\infty}(B(0,N))}\leq C\left(\|f_0\|_{L^\infty(\mathbb{R})}\right).$$

Using Lemma 9.13, we conclude the result. □

9.5 Convergence of the regularized system

Looking at (9.1) we give the following definition:

Definition 9.1. *$f(x,t) \in C([0,T]\times\mathbb{R}) \cap L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$ is a weak solution of (9.1) if, for all $\phi(x,t) \in C_c^\infty([0,T)\times\mathbb{R})$ the following equality holds*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} f(x,t)\partial_t\phi(x,t)dxdt + \int_{\mathbb{R}} f_0(x)\phi(x,0)dx \\ = \frac{\rho^2-\rho^1}{2\pi}\int_0^T\int_{\mathbb{R}}\partial_x\phi(x,t)\left(P.V.\int_{\mathbb{R}}\arctan\left(\frac{\tan\left(\frac{\pi}{2l}\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)}\right)d\eta\right. \\ \left. +P.V.\int_{\mathbb{R}}\arctan\left(\tan\left(\frac{\pi}{2l}\frac{f(x)+f(x-\eta)}{2}\right)\tanh\left(\frac{\pi}{2l}\frac{\eta}{2}\right)\right)d\eta\right)dxdt.\end{aligned}$$

In this section we show the convergence, as $\epsilon\to 0$, of the weak formulation of the problem (9.11).

Proposition 9.5. *Let f be the limit of the regularized solutions f^ϵ. Then, f is a weak solution of (9.1).*

Proof. First, we deal with the linear terms. Using the weak-* convergence in $L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$ and Lemma 9.1, we obtain

$$\int_0^T\int_{\mathbb{R}} f^\epsilon(x,t)\partial_t\phi(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} f(x,t)\partial_t\phi(x,t)dxdt,$$

$$\int_0^T\int_{\mathbb{R}} f^\epsilon(x,t)\phi(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} f(x,t)\phi(x,t)dxdt,$$

$$\int_0^T\int_{\mathbb{R}} f^\epsilon(x,t)\Lambda_l^{1-\epsilon}\phi(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} f(x,t)\Lambda_l\phi(x,t)dxdt,$$

$$\int_0^T\int_{\mathbb{R}} f^\epsilon(x,t)\Lambda_l^{1-3\epsilon}\phi(x,t)dxdt \to \int_0^T\int_{\mathbb{R}} f(x,t)\Lambda_l\phi(x,t)dxdt,$$

$$\int_{\mathbb{R}} f_0^\epsilon(x)\phi(x,0)dx \to \int_{\mathbb{R}} f_0(x)\phi(x,0)dx,$$

where, in the last step, we use the L^2 convergence of the mollifier. To deal with the nonlinear terms, we split the integrals

$$\text{P.V.}\int_{\mathbb{R}} = \text{P.V.}\int_{B(0,\delta)} + \text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)} + \text{P.V.}\int_{B^c(0,N)},$$

for sufficiently small δ and large enough N. These parameters, δ, N, that will be fixed below, can depend on f_0 but they don't depend on ϵ. For the inner part of the integrals, we get

$$\begin{aligned}I_1^\epsilon = \int_0^T\int_{\mathbb{R}}\partial_x\phi(x,t)\left(2\text{P.V.}\int_{B(0,\delta)}\arctan\left(\mu_1(t)\left|\tanh(\eta/2)\right|^\epsilon\right)d\eta\right. \\ \left. +2\text{P.V.}\int_{B(0,\delta)}\arctan\left(\frac{\mu_2(t)}{\left|\tanh(\eta/2)\right|^\epsilon}\right)d\eta\right)dxdt \leq c\delta\|\partial_x\phi\|_{L^1([0,T]\times\mathbb{R})}.\end{aligned}$$

The outer integral goes to zero as N grows. We compute

$$I_3^\epsilon=\int_0^T\int_{\mathbb{R}}\partial_x\phi(x,t)\left(2\text{P.V.}\int_{B^c(0,N)}\arctan\left(\mu_1(t)\left|\tanh(\eta/2)\right|^\epsilon\right)d\eta\right.$$
$$\left.+2\text{P.V.}\int_{B^c(0,N)}\arctan\left(\frac{\mu_2(t)}{|\tanh(\eta/2)|^\epsilon}\right)d\eta\right)dxdt.$$

As $\eta\in B^c(0,N)$, the integrals are not singular and we only have to deal with the decay at infinity. Using (9.12), (9.13), (9.16), the bound $\epsilon<1/10$ and following the same ideas in Section 9.3, we have

$$I_3^\epsilon\to 0,\text{ uniformly in }\epsilon\text{ as }N\to\infty.$$

The only thing to check is the convergence of I_2^ϵ. Due to the compactness of the support of ϕ, we have

$$I_2^\epsilon=\int_0^T\int_{\mathbb{R}}\partial_x\phi(x,t)\left(2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\mu_1(t)\left|\tanh(\eta/2)\right|^\epsilon\right)d\eta\right.$$
$$\left.+2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\frac{\mu_2(t)}{|\tanh(\eta/2)|^\epsilon}\right)d\eta\right)dxdt$$
$$=\int_0^T\int_{B(0,M)}\partial_x\phi(x,t)\left(2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\mu_1(t)\left|\tanh(\eta/2)\right|^\epsilon\right)d\eta\right.$$
$$\left.+2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\frac{\mu_2(t)}{|\tanh(\eta/2)|^\epsilon}\right)d\eta\right)dxdt,$$

with M large enough to ensure $\text{supp}(\phi)\subset B(0,M)$. Since we have (up to a subsequence) that $f^\epsilon\to f$ uniformly on compact sets (see Lemma 9.12), the uniform convergence $|\tanh(\eta/2)|^\epsilon\to 1$ if $|\eta|>\delta$ and the continuity of all the functions in this integral, the limit in ϵ and the integral commute and we get

$$I_2^\epsilon\to\int_0^T\int_{\mathbb{R}}\partial_x\phi(x,t)\left(2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\frac{\tan\left(\frac{f(x)-f(x-\eta)}{2}\right)}{\tanh\left(\frac{\eta}{2}\right)}\right)d\eta\right.$$
$$\left.+2\text{P.V.}\int_{B^c(0,\delta)\cap B(0,N)}\arctan\left(\tan\left(\frac{f(x)+f(x-\eta)}{2}\right)\tanh\left(\frac{\eta}{2}\right)\right)d\eta\right)dxdt=I_2^0.$$

We conclude by taking $\delta<<1$ and $N>>1$ to control the tails and then we send $\epsilon\to 0$. Indeed, if we write $F^\epsilon(\phi)$ for the weak formulation of the regularized system (9.11) and $F^0(\phi)$ for the weak formulation of (9.1) (see Definition 9.1), then we have

$$0=F^\epsilon(\phi)=F^\epsilon(\phi)-F^0(\phi)+F^0(\phi),$$

thus

$$-|F^\epsilon(\phi)-F^0(\phi)|\le F^0(\phi)\le|F^\epsilon(\phi)-F^0(\phi)|.$$

Then, given $\gamma<<1$ we can take δ and N such that $I_1^\epsilon+I_3^\epsilon+I_1^0+I_3^0$ is smaller than $\gamma/3$. Then, taking ϵ small enough we can ensure that the linear terms are smaller than $\gamma/3$. Finally, we can take ϵ small enough such that $|I_2^\epsilon-I_2^0|<\gamma/3$. Thus,

$$-\gamma\le F^0(\phi)\le\gamma.$$

We conclude the proof of the Theorem 9.1. □

Part III

The inhomogeneous Muskat problem

Chapter 10

The Muskat problem with different permeabilities

10.1 The problem

In this part of the thesis we study the evolution of the interface between two different incompressible fluids with the same viscosity coefficient in a porous medium with two different permeabilities. This problem is of practical importance because it is used as a model for a geothermal reservoir (see [11] and references therein). The velocity of a fluid flowing in a porous medium satisfies Darcy's law (see [4, 47, 48])

$$\frac{\mu}{\kappa(\vec{x})}v = -\nabla p - g\rho(\vec{x})(0,1),$$

where μ is the dynamic viscosity, $\kappa(\vec{x})$ is the permeability of the medium, g is the acceleration due to gravity, $\rho(\vec{x})$ is the density of the fluid, $p(\vec{x})$ is the pressure of the fluid and $v(\vec{x})$ is the incompressible velocity field. In our favourite units, we can assume $g = \mu = 1$. We have two immiscible and incompressible fluids with the same viscosity and different densities; ρ^1 fill in the upper domain $S^1(t)$ and ρ^2 fill in the lower domain $S^2(t)$. The curve

$$z(\alpha,t) = \{(z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\}$$

is the interface between the fluids. We have a curve

$$h(\alpha) = \{(h_1(\alpha), h_2(\alpha)) : \alpha \in \mathbb{R}\}$$

separating two regions with different values for the permeability (see Figure 10.1). In the region above the curve $h(\alpha)$ the permeability is $\kappa(\vec{x}) \equiv \kappa^1$, while in the region below the curve $h(\alpha)$ the permeability is $\kappa(\vec{x}) \equiv \kappa^2 \neq \kappa^1$. Notice that the curve $h(\alpha)$ is known and fixed.

Remark 10.1. *For notational simplicity, we denote*

$$\mathcal{K} = \frac{\kappa^1 - \kappa^2}{\kappa^1 + \kappa^2} \tag{10.1}$$

and we drop the t dependence.

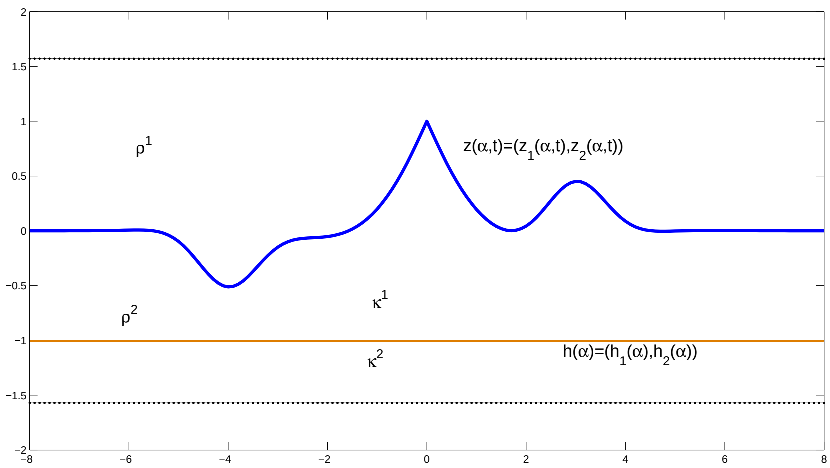

Figure 10.1: The physical situation.

10.2 The equation for the internal wave

In this section we derive the following contour equations, *i.e.* the equations for the interface. First, when the fluids fill the whole space, the equation is:

$$\partial_t f(x) = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(\beta))(x-\beta)}{(x-\beta)^2+(f(x)-f(\beta))^2}d\beta + \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)(x-\beta+\partial_x f(x)(f(x)+h_2))}{(x-\beta)^2+(f(x)+h_2)^2}d\beta, \quad (10.2)$$

with

$$\varpi_2(x) = \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)(h_2+f(\beta))}{(x-\beta)^2+(-h_2-f(\beta))^2}d\beta. \quad (10.3)$$

If the fluids fill the whole space but the initial curve is periodic the equation reduces to

$$\partial_t f(x) = \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{T}}\frac{\sin(x-\beta)(\partial_x f(x)-\partial_x f(\beta))d\beta}{\cosh(f(x)-f(\beta))-\cos(x-\beta)} + \frac{1}{4\pi}\text{P.V.}\int_{\mathbb{T}}\frac{(\partial_x f(x)\sinh(f(x)+h_2)+\sin(x-\beta))\varpi_2(\beta)d\beta}{\cosh(f(x)+h_2)-\cos(x-\beta)}, \quad (10.4)$$

where the second vorticity amplitude can be written as

$$\varpi_2(x) = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(h_2+f(\beta))\partial_x f(\beta)d\beta}{\cosh(h_2+f(\beta))-\cos(x-\beta)}. \quad (10.5)$$

If we consider the confined problem $S=\mathbb{R}\times(-l,l)$ where the depth is $l=\pi/2$, the appropriate equation is

$$\begin{aligned}\partial_t f(x) &= \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)-\cos(f(x)-f(\beta))}d\beta\\ &+\frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)+\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)+\cos(f(x)+f(\beta))}d\beta\\ &+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)(\sinh(x-\beta)+\partial_x f(x)\sin(f(x)+h_2))}{\cosh(x-\beta)-\cos(f(x)+h_2)}d\beta\\ &+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)(-\sinh(x-\beta)+\partial_x f(x)\sin(f(x)-h_2))}{\cosh(x-\beta)+\cos(f(x)-h_2)}d\beta, \end{aligned} \tag{10.6}$$

with

$$\begin{aligned}\varpi_2(x) &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\beta)\frac{\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta\\ &-\mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\beta)\frac{\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta\\ &+\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}}*\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta\\ &-\frac{\mathcal{K}^2}{\sqrt{2\pi}}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}G_{h_2,\mathcal{K}}*\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta, \end{aligned} \tag{10.7}$$

with

$$G_{h_2,\mathcal{K}}(x)=\mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}{1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}\right)$$

a Schwartz function.

Given ω a scalar, γ, z, curves, and a spatial domain $\Omega=\mathbb{T}$ or $\Omega=\mathbb{R}$, we denote the Birkhoff-Rott integral as

$$BR(\omega,z)\gamma=\text{P.V.}\int_{\Omega}\omega(\beta)BS(\gamma_1(\alpha),\gamma_2(\alpha),z_1(\beta),z_2(\beta))d\beta, \tag{10.8}$$

where BS denotes the kernel of $\nabla^{\perp}\Delta^{-1}$ (which depends on the domain) and $(a,b)^{\perp}=(-b,a)$. If the domain is $\mathbb{R}^2$ we have

$$BS(x,y,\mu,\nu)=\frac{1}{2\pi}\left(-\frac{y-\nu}{(y-\nu)^2+(x-\mu)^2},\frac{x-\mu}{(y-\nu)^2+(x-\mu)^2}\right), \tag{10.9}$$

for $\mathbb{T}\times\mathbb{R}$ we have

$$BS(x,y,\mu,\nu)=\frac{1}{4\pi}\left(\frac{-\sinh(y-\nu)}{\cosh(y-\nu)-\cos(x-\mu)},\frac{\sin(x-\mu)}{\cosh(y-\nu)-\cos(x-\mu)}\right), \tag{10.10}$$

and for $\mathbb{R}\times(-\pi/2,\pi/2)$ the kernel is (see Chapter 3)

$$\begin{aligned}BS(x,y,\mu,\nu)=\frac{1}{4\pi}\Bigg(&-\frac{\sin(y-\nu)}{\cosh(x-\mu)-\cos(y-\nu)}-\frac{\sin(y+\nu)}{\cosh(x-\mu)+\cos(y+\nu)},\\ &\frac{\sinh(x-\mu)}{\cosh(x-\mu)-\cos(y-\nu)}-\frac{\sinh(x-\mu)}{\cosh(x-\mu)+\cos(y+\nu)}\Bigg). \end{aligned} \tag{10.11}$$

10.2.1 Infinitely deep, flat at infinity case

Now we study the case where $S=\mathbb{R}^2$. Using the kernel (10.9), we obtain

$$v(\vec{x})=\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_1(\beta)\frac{(\vec{x}-z(\beta))^{\perp}}{|\vec{x}-z(\beta)|^2}d\beta+\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_2(\beta)\frac{(\vec{x}-h(\beta))^{\perp}}{|\vec{x}-h(\beta)|^2}d\beta. \tag{10.12}$$

We have

$$v^{\pm}(z(\alpha))=\lim_{\epsilon\to0}v(z(\alpha)\pm\epsilon\partial_\alpha^{\perp}z(\alpha))=BR(\varpi_1,z)z+BR(\varpi_2,h)z\mp\frac{1}{2}\frac{\varpi_1(\alpha)}{|\partial_\alpha z(\alpha)|^2}\partial_\alpha z(\alpha), \tag{10.13}$$

and

$$v^{\pm}(h(\alpha))=\lim_{\epsilon\to0}v(h(\alpha)\pm\epsilon\partial_\alpha^{\perp}h(\alpha))=BR(\varpi_1,z)h+BR(\varpi_2,h)h\mp\frac{1}{2}\frac{\varpi_2(\alpha)}{|\partial_\alpha h(\alpha)|^2}\partial_\alpha h(\alpha). \tag{10.14}$$

We observe that $v^+(z(\alpha))$ is the limit inside S^1 (the upper subdomain) and $v^-(z(\alpha))$ is the limit inside S^2 (the lower subdomain). The curve $z(\alpha)$ doesn't touch the curve $h(\alpha)$, so, the limit for the curve h are in the same domain S^i.

Using Darcy's Law and assuming that the initial interface $z(\alpha,0)$ is in the region with permeability κ^1, we obtain

$$\begin{aligned}(v^-(z(\alpha))-v^+(z(\alpha)))\cdot\partial_\alpha z(\alpha) &= \kappa^1\left(-\partial_\alpha(p^-(z(\alpha))-p^+(z(\alpha)))\right)-\kappa^1(\rho^2-\rho^1)\partial_\alpha z_1(\alpha)\\ &= 0-\kappa^1(\rho^2-\rho^1)\partial_\alpha z_2(\alpha),\end{aligned}$$

where in the last equality we have used the continuity of the pressure along the interface (see [22]). Using (10.13) we conclude

$$\varpi_1(\alpha)=-\kappa^1(\rho^2-\rho^1)\partial_\alpha z_2(\alpha). \tag{10.15}$$

We need to determine ϖ_2. We consider

$$\begin{aligned}\left[\frac{v}{\kappa}\right] &= \left(\frac{v^-(h(\alpha))}{\kappa^2}-\frac{v^+(h(\alpha))}{\kappa^1}\right)\cdot\partial_\alpha h(\alpha)\\ &= -\partial_\alpha(p^-(h(\alpha))-p^+(h(\alpha)))\\ &= 0,\end{aligned}$$

where, in the first equality, we use Darcy's Law. Using the expression (10.14) we have

$$\left[\frac{v}{\kappa}\right]=\left(\frac{1}{\kappa^2}-\frac{1}{\kappa^1}\right)(BR(\varpi_1,z)h+BR(\varpi_2,h)h)\cdot\partial_\alpha h(\alpha)+\left(\frac{1}{2\kappa^2}+\frac{1}{2\kappa^1}\right)\varpi_2.$$

We take $h(\alpha)=(\alpha,-h_2)$, with $h_2>0$ a fixed constant. Then

$$BR(\varpi_2,h)h\cdot\partial_\alpha h=\left(0,\frac{1}{2}H(\varpi_2)\right)\cdot(1,0)=0,$$

where H denotes the Hilbert transform. Finally, we have

$$\varpi_2(\alpha)=-2\mathcal{K}BR(\varpi_1,z)h\cdot(1,0)=\mathcal{K}\frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_1(\beta)\frac{-h_2-z_2(\beta)}{|h(\alpha)-z(\beta)|^2}d\beta, \tag{10.16}$$

see (10.1) for the definition of $\mathcal{K}$.

The identity

$$\int_{\mathbb{R}} \partial_\beta \log((A - z_1(\beta))^2 + (B - z_2(\beta))^2) = 0,$$

gives us

$$\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}(-\partial_\alpha z_2(\beta))\frac{z_2(\alpha)-z_2(\beta)}{|z(\alpha)-z(\beta)|^2}d\beta = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_1(\beta)\frac{z_1(\alpha)-z_1(\beta)}{|z(\alpha)-z(\beta)|^2}d\beta,$$

$$\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_2(\beta)\frac{h_2+z_2(\beta)}{|h(\alpha)-z(\beta)|^2}d\beta = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_1(\beta)\frac{h_1(\alpha)-z_1(\beta)}{|h(\alpha)-z(\beta)|^2}d\beta.$$

Thus, insterting this in equation (10.16), we have

$$\varpi_2(\alpha) = \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_\alpha z_1(\beta)\frac{h_1(\alpha)-z_1(\beta)}{|h(\alpha)-z(\beta)|^2}d\beta, \tag{10.17}$$

and

$$BR(\varpi_1, z)z = \frac{-\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{z_1(\alpha)-z_1(\beta)}{|z(\alpha)-z(\beta)|^2}\partial_\alpha z(\beta)d\beta.$$

Due to the conservation of mass the curve z is advected by the flow, but we can add any tangential term in the equation for the evolution of the interface without changing the shape of the resulting curve (see [22]).Thus, the equation for the curve is

$$\partial_t z(\alpha) = v(\alpha) + c(\alpha, t)\partial_\alpha z(\alpha).$$

Taking $c(\alpha) = -v_1(\alpha)$, we get

$$\begin{aligned}\partial_t z = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\frac{z_1(\alpha)-z_1(\beta)}{|z(\alpha)-z(\beta)|^2}(\partial_\alpha z(\alpha)-\partial_\alpha z(\beta))d\beta \\ + \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_2(\beta)\frac{(z(\alpha)-h(\beta))^\perp}{|z(\alpha)-h(\beta)|^2}d\beta \\ + \partial_\alpha z(\alpha)\frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}}\varpi_2(\beta)\frac{z_2(\alpha)+h_2}{|z(\alpha)-h(\beta)|^2}d\beta.\end{aligned} \tag{10.18}$$

By choosing this tangential term, if our initial datum can be parametrized as a graph, we have $\partial_t z_1 = 0$. Therefore, the parametrization as a graph propagates.

Finally, we conclude (10.2) as the evolution equation for the interface (which initially is a graph above the line $y \equiv -h_2$). We remark that the second vorticity (10.3) can be written in equivalent formulae

$$\begin{aligned}\varpi_2(x) &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x f(\beta)\frac{h_2+f(\beta)}{(x-\beta)^2+(-h_2-f(\beta))^2}d\beta && (10.19)\\ &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}}\frac{x-\beta}{(x-\beta)^2+(-h_2-f(\beta))^2}d\beta && (10.20)\\ &= \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}}\partial_x\log((x-\beta)^2+(-h_2-f(\beta))^2)d\beta.\end{aligned}$$

10.2.2 Infinitely deep, periodic case

Now, the domain is $S = \mathbb{T}\times\mathbb{R}$. We have that (10.12) is still valid, but now ϖ_i are periodic functions and $z(\alpha+2k\pi) = z(\alpha)+(2k\pi,0)$. Using complex variables notation we have

$$\begin{aligned}\bar{v}(\vec{x}) &= \frac{1}{2\pi i}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_1(\beta)}{\vec{x}-z(\beta)}d\beta + \frac{1}{2\pi i}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)}{\vec{x}-h(\beta)}d\beta\\ &= \frac{1}{2\pi i}\left(\text{P.V.}\int_{-\pi}^{\pi}+\sum_{k\geq 1}\left(\int_{(2k-1)\pi}^{(2k+1)\pi}+\int_{-(2k+1)\pi}^{-(2k-1)\pi}\right)\right)\frac{\varpi_1(\beta)}{\vec{x}-z(\beta)}+\frac{\varpi_2(\beta)}{\vec{x}-h(\beta)}d\beta.\end{aligned}$$

Changing variables and using the identity

$$\frac{1}{z}+\sum_{k\geq 1}\frac{2z}{z^2-(2k\pi)^2} = \frac{1}{2\tan(z/2)},\quad \forall z\in\mathbb{C},$$

we obtain

$$\bar{v}(\vec{x}) = \frac{1}{4\pi i}\left(\text{P.V.}\int_{\mathbb{T}}\frac{\varpi_1(\beta)}{\tan((\vec{x}-z(\beta))/2)}d\beta+\text{P.V.}\int_{\mathbb{T}}\frac{\varpi_2(\beta)}{\tan((\vec{x}-h(\beta))/2)}d\beta\right).$$

Equivalently,

$$\begin{aligned}v(\vec{x}) = \frac{1}{4\pi}&\left(\text{P.V.}\int_{\mathbb{T}}\frac{-\sinh(y-z_2(\beta))\varpi_1(\beta)d\beta}{\cosh(y-z_2(\beta))-\cos(x-z_1(\beta))}\right.\\ &\left.\quad+\text{P.V.}\int_{\mathbb{T}}\frac{-\sinh(y-h_2(\beta))\varpi_2(\beta)d\beta}{\cosh(y-h_2(\beta))-\cos(x-h_1(\beta))}\right)\\ &+\frac{i}{4\pi}\left(\text{P.V.}\int_{\mathbb{T}}\frac{\sin(x-z_1(\beta))\varpi_1(\beta)d\beta}{\cosh(y-z_2(\beta))-\cos(x-z_1(\beta))}\right.\\ &\left.\quad+\text{P.V.}\int_{\mathbb{T}}\frac{\sin(x-h_1(\beta))\varpi_2(\beta)d\beta}{\cosh(y-h_2(\beta))-\cos(x-h_1(\beta))}\right).\end{aligned}$$

Recall that (10.15) and (10.17) are still valid if $h(\alpha) = (\alpha,-h_2)$ for $0<h_2$ a fixed constant. We have

$$\int_{\mathbb{T}}\partial_\beta\log(\cosh(B-z_2(\beta))-\cos(A-z_1(\beta)))d\beta = 0,$$

thus, the velocity in the curve when the correct tangential terms are added is

$$\begin{aligned}\partial_t z(\alpha) = \frac{1}{4\pi}&\left(\kappa^1(\rho^2-\rho^1)\text{P.V.}\int_{\mathbb{T}}\frac{\sin(z_1(\alpha)-z_1(\beta))(\partial_\alpha z(\alpha)-\partial_\alpha z(\beta))d\beta}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))}\right.\\ &\left.+(\partial_\alpha z_1(\alpha)-1)\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(z_2(\alpha)+h_2)\varpi_2(\beta)d\beta}{\cosh(z_2(\alpha)+h_2)-\cos(z_1(\alpha)-h_1(\beta))}\right)\\ &+\frac{i}{4\pi}\text{P.V.}\int_{\mathbb{T}}\frac{(\partial_\alpha z_2(\alpha)\sinh(z_2(\alpha)+h_2)+\sin(z_1(\alpha)-h_1(\beta)))\varpi_2(\beta)d\beta}{\cosh(z_2(\alpha)+h_2)-\cos(z_1(\alpha)-h_1(\beta))}.\end{aligned}\tag{10.21}$$

We can do the same in order to write ϖ_2 as an integral on the torus.

$$\begin{aligned}\varpi_2(\alpha) &= -2\mathcal{K}BR(\varpi_1,z)h\cdot(1,0)\\ &= \frac{1}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(-h_2-z_2(\beta))\varpi_1(\beta)d\beta}{\cosh(-h_2-z_2(\beta))-\cos(h_1(\alpha)-z_1(\beta))}\\ &= \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(h_2+z_2(\beta))\partial_\alpha z_2(\beta)d\beta}{\cosh(-h_2-z_2(\beta))-\cos(h_1(\alpha)-z_1(\beta))}.\end{aligned}\tag{10.22}$$

If the initial datum can be parametrized as a graph the equation for the interface reduces to (10.4), where the second vorticity amplitude (10.5) can be written as

$$\begin{aligned}\varpi_2(x) &= \frac{1}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(-h_2-f(\beta))\varpi_1(\beta)d\beta}{\cosh(-h_2-f(\beta))-\cos(x-\beta)}\\ &= \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sinh(h_2+f(\beta))\partial_x f(\beta)d\beta}{\cosh(h_2+f(\beta))-\cos(x-\beta)} \qquad (10.23)\\ &= \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\mathcal{K}\text{P.V.}\int_{\mathbb{T}}\frac{\sin(x-\beta)d\beta}{\cosh(h_2+f(\beta))-\cos(x-\beta)}. \qquad (10.24)\end{aligned}$$

10.2.3 Finitely deep

Now we consider the bounded porous medium $\mathbb{R}\times(-\pi/2,\pi/2)$ (see Figure 2.4). This regime is equivalent to the case with more than two κ^i because the boundaries can be understood as regions with $\kappa=0$. As before,

$$v(x,y)=\text{P.V.}\int_{\mathbb{R}}\varpi_1(\beta)BS(x,y,z_1(\beta),z_2(\beta))d\beta+\text{P.V.}\int_{\mathbb{R}}\varpi_2(\beta)BS(x,y,h_1(\beta),h_2(\beta))d\beta.$$

We assume that $h(\alpha)=(\alpha,-h_2)$ with $0<h_2<\pi/2$. We have that ϖ_1 is given by (10.15). The main difference between the finite depth and the infinite depth is at the level of ϖ_2. As in the infinite depth case we have

$$0=\left(\frac{1}{\kappa^2}-\frac{1}{\kappa^1}\right)(BR(\varpi_1,z)h+BR(\varpi_2,h)h)\cdot\partial_\alpha h(\alpha)+\left(\frac{1}{2\kappa^2}+\frac{1}{2\kappa^1}\right)\varpi_2,$$

where now BR has the usual definition (10.8) in terms of BS in expression (10.11). In the unbounded case we have an explicit expression (10.17) for ϖ_2 in terms of z and h, but now we have a Fredholm integral equation of second kind:

$$\varpi_2(\alpha)+\frac{\mathcal{K}}{2\pi}\text{ P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)\sin(2h_2)}{\cosh(\alpha-\beta)+\cos(2h_2)}d\beta=-2\mathcal{K}BR(\varpi_1,z)h\cdot(1,0). \qquad (10.25)$$

We consider the Fourier transform as

$$\mathcal{F}(f)(\zeta)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ix\zeta}f(x)dx,$$

and using some of its basic properties, we have

$$\mathcal{F}(\varpi_2)(\zeta)\left(1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)\right)=-2\mathcal{K}\mathcal{F}(BR(\varpi_1,z)h\cdot(1,0))(\zeta).$$

We can solve the equation for ϖ_2 for any $|\mathcal{K}|<\delta(h_2)$ with

$$\delta(h_2)=\min\left\{1,\frac{\sqrt{2\pi}}{\max\limits_{\zeta\in\mathbb{R}}\left|\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)\right|}\right\}. \qquad (10.26)$$

Moreover, we have the following result concerning the range of correct parameters

Proposition 10.1. *Let $0<h_2<\pi/2$ be a constant, then $\delta(h_2)=1$. Thus, equation (10.25) can be solved for every $\mathcal{K}$.*

Proof. We prove the result by computing

$$\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\zeta}\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}dx.$$

Take $\zeta \in \mathbb{R}$, $\zeta < 0$. We consider the complex extension

$$I=\int_{\Gamma} e^{-iz\zeta}\frac{\sin(2h_2)}{\cosh(z)+\cos(2h_2)}dz,$$

where $\Gamma = (-\pi - 2k\pi, \pi + 2k\pi) \times (0, R) \in \mathbb{C}$. The poles of the function are in

$$\gamma_k^- = (\pi - 2h_2 + 2k\pi)i \text{ and } \gamma_k^+ = (\pi + 2h_2 + 2k\pi)i.$$

We have

$$\cosh(z)+\cos(2h_2) = 2\cosh\left((z+2h_2 i)/2\right)\cosh\left((z-2h_2 i)/2\right),$$

thus, $\gamma_k^\pm$ are simple poles. Due to the form of $\gamma_k^\pm$ we take $R = 2k\pi$. We split the contour integral

$$I = I_{\mathbb{R}} + I_{iR} + I_{-\pi-2k\pi} + I_{\pi+2k\pi},$$

with

$$I_{\mathbb{R}} = \int_{-(\pi+2k\pi)}^{\pi+2k\pi} \frac{e^{-ix\zeta}\sin(2h_2)}{\cosh(x)+\cos(2h_2)}dx,$$

$$I_{iR} = \int_{\pi+2k\pi}^{-(\pi+2k\pi)} \frac{e^{-i(x+iR)\zeta}\sin(2h_2)}{\cosh(x+iR)+\cos(2h_2)}dx,$$

$$I_{\pi+2k\pi} = \int_0^{2k\pi} \frac{e^{-i(\pi+2k\pi+iy)\zeta}\sin(2h_2)}{\cosh(\pi+2k\pi+iy)+\cos(2h_2)}dy,$$

$$I_{-\pi-2k\pi} = \int_{2k\pi}^{0} \frac{e^{-i(-\pi-2k\pi+iy)\zeta}\sin(2h_2)}{\cosh(-\pi-2k\pi+iy)+\cos(2h_2)}dy.$$

Using classical trigonometric identities, we get

$$|I_{iR}| \leq \int_{-\infty}^{\infty} \frac{e^{2k\pi\zeta}\sin(2h_2)}{\cosh(x)+\cos(2h_2)}dx \leq c_{h_2}e^{2k\pi\zeta}.$$

For the third integral, we have

$$I_{\pi+2k\pi} \leq \int_0^{2k\pi} \frac{e^{y\zeta}}{|\cosh(\pi+2k\pi)\cos(y)+\sinh(\pi+2k\pi)\sin(y)+\cos(2h_2)|}dy$$
$$\leq \int_0^{\infty} \frac{e^{y\zeta}}{(\cosh(\pi+2k\pi)-1)^2+\cos(2h_2)-2}dy.$$

The same remains valid for $I_{-\pi-2k\pi}$. Then, taking the limit $k \to \infty$ we have

$$I = I_{\mathbb{R}} = \int_{-\infty}^{\infty} \frac{e^{-ix\zeta}\sin(2h_2)}{\cosh(x)+\cos(2h_2)}dx,$$

and, due to the Residue Theorem, we have

$$I = 2\pi i \sum_{k\geq 0} \text{Res}\left(\frac{e^{-iz\zeta}\sin(2h_2)}{\cosh(z)+\cos(2h_2)}, \gamma_k^\pm\right) = 2\pi i \sum_{k\geq 0} i e^{\pi\zeta}(e^{2\pi\zeta})^k 2\sinh(2h_2\zeta) = \frac{2\pi\sinh(2h_2\zeta)}{\sinh(\pi\zeta)}.$$

Finally, we obtain

$$\delta(h_2) = \min\left\{1, \frac{\pi}{2h_2}\right\},$$

and we conclude that $\delta(h_2) = 1$ for every $0 < h_2 < \pi/2$. □

Using the properties of the Fourier Transform, we obtain

$$\varpi_2(\alpha) = -2\mathcal{K}BR(\varpi_1, z)h \cdot (1,0) + \frac{2\mathcal{K}^2}{\sqrt{2\pi}} BR(\varpi_1, z)h \cdot (1,0) * \mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}{1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}\right). \quad (10.27)$$

Now we observe that if $s(\zeta)$ is a function in the Schwartz class, $\mathcal{S}$, such that $1+s(\zeta)>0$, we have that

$$\frac{s(\zeta)}{1+s(\zeta)} \in \mathcal{S},$$

and we obtain

$$G_{h_2,\mathcal{K}}(x) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}{1+\frac{\mathcal{K}}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\sin(2h_2)}{\cosh(x)+\cos(2h_2)}\right)(\zeta)}\right) \in \mathcal{S}.$$

Recall here that in order to obtain ϖ_2 we invert an integral operator. In general this is a delicate issue (compare with [22]), but with our choice of h this point can be addressed in a simpler way. Using

$$\int_{\mathbb{R}} \partial_\beta \log\left(\cosh(x - z_1(\beta)) \pm \cos(y \pm z_2(\beta))\right) d\beta = 0,$$

and adding the correct tangential term, we obtain

$$\begin{aligned}
\partial_t z(\alpha) \quad = \quad & \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\beta))\sinh(z_1(\alpha)-z_1(\beta))}{\cosh(z_1(\alpha)-z_1(\beta)) - \cos(z_2(\alpha)-z_2(\beta))} d\beta \\
& + \frac{\kappa^1(\rho^2-\rho^1)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\beta), \partial_\alpha z_2(\alpha) + \partial_\alpha z_2(\beta))\sinh(z_1(\alpha)-z_1(\beta))}{\cosh(z_1(\alpha)-z_1(\beta)) + \cos(z_2(\alpha)+z_2(\beta))} d\beta \\
& + \frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \varpi_2(\beta) BS(z_1(\alpha), z_2(\alpha), \beta, -h_2) d\beta \\
& + \frac{\partial_\alpha z(\alpha)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \varpi_2(\beta)\frac{\sin(z_2(\alpha)+h_2)}{\cosh(z_1(\alpha)-\beta) - \cos(z_2(\alpha)+h_2)} d\beta \\
& + \frac{\partial_\alpha z(\alpha)}{4\pi}\text{P.V.}\int_{\mathbb{R}} \varpi_2(\beta)\frac{\sin(z_2(\alpha)-h_2)}{\cosh(z_1(\alpha)-\beta) + \cos(z_2(\alpha)-h_2)} d\beta. \qquad (10.28)
\end{aligned}$$

If the initial curve can be parametrized as a graph the equation reduces to (10.6) where ϖ_2 is defined in (10.7).

Chapter 11

Local existence in Sobolev spaces

11.1 Foreword

In this Chapter we obtain a $L^2(\mathbb{R})$ maximum principle and an energy balance and the local solvability in Sobolev spaces for the equation

$$\partial_t f(x) = \frac{\kappa^1(\rho^2-\rho^1)}{2\pi}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_x f(x)-\partial_x f(\beta))(x-\beta)}{(x-\beta)^2+(f(x)-f(\beta))^2}d\beta + \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(\beta)(x-\beta+\partial_x f(x)(f(x)+h_2))}{(x-\beta)^2+(f(x)+h_2)^2}d\beta, \quad (11.1)$$

with

$$\varpi_2(x) = \mathcal{K}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)(h_2+f(\beta))}{(x-\beta)^2+(-h_2-f(\beta))^2}d\beta.$$

The same results remain valid for equations (10.4) and (10.6).

An interesting point in these local existence results is that there is a well-known sign condition in the difference of the densities in order to have a well-posed problem but there is not any sign condition on the difference of the permeabilities. We will further investigate numerically this interesting fact (see Chapter 13).

11.2 Maximum principle for $\|f\|_{L^2(\mathbb{R})}$

Here we obtain an energy balance inequality for the L^2 norm of the solution of equation (2.16). We define

$$\Omega^1 = \{(x,y), f(x,t) < y < \pi/2\},$$
$$\Omega^2 = \{(x,y), -h_2 < y < f(x,t))\}$$

and

$$\Omega^3 = \{(x,y), -\pi/2 < y < -h_2\}.$$

Then, we have the following result:

Theorem 11.1 (Maximum principle for L^2)**.** *For every* $0<\kappa^1,\kappa^2$ *the smooth solutions of* (10.6) *in the stable regime,* i.e. $\rho^2>\rho^1$*, case verifies*

$$\|f(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\frac{\|v\|^2_{L^2(\mathbb{R}\times(-h_2,\pi/2))}}{\kappa^1(\rho^2-\rho^1)}+\frac{\|v\|^2_{L^2(\mathbb{R}\times(-\pi/2,-h_2))}}{\kappa^2(\rho^2-\rho^1)}ds=\|f_0\|^2_{L^2(\mathbb{R})}. \tag{11.2}$$

Proof. We define the potentials

$$\phi^1(x,y,t)=\kappa^1(p(x,y,t)+\rho^1y),\ \text{ if } (x,y)\in\Omega^1,$$

$$\phi^2(x,y,t)=\kappa^1(p(x,y,t)+\rho^2y),\ \text{ if } (x,y)\in\Omega^2,$$

$$\phi^3(x,y,t)=\kappa^2(p(x,y,t)+\rho^2y),\ \text{ if } (x,y)\in\Omega^3.$$

We have $v^i=-\nabla\phi^i$ in each subdomain S^i. Since the velocity is incompressible, we have

$$0=\int_{\Omega^i}\Delta\phi^i\phi^i dxdy=-\int_{\Omega^i}|v^i|^2dxdy+\int_{\partial\Omega^i}\phi^i\partial_n\phi^i ds.$$

Moreover, the normal component of the velocity is continuous through the interface $(x,f(x))$ and the line where permeability changes $(x,-h_2)$. Using the impermeable boundary conditions, we only need to integrate over the curve $(x,f(x,t))$ and $(x,-h_2)$. Indeed, we have

$$0=-\int_{\Omega^1}|v^1|^2dxdy+\kappa^1\int_{\mathbb{R}}(p(x,f(x,t),t)+\rho^1f(x,t))(-v(x,f(x,t),t)\cdot(\partial_xf(x,t),-1))dx, \tag{11.3}$$

$$\begin{aligned}0=-\int_{\Omega^2}|v^2|^2dxdy+\kappa^1\int_{\mathbb{R}}&(p(x,f(x,t),t)+\rho^2f(x,t))(-v(x,f(x,t),t)\cdot(-\partial_xf(x,t),1))dx\\&+\kappa^1\int_{\mathbb{R}}(p(x,-h_2,t)-\rho^2h_2)(-v(x,-h_2,t)\cdot(0,-1))dx,\end{aligned} \tag{11.4}$$

$$0=-\int_{\Omega^3}|v^3|^2dxdy+\kappa^2\int_{\mathbb{R}}(p(x,-h_2,t)-\rho^2h_2)(-v(x,-h_2,t)\cdot(0,1))dx. \tag{11.5}$$

Inserting (11.5) in (11.4), we get

$$\begin{aligned}0=&-\int_{\Omega^2}|v^2|^2dxdy-\frac{\kappa^1}{\kappa^2}\int_{\Omega^3}|v^3|^2dxdy\\&+\kappa^1\int_{\mathbb{R}}(p(x,f(x,t),t)+\rho^2f(x,t))(-v(x,f(x,t),t)\cdot(-\partial_xf(x,t),1))dx.\end{aligned} \tag{11.6}$$

Thus, summing (11.6) and (11.3) together and using the continuity of the pressure and the velocity in the normal direction, we obtain

$$\int_{\Omega^1\cup\Omega^2}|v|^2dxdy+\frac{\kappa^1}{\kappa^2}\int_{\Omega^3}|v|^2dxdy=\kappa^1\int_{\mathbb{R}}(\rho^2-\rho^1)f(x,t)(-\partial_tf(x,t))dx. \tag{11.7}$$

Integrating in time we get the desired result (11.2). □

Notice that for the infinite depth case the proof is similar and the result follows.

11.3 Well-posedness for the infinite depth case

Let Ω be the spatial domain considered, *i.e.* $\Omega=\mathbb{R}$ or $\Omega=\mathbb{T}$. In this section we prove the short time existence of classical solution for both spatial domains. We have the following result:

Theorem 11.2. *Consider $0<h_2$ a fixed constant and the initial datum $f_0(x)=f(x,0)\in H^k(\Omega)$, $k\geq 3$, such that $-h_2<\min_x f_0(x)$. Then, if the Rayleigh-Taylor condition is satisfied,* i.e. $\rho^2-\rho^1>0$, *there exists an unique classical solution of (11.1), or (2.14) in the periodic case, $f\in C([0,T],H^k(\Omega))$ where $T=T(f_0)$. Moreover, we have $f\in C^1([0,T],C(\Omega))\cap C([0,T],C^2(\Omega))$.*

Proof. We prove the result in the case $\Omega=\mathbb{R}$, being the case $\Omega=\mathbb{T}$ similar. Let us consider the usual Sobolev space $H^3(\mathbb{R})$ endowed with the norm

$$\|f\|_{H^3}=\|f\|_{L^2}+\|\Lambda^3 f\|_{L^2},$$

where $\Lambda=\sqrt{-\Delta}$. Define the energy

$$E[f]:=\|f\|_{H^3}+\|d^h[f]\|_{L^\infty}, \tag{11.8}$$

with

$$d^h[f](x,\beta)=\frac{1}{(x-\beta)^2+(f(x)+h_2)^2}. \tag{11.9}$$

To use the classical energy method we need *a priori* estimates. To simplify notation we drop the physical parameters present in the problem by considering $\kappa^1(\rho^2-\rho^1)=2\pi$ and $\mathcal{K}=\frac{1}{2}$. The sign of the difference between the permeabilities will not be important to obtain local existence. We denote c a constant that can changes from one line to another.

Step 1: Estimates on $\|\varpi_2\|_{H^3}$ Given $f(x)$ such that $E[f]<\infty$, we consider ϖ_2 as defined in (10.19). Then we have that $\|\varpi_2\|_{H^3}\leq c(E[f]+1)^k$ for some constants $c,k>0$.

We proceed now to prove this claim. We start with the L^2 norm. Changing variables in (10.19) we have

$$\begin{aligned}\|\varpi_2\|_{L^2}^2\leq c&\left\|\text{P.V.}\int_{B(0,1)}\frac{\partial_x f(x-\beta)(h_2+f(x-\beta))}{\beta^2+(h_2+f(x-\beta))^2}d\beta\right\|_{L^2}^2\\&+c\left\|\text{P.V.}\int_{B^c(0,1)}\frac{\partial_x f(x-\beta)(h_2+f(x-\beta))}{\beta^2+(h_2+f(x-\beta))^2}d\beta\right\|_{L^2}^2\\&=A_1+A_2.\end{aligned}$$

The inner term, A_1, can be bounded as follows

$$\begin{aligned}A_1=\int_{\mathbb{R}}\text{P.V.}\int_{B(0,1)}&\frac{\partial_x f(x-\beta)(h_2+f(x-\beta))}{\beta^2+(h_2+f(x-\beta))^2}d\beta\\&\times\text{P.V.}\int_{B(0,1)}\frac{\partial_x f(x-\xi)(h_2+f(x-\xi))}{\xi^2+(h_2+f(x-\xi))^2}d\xi dx\\&\leq c\|d^h[f]\|_{L^\infty}^2(1+\|f\|_{L^\infty})^2\|\partial_x f\|_{L^2}^2.\end{aligned}$$

In the last inequality we have used Cauchy–Schwarz inequality and Tonelli's Theorem. For the outer part we have

$$A_2 = \int_{\mathbb{R}} \text{P.V.} \int_{B^c(0,1)} \frac{\partial_x f(x-\beta)(h_2 + f(x-\beta))}{\beta^2 + (h_2 + f(x-\beta))^2} d\beta dx$$
$$\times \text{P.V.} \int_{B^c(0,1)} \frac{\partial_x f(x-\xi)(h_2 + f(x-\xi))}{\xi^2 + (h_2 + f(x-\xi))^2} d\xi dx$$
$$\leq c(1 + \|f\|_{L^\infty})^2 \|\partial_x f\|_{L^2}^2,$$

where we have used that $\int_1^\infty \frac{d\beta}{\beta^2} < \infty$ and Cauchy–Schwarz inequality. We change variables in (10.20) to obtain

$$\varpi_2(x) = \text{P.V.} \int_{\mathbb{R}} \frac{\beta}{\beta^2 + (h_2 + f(x-\beta))^2} d\beta.$$

Now it is clear that ϖ_2 is at the level of f in terms of regularity and the inequality follows using the same techniques. Using Sobolev embedding we conclude this step.

Step 2: Estimates on $\|d^h[f]\|_{L^\infty}$ The first integral in (11.1) can be bounded as follows

$$I_1 \leq \left\| \text{P.V.} \int_{\mathbb{R}} \frac{(x-\beta)(\partial_x f(x) - \partial_x f(\beta))}{(x-\beta)^2 + (f(x) - f(\beta))^2} d\beta \right\|_{L^\infty} \leq c(E[f]+1)^k,$$

for some positive and finite k. The new term is the second integral in (11.1).

$$I_2 \leq \left\| \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varpi_2(x-\beta)(\beta + \partial_x f(x)(f(x)+h_2))}{\beta^2 + (f(x)+h_2)^2} d\beta \right\|_{L^\infty}$$
$$\leq \left\| \frac{1}{2\pi} \text{P.V.} \int_{B(0,1)} d\beta \right\|_{L^\infty} + \left\| \frac{1}{2\pi} \text{P.V.} \int_{B^c(0,1)} d\beta \right\|_{L^\infty} = J_1 + J_2.$$

Easily we have

$$J_1 \leq c\|\varpi_2\|_{L^\infty} \|d^h[f]\|_{L^\infty} (1 + \|\partial_x f\|_{L^\infty} (\|f\|_{L^\infty} + 1)).$$

We split $J_2 = B_1 + B_2$

$$B_1 = \frac{1}{2\pi} \text{P.V.} \int_{B^c(0,1)} \frac{\varpi_2(x-\beta)\beta}{\beta^2 + (f(x)+h_2)^2} \pm \frac{\varpi_2(x-\beta)\beta}{\beta^2} d\beta$$
$$\leq c\|\varpi_2\|_{L^\infty}(\|f\|_{L^\infty}+1)^2 + c\|H\varpi_2\|_{L^\infty} + c\|\partial_x \varpi_2\|_{L^\infty},$$

where H denotes the Hilbert transform. Now we conclude the desired bound using the previous estimate on $\|\varpi_2\|_{H^3}$ and Sobolev embedding. The second term can be bounded as

$$B_2 = \frac{1}{2\pi} \text{P.V.} \int_{B^c(0,1)} \frac{\varpi_2(x-\beta)\partial_x f(x)(f(x)+h_2)}{\beta^2 + (f(x)+h_2)^2} d\beta \leq c\|\varpi_2\|_{L^\infty}(\|f\|_{L^\infty}+1)\|\partial_x f\|_{L^\infty}.$$

We obtain the following useful estimate

$$\|\partial_t f\|_{L^\infty} \leq c(E[f]+1)^k. \tag{11.10}$$

We have

$$\frac{d}{dt} d^h[f] = \frac{-\partial_t f(x) 2(f(x)+h_2)}{(\beta^2 + (f(x)+h_2)^2)^2} \leq c d^h[f] \|d^h[f]\|_{L^\infty} (\|f\|_{L^\infty}+1)\|\partial_t f\|_{L^\infty}.$$

Thus, integrating in time and using (11.10),

$$\|d^h[f](t+h)\|_{L^\infty} \leq \|d^h[f](t)\|_{L^\infty} e^{c\int_t^{t+h}(E[f]+1)^k},$$

and we conclude this step

$$\frac{d}{dt}\|d^h[f]\|_{L^\infty} = \lim_{h\to 0}\frac{\|d^h[f](t+h)\|_{L^\infty} - \|d^h[f](t)\|_{L^\infty}}{h} \leq c(E[f]+1)^k.$$

Step 3: Estimates on $\|\partial_x^3 f\|_{L^2}$ As before, the bound for the term coming from the first integral in (11.1) can be obtained as in [23], so it only remains the term coming from the second integral. We have

$$I_2 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\partial_x^3\left(\frac{\varpi_2(x-\beta)(\beta+\partial_x f(x)(f(x)+h_2))}{\beta^2+(f(x)+h_2)^2}\right)d\beta dx.$$

For the sake of brevity we only bound the terms with higher order, being the lower order terms (l.o.t.) analogous. We have

$$I_2 = J_3 + J_4 + J_5 + J_6 + J_7 + \text{l.o.t.},$$

with

$$J_3 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x^3\varpi_2(x-\beta)\beta}{\beta^2+(f(x)+h_2)^2}d\beta dx,$$

$$J_4 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x^3\varpi_2(x-\beta)\partial_x f(x)(f(x)+h_2)}{\beta^2+(f(x)+h_2)^2}d\beta dx,$$

$$J_5 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\frac{2\varpi_2(x-\beta)(\beta+\partial_x f(x)(f(x)+h_2))(-f(x)-h_2)\partial_x^3 f(x)}{(\beta^2+(f(x)+h_2)^2)^2}d\beta dx,$$

$$J_6 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(x-\beta)(f(x)+h_2)\partial_x^4 f(x)}{\beta^2+(f(x)+h_2)^2}d\beta dx,$$

$$J_7 = \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\frac{4\varpi_2(x-\beta)\partial_x f(x)\partial_x^3 f(x)}{\beta^2+(f(x)+h_2)^2}d\beta dx.$$

In order to bound J_3 we use the symmetries in the formulae ($\partial_x = -\partial_\beta$) and we integrate by parts:

$$\begin{aligned} J_3 &= \frac{1}{2\pi}\int_{\mathbb{R}}\partial_x^3 f(x)\text{P.V.}\int_{\mathbb{R}}\partial_x^2\varpi_2(x-\beta)\partial_\beta\left(\frac{\beta}{\beta^2+(f(x)+h_2)^2}\right)d\beta dx \\ &\leq c\|\partial_x^3 f\|_{L^2}\|\partial_x^2\varpi_2\|_{L^2}(\|d^h[f]\|_{L^\infty}^2 + \|d^h[f]\|_{L^\infty}+1). \end{aligned}$$

In J_4 we use Cauchy–Schwarz inequality to obtain

$$J_4 \leq c(\|d^h[f]\|_{L^\infty}+1)\|\partial_x^3 f\|_{L^2}\|\partial_x^3\varpi_2\|_{L^2}\|\partial_x f\|_{L^\infty}(\|f\|_{L^\infty}+h_2).$$

The bounds for J_5 and J_7 are similar:

$$J_5 \leq c(\|d^h[f]\|_{L^\infty}^2+1)\|\partial_x^3 f\|_{L^2}^2\|\varpi_2\|_{L^\infty}(1+\|\partial_x f\|_{L^\infty}(\|f\|_{L^\infty}+h_2))(\|f\|_{L^\infty}+h_2),$$

$$J_7 \leq c(\|d^h[f]\|_{L^\infty}+1)\|\partial_x^3 f\|_{L^2}^2\|\varpi_2\|_{L^\infty}\|\partial_x f\|_{L^\infty}.$$

Finally, we integrate by parts in J_6 and we get

$$\begin{aligned} J_6 \leq\ & c\|\partial_x^3 f\|_{L^2}^2(\|d^h[f]\|_{L^\infty}+1)\left(\|\partial_x\varpi_2\|_{L^\infty}(\|f\|_{L^\infty}+1)+\|\varpi_2\|_{L^\infty}\|\partial_x f\|_{L^\infty}\right) \\ & + c\|\partial_x^3 f\|_{L^2}^2(\|d^h[f]\|_{L^\infty}^2+1)\|\varpi_2\|_{L^\infty}\|\partial_x f\|_{L^\infty}(\|f\|_{L^\infty}+1)^2. \end{aligned}$$

As a conclusion, we obtain

$$\frac{d}{dt}\|\partial_x^3 f\|_{L^2} \leq c(E[f]+1)^k.$$

Putting all the estimates together we get the desired bound for the energy:

$$\frac{d}{dt}E[f] \leq c(E[f]+1)^k. \tag{11.11}$$

Step 4: Regularization This step is classical, so, we only sketch this part (see [44] and Chapter 4 for the details). We regularize the problem and we show that the regularized systems have a solution using Picard's Theorem. Given $\tau < h_2$ and $\varsigma < \infty$, we define the set

$$O_\varsigma^{-\tau} = \{f, f \in H^3, \min_x f > -\tau, \|f\|_{H^3} < \varsigma\}.$$

The set

$$O^{-\tau} = \{f, f \in H^3, \min_x f > -\tau\}$$

is open. Indeed, given $f \in O^\tau$ and $h \in B(0,\delta) \subset H^3$, we consider $g = f + h$. We have

$$g(x) = f(x) + h(x) > -\tau - \|h\|_{L^\infty} > -\tau - c\|h\|_{H^3} > -\tau - c\delta > -h_2,$$

if $\delta << 1$ is small enough. As $O_\varsigma^{-\tau} = O^{-\tau} \cap B(0,\varsigma)$ we conclude that $O_\varsigma^{-\tau}$ is an open set in H^3. In this set we apply Picard's Theorem.

Using the previous energy estimates and the fact that the initial energy is finite, these solutions have the same time of existence (T depending only on the initial datum and the physical parameters present in the model) and we can show that they are a Cauchy sequence in $C([0,T], L^2)$. From here, we obtain $f \in C([0,T], H^s(\Omega)) \cap L^\infty([0,T], H^3(\Omega))$ where $T = T(f_0)$ and $0 < s < 3$, a solution to (11.1) as the limit of these regularized solutions. The continuity of the strongest norm H^3 for positive times follows from the parabolic character of the equation. The continuity of $\|f(t)\|_{H^3}$ at $t = 0$ follows from the fact that $f(t) \rightharpoonup f_0$ in H^3 and from the energy estimates.

Step 5: Uniqueness Only remains to show that the solution is unique. Let us suppose that for the same initial datum f_0 there are two smooth solutions f^1 and f^2 with finite energy as defined in (11.8) and consider $f = f^1 - f^2$. Following the same ideas as in the energy estimates we obtain

$$\frac{d}{dt}\|f\|_{L^2} \leq c(f_0, E[f^1], E[f^2])\|f\|_{L^2}.$$

Now we conclude using Gronwall inequality. □

11.4 Well-posedness for the finite depth case

In this section we prove the short time existence of classical solution in the case where the depth is finite. We have the following result:

Theorem 11.3. *Consider* $0 < h_2 < \pi/2$ *a constant and* $f_0(x) = f(x,0) \in H^k(\mathbb{R})$, $k \geq 3$, *the initial datum such that* $\|f_0\|_{L^\infty} < \pi/2$ *and* $-h_2 < \min_x f_0(x)$. *We assume the Rayleigh-Taylor condition is satisfied,* i.e. $\rho^2 - \rho^1 > 0$. *Then, there exists an unique classical solution of* (10.6) $f \in C([0,T], H^k(\mathbb{R}))$ *where* $T = T(f_0)$. *Moreover, we have*

$$f \in C^1([0,T], C(\mathbb{R})) \cap C([0,T], C^2(\mathbb{R})).$$

Proof. Let us consider the usual Sobolev space $H^3(\mathbb{R})$, being the other cases analogous, and define the energy

$$E[f] = \|f\|_{H^3} + \|d^h[f]\|_{L^\infty} + \|d[f]\|_{L^\infty}, \tag{11.12}$$

with

$$d^h[f](x,\beta) = \frac{1}{\cosh(x-\beta) - \cos(f(x)+h_2)}, \tag{11.13}$$

and

$$d[f](x,\beta) = \frac{1}{\cosh(x-\beta) + \cos(f(x)+f(\beta))}. \tag{11.14}$$

We note that $d^h[f]$ represents the distance between f and h and $d[f]$ the distance between f and the boundaries. To simplify notation, we drop the physical parameters present in the problem by considering $\kappa^1(\rho^2-\rho^1) = 4\pi$ and $\mathcal{K} = \frac{1}{2}$. Again, the sign of the difference between the permeabilities will not be important to obtain local existence. We write (2.16) as $\partial_t f = I_1 + I_2 + I_3 + I_4$, being I_1, I_2 the integrals corresponding ϖ_1 and I_3, I_4 the integrals involving ϖ_2. We denote c a constant that can changes from one line to another.

Step 1: Estimates on $\|\varpi_2\|_{H^3}$ Given $f(x)$ such that $E[f] < \infty$ and consider ϖ_2 as defined in (10.7). Then we have that $\|\varpi_2\|_{H^3} \leq c(E[f]+1)^k$. We need to bound $\|J_1\|_{H^3}$ and $\|J_2\|_{H^3}$ with

$$J_1 = \text{P.V.}\int_{\mathbb{R}} \partial_x f(x-\beta)\frac{\sin(h_2+f(x-\beta))}{\cosh(\beta)-\cos(h_2+f(x-\beta))}d\beta,$$

$$J_2 = -\text{P.V.}\int_{\mathbb{R}} \partial_x f(x-\beta)\frac{\sin(-h_2+f(x-\beta))}{\cosh(\beta)+\cos(-h_2+f(x-\beta))}d\beta.$$

We have

$$\begin{aligned}\|J_1\|_{L^2} \leq & \left\|\text{P.V.}\int_{B(0,1)} \frac{\partial_x f(x-\beta)\sin(h_2+f(x-\beta))}{\cosh(\beta)-\cos(h_2+f(x-\beta))}d\beta\right\|_{L^2} \\ &+ \left\|\text{P.V.}\int_{B^c(0,1)} \frac{\partial_x f(x-\beta)\sin(h_2+f(x-\beta))}{\cosh(\beta)-\cos(h_2+f(x-\beta))}d\beta\right\|_{L^2} \\ &\leq c\|\partial_x f\|_{L^2}\|d^h[f]\|_{L^\infty} + c\|\partial_x f\|_{L^2},\end{aligned}$$

where we have used Tonelli's Theorem and Cauchy–Schwarz inequality. Recall that $f - h_2 \in \left(-2h_2, \frac{\pi}{2} - h_2\right)$, thus

$$\frac{1}{\cosh(x-\beta)+\cos(f(x)-h_2)} < \frac{1}{\cosh(x-\beta)-c(h_2)},$$

and the kernel corresponding to ϖ_2 can not be singular and we also obtain

$$\|J_2\|_{L^2} \leq c\|\partial_x f\|_{L^2}.$$

Now, as $G_{h_2,\mathcal{K}} \in \mathcal{S}$, we can use the Young's inequality for the convolution terms obtaining bounds with an universal constant depending on h_2 and $\mathcal{K}$. Indeed, we have

$$\|G_{h_2,\mathcal{K}} * J_i\|_{L^2} \leq c\|J_i\|_{L^2},$$

and we obtain

$$\|\varpi_2\|_{L^2} \leq c(E[f]+1)^k.$$

Now we observe that we can write

$$J_1 = \text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\beta)}{\cosh(\beta)-\cos(h_2+f(x-\beta))}d\beta,$$

$$J_2 = \text{P.V.}\int_{\mathbb{R}} \frac{\sinh(\beta)}{\cosh(\beta)+\cos(-h_2+f(x-\beta))}d\beta,$$

and we obtain $\|\partial_x^3 J_i\|_{L^2} \leq c(E[f]+1)^k$. Using Young inequality, we conclude

$$\|\varpi_2\|_{H^3} \leq c(E[f]+1)^k.$$

Step 2: Estimates on $\|d^h[f]\|_{L^\infty}$ and $\|d[f]\|_{L^\infty}$ The integrals corresponding to ϖ_1 in (10.6) can be bounded (see Chapter 4) as

$$|I_1+I_2| \leq c(E[f]+1)^k.$$

The new terms are the integrals I_3 and I_4, those involving ϖ_2, in (10.6). We have, when splitted accordingly to the decay at infinity,

$$I_3+I_4 = J_3+J_4,$$

and we obtain the following estimates:

$$|J_3| \leq \left\|\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(x-\beta)\sinh(\beta)}{\cosh(\beta)-\cos(f(x)+h_2)} - \frac{\varpi_2(x-\beta)\sinh(\beta)}{\cosh(\beta)+\cos(f(x)-h_2)}d\beta\right\|_{L^\infty} \leq c\|\varpi_2\|_{L^\infty}\left(\|d^h[f]\|_{L^\infty}+1\right),$$

$$|J_4| \leq \left\|\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2(x-\beta)\partial_x f(x)\sin(f(x)+h_2)}{\cosh(\beta)-\cos(f(x)+h_2)} + \frac{\varpi_2(x-\beta)\partial_x f(x)\sin(f(x)-h_2)}{\cosh(\beta)+\cos(f(x)-h_2)}d\beta\right\|_{L^\infty} \leq c\|\varpi_2\|_{L^\infty}\|\partial_x f\|_{L^\infty}\left(\|d^h[f]\|_{L^\infty}+1\right).$$

We conclude the following useful estimate

$$\|\partial_t f\|_{L^\infty} \leq c(E[f]+1)^k. \tag{11.15}$$

We have

$$\frac{d}{dt}d^h[f] = -\frac{\sin(f(x)+h_2)\partial_t f(x)}{(\cosh(x-\beta)-\cos(f(x)+h_2))^2} \leq d^h[f]\|d^h[f]\|_{L^\infty}\|\partial_t f\|_{L^\infty}.$$

Thus, using (11.15) and integrating in time, we obtain the desired bound for $d^h[f]$:

$$\frac{d}{dt}\|d^h[f]\|_{L^\infty} = \lim_{h\to 0}\frac{\|d^h[f](t+h)\|_{L^\infty}-\|d^h[f](t)\|_{L^\infty}}{h} \leq c(E[f]+1)^k.$$

To obtain the corresponding bound for $d[f]$ we proceed in the same way and we use (11.15) (see Chapter 4 for the details)

Step 3: Estimates on $\|\partial_x^3 f\|_{L^2}$ As before, see Chapter 4 for the details concerning the terms coming from ϖ_1 in (10.6). It only remains the terms coming from ϖ_2:

$$I = \int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\partial_x^3 f(x)\partial_x^3\left(\frac{\varpi_2(\beta)(\sinh(x-\beta)+\partial_x f(x)\sin(f(x)+h_2))}{\cosh(x-\beta)-\cos(f(x)+h_2)} + \frac{\varpi_2(\beta)(-\sinh(x-\beta)+\partial_x f(x)\sin(f(x)-h_2))}{\cosh(x-\beta)+\cos(f(x)-h_2)}\right)d\beta dx.$$

We split

$$I = J_7+J_8+J_9+\text{l.o.t.}.$$

The lower order terms (l.o.t.) can be obtained in a similar way, so we only study the terms J_i. We have

$$J_7 \leq \int_{\mathbb{R}}\text{P.V.}\int_{\mathbb{R}}\frac{\partial_x^3 f(x)\partial_x^3\varpi_2(x-\beta)\sinh(\beta)}{\cosh(\beta)-\cos(f(x)+h_2)} - \frac{\partial_x^3 f(x)\partial_x^3\varpi_2(x-\beta)\sinh(\beta)}{\cosh(\beta)+\cos(f(x)-h_2)}d\beta dx \leq c\|\partial_x^3 f\|_{L^2}\|\partial_x^3\varpi_2\|_{L^2}(\|d^h[f]+1\|),$$

$$J_8 \leq \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \frac{\partial_x^3 f(x) \partial_x^3 \varpi_2(x-\beta) \partial_x f(x) \sin(f(x)+h_2)}{\cosh(\beta) - \cos(f(x)+h_2)} - \frac{\partial_x^3 f(x) \partial_x^3 \varpi_2(x-\beta) \partial_x f(x) \sin(f(x)+h_2)}{\cosh(\beta) + \cos(f(x)-h_2)} d\beta dx$$
$$\leq c\|\partial_x^3 f\|_{L^2} \|\partial_x^3 \varpi_2\|_{L^2} \|\partial_x f\|_{L^\infty} (\|d^h[f] + 1\|).$$

The term J_9 is given by

$$J_9 = \frac{1}{2} \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \partial_x (\partial_x^3 f(x))^2 \left(\frac{\varpi_2(\beta) \sin(f(x)+h_2)}{\cosh(x-\beta) - \cos(f(x)+h_2)} + \frac{\varpi_2(\beta) \sin(f(x)-h_2)}{\cosh(x-\beta) + \cos(f(x)-h_2)} \right) d\beta dx.$$

Integrating by parts

$$|J_9| \leq c\|\partial_x^3 f\|_{L^2} (\|d^h[f]\|_{L^\infty} + 1)(\|\partial_x \varpi_2\|_{L^\infty} + \|\varpi_2\|_{L^\infty} \|\partial_x f\|_{L^\infty})$$
$$+ c\|\partial_x^3 f\|_{L^2} (\|d^h[f]\|_{L^\infty}^2 + 1) \|\varpi_2\|_{L^\infty} (1 + \|\partial_x f\|_{L^\infty})$$

Step 4: Regularization and uniqueness These steps follow the same lines as in Theorem 11.2. This concludes the result. □

Chapter 12

Turning waves

In this Chapter we prove finite time singularities for equations (10.2), (10.4) and (10.6) with some conditions on the physical parameters present in the problem. These singularities mean that the curve turns over or, equivalently, in finite time they can not be parametrized as graphs. The proof of turning waves follows the steps and ideas in [10] for the homogeneus infinitely deep case where here we have to deal with the difficulties coming from the boundaries and the delta coming from the jump in the permeabilities. The result for the confined and homogeneous Muskat problem is contained in Chapter 8.

To obtain these results, we construct curves such that the velocity has the correct property. These curves depends on the physycal parameters $\mathcal{K}$ defined in (10.1) and $-h_2$, the line where the permeability changes. In the infinitely deep case the curve has an amplitude of order h_2 in the periodic case and of order $(h_2)^\delta$ with $\delta < 1/4$ in the flat at infinity case. So, even if $h_2 >> 1$, this result is not some kind of *linearization* (compare with Theorem 13.2 in Chapter 13). In the confined and inhomogeneous case we have less degress of freedom (we can not take a very big amplitude for the curve) and we construct the turning wave by taking $\mathcal{K}$ close to zero. The sharpness of these conditions for the physical parameters is studied in Chapter 13.

12.1 Infinite depth

Let Ω be the spatial domain considered, *i.e.* $\Omega = \mathbb{R}$ or $\Omega = \mathbb{T}$. We have

Theorem 12.1. *Let us suppose that the Rayleigh-Taylor condition is satisfied,* i.e. $\rho^2 - \rho^1 > 0$. *Then, there are initial data,* $f_0(x) \in H^3(\Omega)$, *under the hypothesis of Theorem 11.2, such that, for any possible choice of* $\kappa^1, \kappa^2 > 0$ *and* $h_2 >> 1$, *there exists a solution of* (10.2) *or* (10.4) *and a time* T^* *for which*

$$\lim_{t \to T^*} \|\partial_x f(t)\|_{L^\infty} = \infty.$$

For short time $t > T^*$, *the solution can be continued but it is not a graph.*

Proof. To simplify notation we drop the physical parameters present in the problem by considering $\kappa^1(\rho^2 - \rho^1) = 2\pi$. The proof has three steps. First, we consider solutions which are arbitrary curves (not necessary graphs) and we *translate* the singularity formation to the fact $\partial_\alpha v_1(0) = \partial_t \partial_\alpha z_1(0) < 0$. The second step is to construct a family of curves such that $\partial_\alpha v_1(0)$ is negative. Thus, we have that *if there exists, forward and backward in time, a solution corresponding to initial data which are arbitrary curves*, then, we have proved that there is a singularity in finite time. The last step is to prove, using a Cauchy-Kovalevsky theorem, that there exists local in time solutions in this unstable case.

Step 1: Obtaining the appropriate expression Consider the case $\Omega = \mathbb{R}$. Due to (10.18) we have

$$\partial_\alpha \partial_t z_1(\alpha) = I_1 + I_2 + I_3,$$

where

$$I_1(\alpha) = \partial_\alpha \text{P.V.} \int_{\mathbb{R}} \frac{z_1(\alpha) - z_1(\alpha-\beta)}{|z(\alpha) - z(\alpha-\beta)|^2} (\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta)) d\beta,$$

$$I_2(\alpha) = \partial_\alpha \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \varpi_2(\alpha-\beta) \frac{-z_2(\alpha) - h_2}{|z(\alpha) - h(\alpha-\beta)|^2} d\beta,$$

$$I_3(\alpha) = \partial_\alpha \left(\partial_\alpha z_1(\alpha) \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \varpi_2(\alpha-\beta) \frac{z_2(\alpha) + h_2}{|z(\alpha) - h(\alpha-\beta)|^2} d\beta \right).$$

Assume now that the following conditions for $z(\alpha)$ holds:

- $z_i(\alpha)$ are odd functions,
- $\partial_\alpha z_1(0) = 0, \partial_\alpha z_1(\alpha) > 0 \ \forall \alpha \neq 0$, and $\partial_\alpha z_2(0) > 0$,
- $z(\alpha) \neq h(\alpha) \ \forall \alpha$.

The previous hypotheses mean that z is a curve satisfying the arc-chord condition and $\partial_\alpha z(0)$ only has vertical component. Due to these conditions on z, we have $\partial_\alpha z_1(0) = 0$ and $\partial_\alpha^2 z_1$ is odd. Then, the second derivative at zero is zero and we get that $I_3(0) = 0$. For I_1 we get

$$I_1(0) = \text{P.V.} \int_{\mathbb{R}} \frac{\partial_\alpha^2 z_1(\beta) z_1(\beta) + (\partial_\alpha z_1(\beta))^2}{(z_1(\beta))^2 + (z_2(\beta))^2} d\beta - 2\text{P.V.} \int_{\mathbb{R}} \frac{(\partial_\alpha z_1(\beta) z_1(\beta))^2}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta$$
$$+ 2\text{P.V.} \int_{\mathbb{R}} \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta) \left(\partial_\alpha z_2(0) - \partial_\alpha z_2(\beta)\right)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta.$$

We integrate by parts and we obtain, after some lengthy computations,

$$I_1(0) = 4\partial_\alpha z_2(0) \text{P.V.} \int_0^\infty \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta. \tag{12.1}$$

For the term with the second vorticity, ϖ_2, we have

$$I_2(0) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\partial_\beta \varpi_2(-\beta) h_2}{\beta^2 + h_2^2} d\beta + \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{-\varpi_2(-\beta) \partial_\alpha z_2(0)}{\beta^2 + h_2^2} d\beta$$
$$- \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{2\varpi_2(-\beta)\beta h_2}{(\beta^2 + h_2^2)^2} d\beta - \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\partial_\alpha z_2(0) \varpi_2(-\beta)(-h_2^2)}{(\beta^2 + h_2^2)^2} d\beta,$$

and, after an integration by parts we obtain

$$I_2(0) = -\frac{\partial_\alpha z_2(0)}{2\pi} \text{P.V.} \int_0^\infty \frac{(\varpi_2(\beta) + \varpi_2(-\beta))\beta^2}{(\beta^2 + h_2^2)^2} d\beta. \tag{12.2}$$

Putting all together we obtain that in the flat at infinity case the important quantity for the singularity is

$$\partial_\alpha v_1(0) = \partial_\alpha z_2(0) \left(4\text{P.V.} \int_0^\infty \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta \right.$$
$$\left. - \frac{1}{2\pi} \text{P.V.} \int_0^\infty \frac{(\varpi_2(\beta) + \varpi_2(-\beta))\beta^2}{(\beta^2 + h_2^2)^2} d\beta \right), \tag{12.3}$$

where, due to (10.17), ϖ_2 is defined as

$$\varpi_2(\beta) = 2\mathcal{K}\text{P.V.}\int_{\mathbb{R}} \frac{(h_2 + z_2(\gamma))\partial_\alpha z_2(\gamma)}{(h_2 + z_2(\gamma))^2 + (\beta - z_1(\gamma))^2} d\gamma. \tag{12.4}$$

We apply the same procedure to equation (10.21) and we get the importat quantity in the periodic setting[1]:

$$\begin{aligned}\partial_\alpha v_1^p(0) = \partial_\alpha z_2(0) \Bigg(\int_0^\pi \frac{\partial_\alpha z_1(\beta)\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} d\beta \\ + \frac{1}{4\pi}\int_0^\pi \frac{(\varpi_2^p(\beta) + \varpi_2^p(-\beta))(-1 + \cosh(h_2)\cos(\beta))}{(\cosh(h_2) - \cos(\beta))^2} d\beta \Bigg),\end{aligned} \tag{12.5}$$

and, due to (10.22),

$$\varpi_2^p(\beta) = \mathcal{K}\int_{\mathbb{T}} \frac{\sin(\beta - z_1(\gamma))\partial_\alpha z_1(\gamma)}{\cosh(h_2 + z_2(\gamma)) - \cos(\beta - z_1(\gamma))} d\gamma. \tag{12.6}$$

Step 2: Taking the appropriate curve To clarify the proof, let us consider first the periodic setting. Given $1 < h_2$, we consider a, b, constants such that $2 < b \leq a$ and let us define

$$z_1(\alpha) = \alpha - \sin(\alpha),$$

and

$$z_2(\alpha) = \begin{cases} \frac{\sin(a\alpha)}{a} & \text{if } 0 \leq \alpha \leq \frac{\pi}{a}, \\ \frac{\sin\left(\pi\frac{\alpha - (\pi/a)}{(\pi/a) - (\pi/b)}\right)}{b} & \text{if } \frac{\pi}{a} < \alpha < \frac{\pi}{b}, \\ \left(\frac{-h_2/2}{\frac{\pi}{2} - \frac{\pi}{b}}\right)\left(\alpha - \frac{\pi}{b}\right) & \text{if } \frac{\pi}{b} \leq \alpha < \frac{\pi}{2}, \\ -\left(\frac{-h_2/2}{\frac{\pi}{2} - \frac{\pi}{b}}\right)\left(\alpha - \pi + \frac{\pi}{b}\right) & \text{if } \frac{\pi}{2} \leq \alpha < \pi(1 - \frac{1}{b}), \\ 0 & \text{if } \pi(1 - \frac{1}{b}) \leq \alpha. \end{cases} \tag{12.7}$$

Due to the definition of z_2, we have

$$\frac{h_2}{2} \leq h_2 + z_2(\alpha) \leq \frac{3h_2}{2},$$

and using (12.6), we get

$$|\varpi_2^p(\beta)| \leq \frac{4\pi}{\cosh(h_2/2) - 1}.$$

Inserting this curve in (12.5) we obtain

$$\partial_\alpha v_1^p(0) \leq I_a + I_b^{h_2} + I_2^{h_2},$$

with

$$I_a = \int_0^{\pi/a} \frac{(1 - \cos(\beta))\sin(\beta - \sin(\beta))\sinh\left(\frac{\sin(a\beta)}{a}\right)}{\left(\cosh\left(\frac{\sin(a\beta)}{a}\right) - \cos(\beta - \sin(\beta))\right)^2} d\beta,$$

[1]Recall the superscript p in the notation denoting that we are in the periodic setting

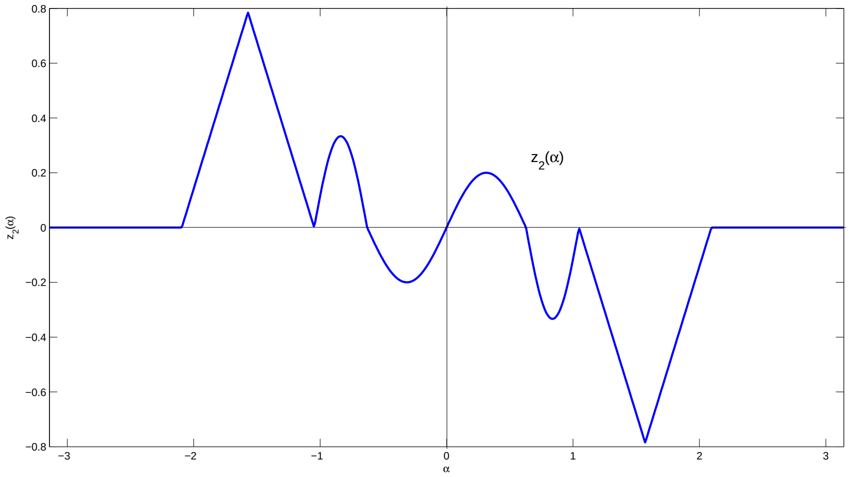

Figure 12.1: z_2 for $a = 5, b = 3, h_2 = \pi/2$

$$I_b^{h_2} = \int_{(\pi/b+\pi)/3}^{\pi/2} \frac{(1-\cos(\beta))\sin(\beta-\sin(\beta))\sinh\left(\left(\frac{-h_2/2}{\frac{\pi}{2}-\frac{\pi}{b}}\right)\left(\beta-\frac{\pi}{b}\right)\right)}{\left(\cosh\left(\left(\frac{-h_2/2}{\frac{\pi}{2}-\frac{\pi}{b}}\right)\left(\beta-\frac{\pi}{b}\right)\right)-\cos(\beta-\sin(\beta))\right)^2}d\beta$$
$$+\int_{\pi/2}^{(2\pi-\pi/b)/3)} \frac{(1-\cos(\beta))\sin(\beta-\sin(\beta))\sinh\left(-\left(\frac{-h_2/2}{\frac{\pi}{2}-\frac{\pi}{b}}\right)\left(\alpha-\pi+\frac{\pi}{b}\right)\right)}{\left(\cosh\left(-\left(\frac{-h_2/2}{\frac{\pi}{2}-\frac{\pi}{b}}\left(\alpha-\pi+\frac{\pi}{b}\right)\right)\right)-\cos(\beta-\sin(\beta))\right)^2}d\beta,$$

and $I_2^{h_2}$ is the integral involving the second vorticity ϖ_2^p. We remark that I_a does not depend on h_2. The sign of $I_b^{h_2}$ is the same as the sign of z_2, thus we get $I_b^{h_2} < 0$ and this is independent of the choice of a and h_2.

Now, we fix b and we take h_2 sufficiently large such that

$$I_b^{h_2} + I_2^{h_2} \leq I_b^{h_2} + \frac{2\pi}{\cosh(h_2/2)-1}\frac{1+\cosh(h_2)}{(\cosh(h_2)-1)^2} < 0.$$

We can do that because

$$\frac{c_b\sinh(h_2/3)}{(\cosh(h_2/2)+1)^2} \leq |I_b^{h_2}|$$

or, equivalently,

$$I_b^{h_2} + I_2^{h_2} = -|I_b^{h_2}| + I_2^{h_2} \leq -\frac{c_b\sinh(h_2/3)}{(\cosh(h_2/2)-1)^2} + \frac{2\pi}{\cosh(h_2/2)-1}\frac{1+\cosh(h_2)}{(\cosh(h_2)-1)^2} < 0,$$

if h_2 is large enough. The integral I_a is well defined and positive, but goes to zero as a grows. Then, fixed b and h_2 in such a way $I_b^{h_2} + I_2^{h_2} < 0$, we take a sufficiently large such that $I_a + I_b^{h_2} + I_2^{h_2} < 0$. We are done with the periodic case.

We proceed with the flat at infinity case. We take $2 < b \leq a$ as before and $0 < \delta < 1$ and define

$$z_1(\alpha) = \alpha - \sin(\alpha)\exp(-\alpha^2), \tag{12.8}$$

and

$$z_2(\alpha) = \begin{cases} \dfrac{\sin(a\alpha)}{a} & \text{if } 0 \leq \alpha \leq \dfrac{\pi}{a}, \\ \dfrac{\sin\left(\pi \frac{\alpha-(\pi/a)}{(\pi/a)-(\pi/b)}\right)}{b} & \text{if } \dfrac{\pi}{a} < \alpha < \dfrac{\pi}{b}, \\ \left(\dfrac{-h_2^\delta}{\frac{\pi}{2}-\frac{\pi}{b}}\right)\left(\alpha - \dfrac{\pi}{b}\right) & \text{if } \dfrac{\pi}{b} \leq \alpha < \dfrac{\pi}{2}, \\ -\left(\dfrac{-h_2^\delta}{\frac{\pi}{2}-\frac{\pi}{b}}\right)\left(\alpha - \pi + \dfrac{\pi}{b}\right) & \text{if } \dfrac{\pi}{2} \leq \alpha < \pi(1-\dfrac{1}{b}), \\ 0 & \text{if } \pi(1-\dfrac{1}{b}) \leq \alpha. \end{cases} \tag{12.9}$$

We have

$$h_2 - h_2^\delta < h_2 + z_2(\beta) < h_2 + h_2^\delta,$$

and we assume $1 < h_2 - h_2^\delta$. Inserting the curve (12.8) and (12.9) in (12.4) and changing variables, we obtain

$$|\varpi_2(\beta)| \leq 2\text{P.V.} \int_{\mathbb{R}} \frac{(h_2+h_2^\delta)h_2^\delta\left(\frac{\pi}{2}-\frac{\pi}{b}\right)^{-1}}{(h_2-h_2^\delta)^2 + (\gamma - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})^2} d\gamma.$$

We split the integral in two parts:

$$J_1 = 2\text{P.V.} \int_{B(0,2(h_2-h_2^\delta))} \frac{(h_2+h_2^\delta)h_2^\delta\left(\frac{\pi}{2}-\frac{\pi}{b}\right)^{-1}}{(h_2-h_2^\delta)^2 + (\gamma - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})^2} d\gamma \leq 8h_2^\delta\left(\frac{\pi}{2}-\frac{\pi}{b}\right)^{-1},$$

$$J_2 = 2\text{P.V.} \int_{B^c(0,2(h_2-h_2^\delta))} \frac{(h_2+h_2^\delta)h_2^\delta\left(\frac{\pi}{2}-\frac{\pi}{b}\right)^{-1}}{(h_2-h_2^\delta)^2 + (\gamma - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})^2} d\gamma.$$

We split the integral J_2 in its positive and negative regions, K_1 and K_2. We have

$$\begin{aligned} K_1 &= \text{P.V.} \int_{2(h_2-h_2^\delta)}^{\infty} \frac{1}{(h_2-h_2^\delta)^2 + (\gamma - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})^2} d\gamma \\ &\leq \text{P.V.} \int_{2(h_2-h_2^\delta)}^{\infty} \frac{1}{(h_2-h_2^\delta)^2 + \gamma^2 - 2\gamma\sin(\beta-\gamma)e^{-(\beta-\gamma)^2}} d\gamma \\ &\leq \text{P.V.} \int_{2(h_2-h_2^\delta)}^{\infty} \frac{1}{(h_2-h_2^\delta-\gamma)^2 + 2\gamma(h_2-h_2^\delta - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})} d\gamma, \end{aligned}$$

and using that h_2 is such that $1 < h_2 - h_2^\delta$, we get

$$K_1 \leq \text{P.V.} \int_{2(h_2-h_2^\delta)}^{\infty} \frac{1}{(h_2-h_2^\delta-\gamma)^2} d\gamma = \frac{1}{h_2-h_2^\delta}.$$

The remaining integral is

$$\begin{aligned} K_2 &= \text{P.V.} \int_{-\infty}^{-2(h_2-h_2^\delta)} \frac{1}{(h_2-h_2^\delta)^2 + (\gamma - \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})^2} d\gamma \\ &\leq \text{P.V.} \int_{-\infty}^{-2(h_2-h_2^\delta)} \frac{1}{(h_2-h_2^\delta)^2 + \gamma^2 - 2\gamma\sin(\beta-\gamma)e^{-(\beta-\gamma)^2}} d\gamma \\ &\leq \text{P.V.} \int_{-\infty}^{-2(h_2-h_2^\delta)} \frac{1}{(h_2-h_2^\delta+\gamma)^2 - 2\gamma(h_2-h_2^\delta + \sin(\beta-\gamma)e^{-(\beta-\gamma)^2})} d\gamma, \end{aligned}$$

and using that h_2 is such that $1 < h_2 - h_2^\delta$, we get

$$K_2 \leq \text{P.V.} \int_{-\infty}^{-2(h_2-h_2^\delta)} \frac{1}{(h_2 - h_2^\delta + \gamma)^2} d\gamma = \frac{1}{h_2 - h_2^\delta}.$$

Collecting all the estimates, we get

$$J_2 \leq 4h_2^\delta \left(\frac{\pi}{2} - \frac{\pi}{b}\right)^{-1},$$

and

$$|\varpi_2(\beta)| \leq 12h_2^\delta \left(\frac{\pi}{2} - \frac{\pi}{b}\right)^{-1}.$$

Using this bound in (12.3) we get

$$|I_2^{h_2}| \leq 3h_2^{\delta-1} \left(\frac{\pi}{2} - \frac{\pi}{b}\right)^{-1}.$$

Then, as before,

$$\partial_\alpha v_1(0) \leq I_a + I_b^{h_2} + I_2^{h_2},$$

where $I_a, I_b^{h_2}$ are the integrals $I_1(0)$ on the intervals $(0, \pi/a)$ and $((\pi/b + \pi)/3, (2\pi - \pi/b)/3)$, respectively. We have

$$c_b \frac{2h_2^\delta}{3(h_2^\delta)^4} \leq |I_b^{h_2}|,$$

thus, we get

$$I_b^{h_2} + I_2^{h_2} = -|I_b^{h_2}| + I_2^{h_2} \leq -c_b \frac{2h_2^{-3\delta}}{3} + 3h_2^{\delta-1} \left(\frac{\pi}{2} - \frac{\pi}{b}\right)^{-1}.$$

To ensure that the decay of $I_2^{h_2}$ is faster than the decay of $I_b^{h_2}$, we take $\delta < 1/4$. Now, fixing b, we can obtain $1 < h_2$ and $0 < \delta < 1/4$ such that $1 < h_2 - h_2^\delta$ and $I_b^{h_2} + I_2^{h_2} < 0$. Taking $a >> b$ we obtain a curve such that $\partial_\alpha v_1(0) < 0$. In order to conclude the argument, it is enough to approximate these curves (12.7) and (12.9) by analytic functions. We are done with this step of the proof.

Step 3: Showing the forward and backward solvability At this point, we need to prove that there is a solution forward and backward in time corresponding to these curves (12.7) and (12.9). Indeed, if this solution exists then, due to the previous step, we obtain that, for a short time $t < 0$, the solution is a graph with finite $H^3(\Omega)$ energy (in fact, it is analytic). This graph at time $t = 0$ has a blow up for $\|\partial_x f\|_{L^\infty}$ and, for a short time $t > 0$, the solution can not be parametrized as a graph. We show the result corresponding to the flat at infinity case, being the periodic one analogous. We consider curves z satisfying the arc-chord condition and such that

$$\lim_{|\alpha|\to\infty} |z(\alpha) - (\alpha, 0)| = 0.$$

We define the complex strip $\mathbb{B}_r = \{\zeta + i\xi, \zeta \in \mathbb{R}, |\xi| < r\}$, and the spaces

$$X_r = \{z = (z_1, z_2) \text{ analytic curves satisfying the arc-chord condition on } \mathbb{B}_r\}, \tag{12.10}$$

with norm

$$\|z\|_r^2 = \|z(\gamma) - (\gamma, 0)\|_{H^3(\mathbb{B}_r)}^2,$$

where $H^3(\mathbb{B}_r)$ denotes the Hardy-Sobolev space on the strip (see [2]) with the norm

$$\|f\|_r^2 = \sum_{\pm} \int_{\mathbb{R}} |f(\zeta \pm ri)|^2 d\zeta + \int_{\mathbb{R}} |\partial_\alpha^3 f(\zeta \pm ri)|^2 d\zeta. \tag{12.11}$$

These spaces form a Banach scale. For notational convenience, we write $\gamma = \alpha \pm ir$, $\gamma' = \alpha \pm ir'$. Recall that, for $0 < r' < r$,

$$\|\partial_\alpha \cdot \|_{L^2(\mathbb{B}_{r'})} \leq \frac{C}{r - r'} \| \cdot \|_{L^2(\mathbb{B}_r)}. \tag{12.12}$$

We consider the complex extension of (10.17) and (10.18), which is given by

$$\begin{aligned}\partial_t z(\gamma) = \text{P.V.} \int_{\mathbb{R}} \frac{(z_1(\gamma) - z_1(\gamma-\beta))(\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma-\beta))}{(z_1(\gamma) - z_1(\gamma-\beta))^2 + (z_2(\gamma) - z_2(\gamma-\beta))^2} d\beta \\ + \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varpi_2(\gamma-\beta)(z(\gamma) - h(\gamma-\beta))^\perp}{(z_1(\gamma) - (\gamma-\beta))^2 + (z_2(\gamma) + h_2)^2} d\beta \\ + \partial_\alpha z(\gamma) \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{(z_2(\gamma) + h_2)\varpi_2(\gamma-\beta)}{(z_1(\gamma) - (\gamma-\beta))^2 + (z_2(\gamma) + h_2)^2} d\beta, \end{aligned} \tag{12.13}$$

with

$$\varpi_2(\gamma) = 2\mathcal{K}\, \text{P.V.} \int_{\mathbb{R}} \frac{(h_2 + z_2(\gamma-\beta))\partial_\alpha z_2(\gamma-\beta)}{(\gamma - z_1(\gamma-\beta))^2 + (h_2 + z_2(\gamma-\beta))^2} d\beta. \tag{12.14}$$

Recall the fact that in the case of a real variable graph ϖ_2 has the same regularity as f, but in the case of an arbitrary curve ϖ_2 is, roughly speaking, at the level of the first derivative of the interface. This fact will be used below. We define

$$d^-[z](\gamma, \beta) = \frac{\beta^2}{(z_1(\gamma) - z_1(\gamma-\beta))^2 + (z_2(\gamma) - z_2(\gamma-\beta))^2}, \tag{12.15}$$

$$d^h[z](\gamma, \beta) = \frac{1 + \beta^2}{(z_1(\gamma) - (\gamma-\beta))^2 + (z_2(\gamma) + h_2)^2}. \tag{12.16}$$

The function d^- is the complex extension of the *arc chord condition* and we need it to bound the terms with ϖ_1. The function d^h comes from the different permeabilities and we use it to bound the terms with ϖ_2. We observe that both are bounded functions for the considered curves. Consider $0 < r' < r$ and the set

$$O_R = \{z \in X_r \text{ such that } \|z\|_r < R, \|d^-[z]\|_{L^\infty(\mathbb{B}_r)} < R, \|d^h[z]\|_{L^\infty(\mathbb{B}_r)} < R\},$$

where $d^-[z]$ and $d^h[z]$ are defined in (12.15) and (12.16). Then we claim that, for $z, w \in O_R$, the righthand side of (12.13), $F : O_R \to X_{r'}$ is continuous and the following inequalities holds:

$$\|F[z]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z\|_r, \tag{12.17}$$

$$\|F[z] - F[w]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z - w\|_{H^3(\mathbb{B}_r)}, \tag{12.18}$$

$$\sup_{\gamma \in \mathbb{B}_r, \beta \in \mathbb{R}} |F[z](\gamma) - F[z](\gamma - \beta)| \leq C_R |\beta|. \tag{12.19}$$

The claim for the spatial operator corresponding to ϖ_1 has been studied in [10], thus, we only deal with the new terms containing ϖ_2. For the sake of brevity we only bound some terms, being the other analogous. Using Tonelli's theorem and Cauchy–Schwarz inequality we have that

$$\|\varpi_2\|_{L^2(\mathbb{B}_{r'})} \leq c \|d^h[z]\|_{L^\infty} (1 + \|z_2\|_{L^\infty(\mathbb{B}_{r'})}) \|\partial_\alpha z_2\|_{L^2(\mathbb{B}_{r'})}.$$

Moreover, we get

$$\|\varpi_2\|_{H^2(\mathbb{B}_r)} \leq C_R \|z\|_r. \tag{12.20}$$

For $\partial_\alpha^3 \varpi_2$ the procedure is similar but we lose one derivative. Using (12.12) and Sobolev embedding we conclude

$$\|\varpi_2\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r - r'} \|z\|_r. \tag{12.21}$$

From here inequality (12.17) follows. Inequality (12.18), for the terms involving ϖ_1, can be obtained using the properties of the Hilbert transform as in [10]. Let's change slightly the notation and write $\varpi_2[z](\gamma)$ for the integral in (12.14). We split

$$\begin{aligned} A_1 &= \text{P.V.}\int_{\mathbb{R}} \frac{(\varpi_2[z](\gamma'-\beta)-\varpi_2[w](\gamma'-\beta))(z(\gamma')-h(\gamma'-\beta))^{\perp}}{(z_1(\gamma')-(\gamma'-\beta))^2+(z_2(\gamma')+h_2)^2}d\beta \\ &+\text{P.V.}\int_{\mathbb{R}} \frac{\varpi_2[w](\gamma'-\beta)\left((z(\gamma')-h(\gamma'-\beta))^{\perp}-(w(\gamma')-h(\gamma'-\beta))^{\perp}\right)}{(z_1(\gamma')-(\gamma'-\beta))^2+(z_2(\gamma')+h_2)^2}d\beta \\ &+\text{P.V.}\int_{\mathbb{R}} \varpi_2[w](\gamma'-\beta)(w(\gamma')-h(\gamma'-\beta))^{\perp}\frac{d^h[z](\gamma',\beta)-d^h[w](\gamma',\beta)}{1+\beta^2}d\beta \\ &= B_1+B_2+B_3. \end{aligned}$$

In B_3 we need some extra decay at infinity to ensure the finiteness of the integral. We compute

$$|d^h[z]-d^h[w]|\leq C_R\frac{|d^h[z]d^h[w]|}{1+\beta^2}\,|(1+\beta)(z_1-w_1)+z_2-w_2|<C_R|z-w|\frac{|1+\beta|}{1+\beta^2},$$

and, due to Sobolev embedding, we get

$$\|B_3\|_{L^2(\mathbb{B}_{r'})}\leq C_R\|\varpi_2[w]\|_{L^2(\mathbb{B}_{r'})}\|z-w\|_{L^\infty(\mathbb{B}_{r'})}\leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

For the second term, B_2, we obtain the same bound

$$\|B_2\|_{L^2(\mathbb{B}_{r'})}\leq C_R\|\varpi_2[w]\|_{L^2(\mathbb{B}_{r'})}\|z-w\|_{L^\infty(\mathbb{B}_{r'})}\leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

We split B_1 componentwise. In the first coordinate we have

$$\begin{aligned} \|C_1\|_{L^2(\mathbb{B}_{r'})} = \left\|\text{P.V.}\int_{\mathbb{R}}\frac{(\varpi_2[z](\gamma'-\beta)-\varpi_2[w](\gamma'-\beta))(-z_2(\gamma')-h_2)}{(z_1(\gamma')-(\gamma'-\beta))^2+(z_2(\gamma')+h_2)^2}d\beta\right\|_{L^2(\mathbb{B}_{r'})} \\ \leq C_R\|\varpi_2[z]-\varpi_2[w]\|_{L^2(\mathbb{B}_{r'})}. \end{aligned}$$

In the second coordinate we need to ensure the integrability at infinity. We get

$$\begin{aligned} C_2 &= \text{P.V.}\int_{\mathbb{R}}\frac{(\varpi_2[z](\gamma'-\beta)-\varpi_2[w](\gamma'-\beta))(z_1(\gamma')-\gamma')}{(z_1(\gamma')-(\gamma'-\beta))^2+(z_2(\gamma')+h_2)^2}d\beta \\ &+\text{P.V.}\int_{\mathbb{R}}(\varpi_2[z](\gamma'-\beta)-\varpi_2[w](\gamma'-\beta))\left(\frac{\beta d^h[z]}{1+\beta^2}-\frac{1}{\beta}\right)d\beta \\ &+H\varpi_2[z](\gamma')-H\varpi_2[w](\gamma'), \end{aligned}$$

and, with this splitting and the properties of the Hilbert transform, we obtain

$$\|C_2\|_{L^2(\mathbb{B}_{r'})}\leq C_R\|\varpi_2[z]-\varpi_2[w]\|_{L^2(\mathbb{B}_{r'})}.$$

We get

$$\varpi_2[z]-\varpi_2[w]=C_3+C_4+C_5,$$

where

$$C_3=2\mathcal{K}\,\text{P.V.}\int_{\mathbb{R}}\frac{(z_2(\gamma-\beta)-w_2(\gamma-\beta))\partial_\alpha z_2(\gamma-\beta)}{(\gamma-z_1(\gamma-\beta))^2+(h_2+z_2(\gamma-\beta))^2}d\beta,$$

$$C_4=2\mathcal{K}\,\text{P.V.}\int_{\mathbb{R}}\frac{(h_2+w_2(\gamma-\beta))(\partial_\alpha z_2(\gamma-\beta)-\partial_\alpha w_2(\gamma-\beta))}{(\gamma-z_1(\gamma-\beta))^2+(h_2+z_2(\gamma-\beta))^2}d\beta,$$

$$C_5 = 2\mathcal{K}\,\text{P.V.}\int_{\mathbb{R}}(h_2+w_2(\gamma-\beta))\partial_\alpha w_2(\gamma-\beta)\frac{d^h[z](\gamma-\beta,-\beta)-d^h[w](\gamma-\beta,-\beta)}{1+\beta^2}d\beta.$$

From these expressions we obtain

$$\|C_3\|_{L^2(\mathbb{B}_{r'})} \le C_R\|z-w\|_{L^\infty}\|\partial_\alpha z_2\|_{L^2(\mathbb{B}_{r'})},$$
$$\|C_4\|_{L^2(\mathbb{B}_{r'})} \le C_R\|\partial_\alpha(z-w)\|_{L^2(\mathbb{B}_{r'})},$$
$$\|C_5\|_{L^2(\mathbb{B}_{r'})} \le C_R\|z-w\|_{L^\infty}\|\partial_\alpha z_2\|_{L^2(\mathbb{B}_{r'})}.$$

Collecting all these estimates, thanks to Sobolev embedding and (12.12), we obtain

$$\|B_1\|_{L^2(\mathbb{B}_{r'})}\frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)}.$$

We are done with (12.18). Inequality (12.19) is equivalent to the bound $|\partial_t\partial_\alpha z| < C_R$. Such a bound for the terms involving ϖ_2 can be obtained from (12.16) and (12.20). For instance

$$\begin{aligned}A_2 = \text{P.V.}\int_{\mathbb{R}}\frac{\partial_\alpha\varpi_2(\gamma-\beta)(z(\gamma)-h(\gamma-\beta))^\perp}{(z_1(\gamma)-(\gamma-\beta))^2+(z_2(\gamma)+h_2)^2}d\beta =\\ -\text{P.V.}\int_{\mathbb{R}}\varpi_2(\gamma-\beta)\partial_\beta\left(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{(z_1(\gamma)-(\gamma-\beta))^2+(z_2(\gamma)+h_2)^2}\right)d\beta\\ \le C_R\|\varpi_2\|_{H^2(\mathbb{B}_r)}\|d^h[z]\|_{L^\infty}.\end{aligned}$$

The remaining terms can be handled in a similar way. Now we can finish with the forward and backward solvability step. Take $z(0)$ the analytic extension of z in (12.9) ((12.7) for the periodic case). We have $z(0)\in X_{r_0}$ for some $r_0>0$, it satisfies the arc-chord condition and does not reach the curve h, thus, there exists R_0 such that $z(0)\in O_{R_0}$. We take $r<r_0$ and $R>R_0$ in order to define O_R and we consider the iterates

$$z_{n+1} = z(0)+\int_0^t F[z_n]ds,\ z_0 = z(0),$$

and assume by induction that $z_k\in O_R$ for $k\le n$. Then, following the proofs in [10, 26, 49, 50], we obtain a time $T_{CK}>0$ of existence. It remains to show that

$$\|d^-[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)}, \|d^h[z_{n+1}]\|_{L^\infty(\mathbb{B}_r)} < R,$$

for some times $T_A, T_B > 0$ respectively. Then, we choose $T=\min\{T_{CK},T_A,T_B\}$ and we finish the proof. As d^- has been studied in [10] we only deal with d^h. Due to (12.17) and the definition of $z(0)$, we have

$$(d^+[z_{n+1}])^{-1} > \frac{1}{R_0} - C_R(t^2+t),$$

and, if we take a sufficiently small T_B we can ensure that for $t<T_B$ we have $d^h[z_{n+1}]<R$. We conclude the proof of the Theorem. □

12.2 Finite depth

In this section we show the existence of finite time singularities for some curves solutions of (10.6). These singularities appear in an explicit range of parameters κ^i. This result is a consequence of Theorem 8.1 in Chapter 8 where we proved the existence of finite time singularities when $\kappa^1=\kappa^2$. Then, the result of this section is a consequence of the continuous dependence on the physical parameters κ^i.

Remark 12.1. *We define the set*

$$\mathcal{V} = \{w = (w_1, w_2) \text{ satisfying the hypotheses C1-C3 in Section 8.3} \\ \text{and such that } \partial_\alpha v_1(0) < 0.\}. \quad (12.22)$$

Notice that this set is non-empty (see Theorem 8.1).

Now, we have

Theorem 12.2. *Let us suppose that the Rayleigh-Taylor condition is satisfied,* i.e. $\rho^2 - \rho^1 > 0$, *and take* $0 < h_2 < \frac{\pi}{2}$. *Then, for each* $w \in \mathcal{V}$, *there are initial data,* $f_0(x) \in H^3(\mathbb{R})$, *under the hypotheses of Theorem 11.3, such that, for any* $|\mathcal{K}| < \mathcal{K}_1$ *with*

$$\mathcal{K}_1(w, h_2) = \frac{(C(h_2) 8\partial_\alpha w_2 \|\partial_\alpha w_2\|_{L^\infty})^{-1}}{\left(\|d_1^h[w]\|_{L^\infty} + \|d_2^h[w]\|_{L^\infty}\right)\left(1 + \frac{1}{\sqrt{2\pi}}\sup_{|\mathcal{K}|<1}\|G_{h_2,\mathcal{K}}\|_{L^1}\right)} \\ \times \left| 2\partial_\alpha w_2 \int_0^\infty \frac{\partial_\alpha w_1(\beta)\sinh(w_1(\beta))\sin(w_2(\beta))}{\left(\cosh(w_1(\beta)) - \cos(w_2(\beta))\right)^2} \right. \\ \left. + \frac{\partial_\alpha w_1(\beta)\sinh(w_1(\beta))\sin(w_2(\beta))}{\left(\cosh(w_1^0(\beta)) + \cos(w_2(\beta))\right)^2} d\beta \right|,$$

there exists a solution of (10.6) *and a time* T^* *such that*

$$\lim_{t \to T^*} \|\partial_x f(t)\|_{L^\infty} = \infty.$$

For short time $t > T^*$, *the solution can be continued but it is not a graph.*

Proof. The proof is similar to the proof in Theorem 12.1. First, using Theorem 8.1 in Chapter 8, we obtain a curve, $z(0)$, such that the integrals in $\partial_\alpha v_1(0)$ coming from ϖ_1 have a negative contribution. The second step is to take $\mathcal{K}$ small enough, when compared with some quantities depending on the curve $z(0)$, such that the contribution of the terms involving ϖ_2 is small enough to ensure the singularity. Now, the third step is to prove, using a Cauchy-Kovalevsky theorem, that there exists local in time solutions corresponding to the initial datum $\bar{z}$. To simplify notation we take $\kappa^1(\rho^2 - \rho^1) = 4\pi$. Then the parameters present in the problem are h_2 and $\mathcal{K}$.

Step 1: Obtaining the appropriate expression As in Theorem 12.1 we obtain

$$\partial_\alpha v_1(0) = \partial_t \partial_\alpha z_1(0) = I_1 + I_2,$$

where

$$I_1 = 2\partial_\alpha z_2(0) \int_0^\infty \frac{\partial_\alpha z_1(\beta)\sinh(z_1(\beta))\sin(z_2(\beta))}{\left(\cosh(z_1(\beta)) - \cos(z_2(\beta))\right)^2} + \frac{\partial_\alpha z_1(\beta)\sinh(z_1(\beta))\sin(z_2(\beta))}{\left(\cosh(z_1(\beta)) + \cos(z_2(\beta))\right)^2} d\beta,$$

and

$$I_2 = \frac{\partial_\alpha z_2(0)}{4\pi} \int_{\mathbb{R}} \frac{\varpi_2(-\beta)(-\cosh(\beta)\cos(h_2) + 1)}{(\cosh(\beta) - \cos(h_2))^2} d\beta \\ + \frac{\partial_\alpha z_2(0)}{4\pi} \int_{\mathbb{R}} \frac{\varpi_2(-\beta)(-\cosh(\beta)\cos(h_2) - \cos^2(h_2) + \sin^2(h_2))}{(\cosh(\beta) + \cos(h_2))^2} d\beta.$$

Step 2: Taking the appropriate curve and $\mathcal{K}$ From Theorem 8.1 in Chapter 8, we know that there are initial curves w_0 such that I_1 is negative. We take one of this curves and we denote this smooth, fixed curve as $\bar{z}$. We need to obtain

$$\partial_\alpha v_1(0) = -a^2 + I_2 < 0.$$

As in (12.16), we define

$$d_1^h[\bar{z}](\gamma,\beta) = \frac{\cosh^2(\beta/2)}{\cosh(\bar{z}_1(\gamma)-(\gamma-\beta)) - \cos(\bar{z}_2(\gamma)+h_2)}, \tag{12.23}$$

$$d_2^h[\bar{z}](\gamma,\beta) = \frac{\cosh^2(\beta/2)}{\cosh(\bar{z}_1(\gamma)-(\gamma-\beta)) + \cos(\bar{z}_2(\gamma)-h_2)}. \tag{12.24}$$

From the definition of I_2 it is easy to obtain

$$|I_2| \leq C(h_2)\partial_\alpha \bar{z}_2(0)\|\varpi_2\|_{L^\infty},$$

where

$$C(h_2) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\cosh(\beta)\cos(h_2)+1}{(\cosh(\beta)-\cos(h_2))^2}d\beta + \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\cosh(\beta)\cos(h_2)+\cos(2h_2)}{(\cosh(\beta)+\cos(h_2))^2}d\beta.$$

From the definition of ϖ_2 for curves (which follows from (2.17) in a straightforward way) we obtain

$$\|\varpi_2\|_{L^\infty} \leq 8\mathcal{K}\|\partial_\alpha \bar{z}_2\|_{L^\infty}\left(\|d_1^h[\bar{z}]\|_{L^\infty} + \|d_2^h[\bar{z}]\|_{L^\infty}\right)\left(1+\frac{\mathcal{K}}{\sqrt{2\pi}}\|G_{h_2,\mathcal{K}}\|_{L^1}\right).$$

Fixing $0 < h_2 < \pi/2$ and collecting all the estimates we obtain

$$|I_2| \leq C(h_2)8\partial_\alpha \bar{z}_2(0)\mathcal{K}\|\partial_\alpha \bar{z}_2\|_{L^\infty}\left(\|d_1^h[\bar{z}]\|_{L^\infty} + \|d_2^h[\bar{z}]\|_{L^\infty}\right)\left(1+\frac{\mathcal{K}}{\sqrt{2\pi}}\sup_{|\mathcal{K}|<1}\|G_{h_2,\mathcal{K}}\|_{L^1}\right).$$

Now it is enough to take $|\mathcal{K}| < \mathcal{K}_1$ with

$$\mathcal{K}_1(\bar{z},h_2) = \frac{(C(h_2)8\partial_\alpha \bar{z}_2(0)\|\partial_\alpha \bar{z}_2\|_{L^\infty})^{-1}a^2}{\left(\|d_1^h[\bar{z}]\|_{L^\infty} + \|d_2^h[\bar{z}]\|_{L^\infty}\right)\left(1+\frac{1}{\sqrt{2\pi}}\sup_{|\mathcal{K}|<1}\|G_{h_2,\mathcal{K}}\|_{L^1}\right)}, \tag{12.25}$$

to ensure that $\partial_\alpha v_1(0) < 0$ for this curve $\bar{z}$.

Step 3: Showing the forward and backward solvability We define

$$d^-[z](\gamma,\beta) = \frac{\sinh^2(\beta/2)}{\cosh(z_1(\gamma)-z_1(\gamma-\beta)) - \cos(z_2(\gamma)-z_2(\gamma-\beta))}, \tag{12.26}$$

and

$$d^+[z](\gamma,\beta) = \frac{\cosh^2(\beta/2)}{\cosh(z_1(\gamma)-z_1(\gamma-\beta)) + \cos(z_2(\gamma)-z_2(\gamma-\beta))}. \tag{12.27}$$

Using the equations (12.23), (12.24), (12.26) and (12.27), the proof of this step mimics the proof in Theorem 12.1 and so we only sketch it. As before, we consider curves z satisfying the arc-chord condition and such that

$$\lim_{|\alpha|\to\infty} |z(\alpha)-(\alpha,0)| = 0.$$

We define the complex strip $\mathbb{B}_r = \{\zeta+i\xi, \zeta\in\mathbb{R}, |\xi|<r\}$, and the spaces (12.10) with norm (12.11) (see [2]). We define the set

$$O_R = \{z \in X_r \text{ such that } \|z\|_r < R, \|d^-[z]\|_{L^\infty(\mathbb{B}_r)} < R, \|d^+[z]\|_{L^\infty(\mathbb{B}_r)} < R, \\ \|d_1^h[z]\|_{L^\infty(\mathbb{B}_r)} < R, \|d_2^h[z]\|_{L^\infty(\mathbb{B}_r)} < R\},$$

where $d_i^h[z]$ and $d^{\pm}[z]$ are defined in (12.23), (12.24), (12.26) and (12.27), respectively. As before, we have that, for $z, w \in O_R$, complex extension of (10.28), $F : O_R \to X_{r'}$ is continuous and the following inequalities holds:

$$\begin{aligned}
&\|F[z]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z\|_r,\\
&\|F[z]-F[w]\|_{H^3(\mathbb{B}_{r'})} \leq \frac{C_R}{r-r'}\|z-w\|_{H^3(\mathbb{B}_r)},\\
&\sup_{\gamma\in\mathbb{B}_r,\beta\in\mathbb{R}}|F[z](\gamma)-F[z](\gamma-\beta)| \leq C_R|\beta|.
\end{aligned}$$

We consider

$$z_{n+1} = z(0) + \int_0^t F[z_n]ds, \ z(0) = \bar{z}.$$

Using the previous properties of F we obtain that, for $T = T(z(0), R)$ small enough, $z^{n+1} \in O_R$, for all n. The rest of the proof follows in the same way as in [49, 50]. □

Part IV

Comparing the models

Chapter 13

Numerical study

13.1 Foreword

In this Chapter we use the computer to further investigate (2.5), (2.12) and (2.14). First we obtain numerical evidence showing that the confined problem is more singular than the problem with infinite depth (2.10) (see Numerical evidence 13.1). We also study the evolution of $\|f(t)\|_{L^\infty(\mathbb{R})}$ for (2.10) and (2.5) with the same initial datum (see Figures 13.2-13.7). We also show by a computer assisted proof the following result:

Theorem 13.1. *There are initial data z_0 such that the following statements hold:*

- *the solutions of* (2.9) *with $l = \pi/2$, $z^{\pi/2}$, corresponding with these initial data turn over,*
- *the solutions of* (2.9) *with $l = \infty$, z^∞, corresponding with these initial data become graphs.*

Moreover, we get the following corollary

Corollary 13.1. *There are initial data z_0 such that the following statements hold:*

- *the solutions of the water waves problem with $l = \pi/2$, $z^{\pi/2}$, corresponding with these initial data turn over,*
- *the solutions of water waves problem with $l = \infty$, z^∞, corresponding with these initial data become graphs.*

For (2.12) and (2.14) the simulations show that, if $\mathcal{K} < 0$, $\|f(t)\|_{L^\infty(\mathbb{R})}$ decays faster but $\|\partial_x f(t)\|_{L^\infty(\mathbb{R})}$ decays slower. Actually, $\mathcal{K} > 0$ seems to be *better* in the sense that it could prevent the turning for some h_2. Recall that in Theorem 12.1 we obtain (by an analytical proof) that if $h_2 >> 1$ and any $\mathcal{K}$ there exist turning waves. To take to pieces this argument, we first obtain firm numerical evidence showing the existence of turning waves for every $\mathcal{K}$ (see numerical evidence 13.2). This numerical evidence becomes a rigorous result in Section 13.4 by a computer assisted proof. Furthermore, we prove that if $h_2 = O(1)$ and any $\mathcal{K}$ there exist turning waves. The precise statement is as follows:

Theorem 13.2. *There are curves, which are solutions of* (2.12) *and* (2.14)*, such that for every $|\mathcal{K}| < 1$ and $h_2 = \pi/2$, they turn over.*

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13.2 Finite vs. Infinite depth

13.2.1 Turning waves

In this Section we obtain firm numerical evidence showing that the confined problem is more singular than the problem with infinite depth (2.10). The precise statement of this fact is the following

Numerical evidence 13.1. *There exists initial data $z_0(\alpha) = (z_1(\alpha), z_2(\alpha))$ such that, the solutions of (2.9) with $l = \pi/2$ corresponding with these initial data turn over and, the solutions of (2.9) with $l = \infty$ corresponding with the same initial data become graphs.*

We can assume that $\rho^2 - \rho^1 = 4\pi$ and take $l = \pi/2$. It is enough to show that there exist smooth curves $z(\alpha, 0) = (z_1(\alpha, 0), z_2(\alpha, 0))$ satisfying arc-chord condition and such that $\partial_\alpha z_1(0, 0) = 0$ and the following holds:

1. $\partial_\alpha v_1(0, 0) = \partial_\alpha \partial_t z_1(0, 0) > 0$ in the deep water regime,
2. $\partial_\alpha v_1(0, 0) = \partial_\alpha \partial_t z_1(0, 0) < 0$ when the strip is considered.

Indeed, if $\partial_\alpha v_1(0, 0) = \partial_\alpha \partial_t z_1(0, 0) > 0$ then, as before, if we denote $m(t) = \min_\alpha \partial_\alpha z_1(\alpha, t)$ we have $m(0) = \partial_\alpha z_1(0, 0) = 0$ and $\frac{d}{dt} m(t) > 0$ for $0 < t$ small enough. This implies $m(\delta) > 0$ for a small enough $\delta > 0$ and the curve can be parametrized as a graph. On the other hand, if $\partial_\alpha v_1(0, 0) = \partial_\alpha \partial_t z_1(0, 0) < 0$ then $m(\delta) < 0$ for δ small enough, and the curve can not be parametrized as a graph.

We construct a piecewise smooth curve such that both conditions holds (see Figure 8.2). We take z_1 defined as follows

$$z_1(\alpha) = \alpha - e^{-\alpha^2 k} \sin(\alpha),$$

with $k = 1e - 4$. The idea is to take $k << 1$ such that $e^{-\alpha^2 k} \approx 1$, for $-\pi < \alpha < \pi$. Moreover, we take z_2 as in (8.5) with $a = b = 3$, i.e.,

$$z_2(\alpha) = \begin{cases} \frac{1}{3}\sin(3\alpha) & \text{if } 0 \leq \alpha \leq \frac{\pi}{3}, \\ -\alpha + \frac{\pi}{3} & \text{if } \frac{\pi}{3} \leq \alpha < \frac{\pi}{2}, \\ \alpha - \frac{2\pi}{3} & \text{if } \frac{\pi}{2} \leq \alpha < \frac{2\pi}{3}, \\ 0 & \text{if } \frac{2\pi}{3} \leq \alpha. \end{cases}$$

Notice that, in the deep water regime, the expression (8.4) takes the form

$$\frac{\partial_\alpha v_1(0)}{2} = 4\partial z_2(0) \int_0^\infty \frac{\partial_\alpha z_1(\eta) z_1(\eta) z_2(\eta)}{(z_1(\eta))^2 + (z_2(\eta))^2)^2} d\eta.$$

Substituting the choice of z, we need to compute

$$\frac{\partial_\alpha v_1(0)}{2} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{13.1}$$

where

$$\mathcal{I}_1 = \frac{4}{3} \int_0^{\frac{\pi}{3}} \frac{(1 - \cos(\eta) e^{-\eta^2 k} + 2k\eta e^{-\eta^2 k} \sin(\eta))(\eta - e^{-\eta^2 k} \sin(\eta)) \sin(3\eta)}{(\eta - e^{-\eta^2 k} \sin(\eta))^2 + (\sin(3\eta)/3)^2)^2} d\eta,$$

$$\mathcal{I}_2 = 4 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(1 - \cos(\eta) e^{-\eta^2 k} + 2k\eta e^{-\eta^2 k} \sin(\eta))(\eta - e^{-\eta^2 k} \sin(\eta))(-\eta + \pi/3)}{(\eta - e^{-\eta^2 k} \sin(\eta))^2 + (-\eta + \pi/3)^2)^2} d\eta,$$

$$\mathcal{I}_3 = 4\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))(\eta - e^{-\eta^2 k}\sin(\eta))(\eta-2\pi/3)}{(\eta - e^{-\eta^2 k}\sin(\eta))^2+(\eta-2\pi/3)^2)^2}d\eta.$$

In the finite depth case, the integrals appearing in (8.4) are

$$\frac{\partial_\alpha v_1(0)}{2} = \mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6, \tag{13.2}$$

where

$$\begin{aligned}\mathcal{I}_4 = \int_0^{\frac{\pi}{3}} &(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))\sinh(\eta - e^{-\eta^2 k}\sin(\eta))\sin(\sin(3\eta)/3)\\ &\cdot\left(\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))-\cos(\sin(3\eta)/3))^2}\right.\\ &\left.+\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))+\cos(\sin(3\eta)/3))^2}\right)d\eta,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_5 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} &(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))\sinh(\eta - e^{-\eta^2 k}\sin(\eta))\sin(-\eta+\pi/3)\\ &\cdot\left(\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))-\cos(-\eta+\pi/3))^2}\right.\\ &\left.+\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))+\cos(-\eta+\pi/3))^2}\right)d\eta,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_6 = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} &(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))\sinh(\eta - e^{-\eta^2 k}\sin(\eta))\sin(\eta-2\pi/3)\\ &\cdot\left(\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))-\cos(\eta-2\pi/3))^2}\right.\\ &\left.+\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))+\cos(\eta-2\pi/3))^2}\right)d\eta.\end{aligned}$$

In order to obtain the sign of (13.1) and (13.2), we compute the integrals $\mathcal{I}_i,\ i=2,3,5,6$ using the trapezoidal rule with a fine enough mesh, $dx = 10^{-9}$ (see Figure 13.1). The integrals $\mathcal{I}_i,\ i=1,4$ are approximated by

$$\mathcal{I}_1' = \frac{4}{3}\int_{0.1}^{\frac{\pi}{3}} \frac{(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))(\eta - e^{-\eta^2 k}\sin(\eta))\sin(3\eta)}{(\eta - e^{-\eta^2 k}\sin(\eta))^2+(\sin(3\eta)/3)^2)^2}d\eta,$$

and

$$\begin{aligned}\mathcal{I}_4' = \int_{0.1}^{\frac{\pi}{3}} &(1-\cos(\eta)e^{-\eta^2 k}+2k\eta e^{-\eta^2 k}\sin(\eta))\sinh(\eta - e^{-\eta^2 k}\sin(\eta))\sin(\sin(3\eta)/3)\\ &\cdot\left(\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))-\cos(\sin(3\eta)/3))^2}\right.\\ &\left.+\frac{1}{(\cosh(\eta - e^{-\eta^2 k}\sin(\eta))+\cos(\sin(3\eta)/3))^2}\right)d\eta.\end{aligned}$$

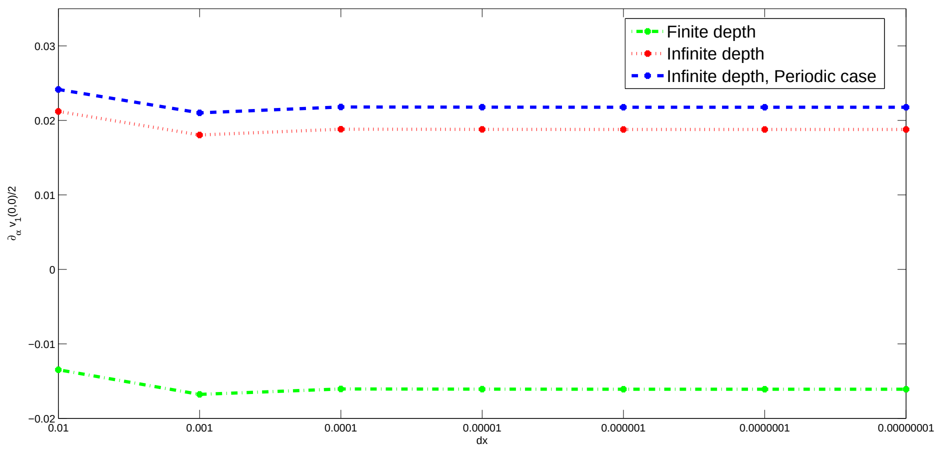

Figure 13.1: Approximating $\partial_\alpha v_1(0)/2$ with different spatial step dx.

The truncation of the integral domains in $\mathcal{I}_i'$, $i = 1, 4$ gives us an error $E_{PV} \leq 0.72 \cdot 10^{-3}$. To obtain this bound we notice that, due to the particular choice of z_i,

$$\int_0^x \frac{\partial_\alpha z_1(\eta) z_1(\eta) z_2(\eta)}{(z_1(\eta))^2 + (z_2(\eta))^2)^2} d\eta = O(x^3),$$

and the same is valid for the relevant integral in the presence of boundaries (8.4).

The other error is coming from the method used in the numerical quadrature. We use the trapezoidal rule, obtaining $E_I \leq 1.1 \cdot 10^{-3}$. We conclude that, if $\partial_\alpha \hat{v_1}(0)$ denotes the numerical approximation of $\partial_\alpha v_1(0)$ defined in (13.2), we have

$$\partial_\alpha v_1(0) \leq \partial_\alpha \hat{v_1}(0) + |E_{PV}| + |E_I| < 0,$$

and, analogously, in the case where $\partial_\alpha v_1(0)$ is defined in (13.1), we get

$$0 < \partial_\alpha \hat{v_1}(0) - |E_{PV}| - |E_I| \leq \partial_\alpha v_1(0).$$

Finally, we approximate this z_0 by analytic functions.

In order to complete a rigorous enclosure of the integral, we are left with the bounding of the errors coming from the floating point representation and the computer operations and their propagation. In Section 13.4 we deal with this matter. By using interval arithmetics we give a computer assisted proof of this result.

Remark 13.1. *This shows that the problem with finite depth appears to be, in this precise sense, more singular than the case $\mathcal{A} = 0$.*

13.2.2 Decay in L^∞

In this section we perform numerical simulations to study the decay of $\|f(t)\|_{L^\infty}$ when the depth is finite and to compare this decay with the case where the depth is infinite. We consider equation (2.5) where $\rho^2 - \rho^1 = 4\pi$ and $l = \pi/2$ and equation (2.10). For each initial datum we approximate the solutions of (2.5) and (2.10) with the same numerical and physical parameters.

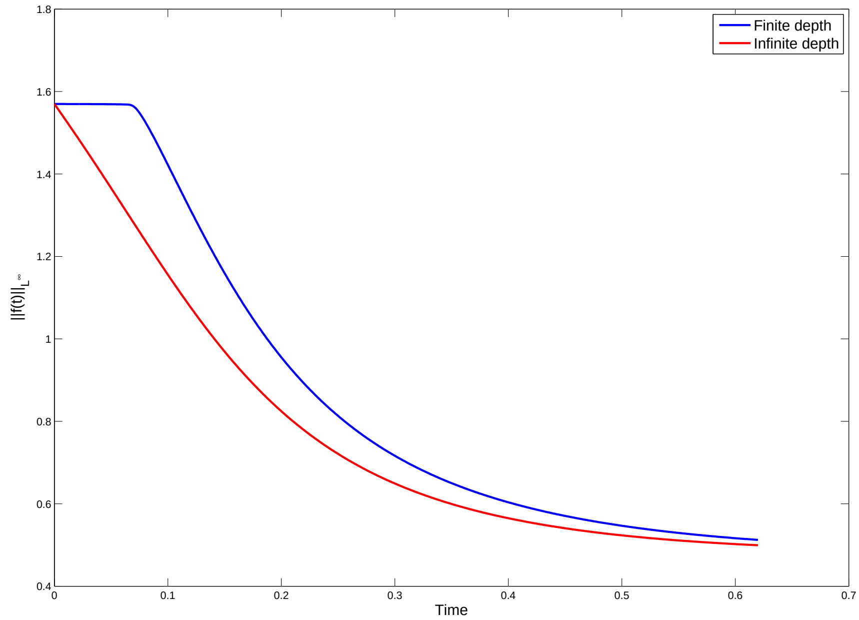

Figure 13.2: Evolution of $\|f^{\pi/2}(t)\|_{L^\infty}$ (blue) and $\|f^\infty(t)\|_{L^\infty}$ (red) and initial datum given by (13.3).

To perform the simulations we follow the ideas in [25]. The interface is approximated using cubic splines with N spatial nodes. The spatial operator is approximated with Lobatto quadrature (using the function *quadl* in Matlab). Then, three different integrals appear for a fixed node x_i: the integral between x_{i-1} and x_i, the integral between x_i and x_{i+1} and the nonsingular ones. In the two first integrals we use Taylor series to remove the singularity. In the nonsingular integrals the integrand is made explicit using the splines. We use a classical explicit Runge-Kutta method of order 4 to integrate in time. In the simulations we take $N = 300$ and $dt = 10^{-3}$. In what follows we change slightly the notation and write $f^l(x,t)$ for the solution of (2.5) with depth equal to l. Then, given an initial datum $f(x,0) = f_0(x)$, we are computing a numerical approximation for $f^{\pi/2}(x,t)$ and $f^\infty(x,t)$. There are three different examples:

- Case 1: The initial datum considered is

$$f_0(x) = \left(\frac{\pi}{2} - 0.001\right)\exp\left(-x^{12}\right). \tag{13.3}$$

We obtain Figures 13.2 and 13.3. We can see that the decay is slower in the finite depth case and the existence of a big time interval with a very small decay.

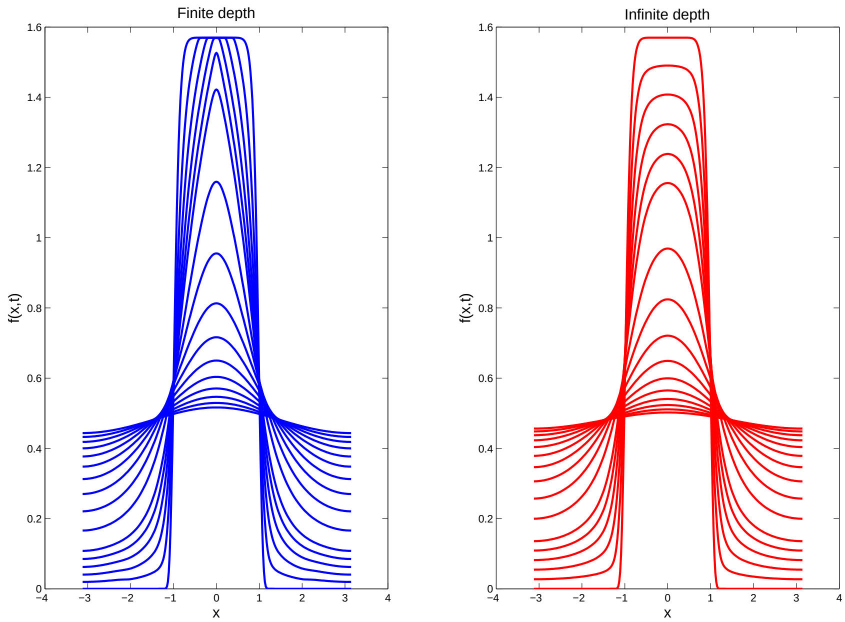

Figure 13.3: $f^{\pi/2}(x,t)$ (blue) and $f^{\infty}(x,t)$ (red) for the same times t_i and initial datum given by (13.3).

- Case 2: The second initial datum considered is

$$f_0(x) = \begin{cases} -\pi/2 + 0.0001 & \text{if } x \leq -\frac{\pi}{2}, \\ x\frac{-\pi+0.0002}{-\pi/2+1} + \pi/2 - 0.0001 + \frac{-\pi+0.0002}{-\pi/2+1} & \text{if } -\frac{\pi}{2} \leq x < -1, \\ \pi/2 - 0.0001 & \text{if } -1 \leq x < 1, \\ x\frac{-\pi+0.0002}{\pi/2-1} + \pi/2 - 0.0001 - \frac{-\pi+0.0002}{\pi/2-1} & \text{if } 1 \leq x < \frac{\pi}{2}. \\ -\pi/2 + 0.0001 & \text{if } \frac{\pi}{2} < x. \end{cases} \tag{13.4}$$

  We obtain Figures 13.4 and 13.5. This second case is similar to the first one and we obtain even a slower decay in the finite depth case.

- Case 3: We consider

$$g(x) = \exp(-x^4)(\cos(x) + \sin(x)).$$

  The third initial datum considered is

$$f_0(x) = (\pi/2 - 0.001)\frac{g(x)}{\|g\|_{L^\infty(\mathbb{R})}}. \tag{13.5}$$

  We obtain Figures 13.6 and 13.7. In this case the initial datum is not symmetric and the evolution is similar in both cases, finite and infinite depth.

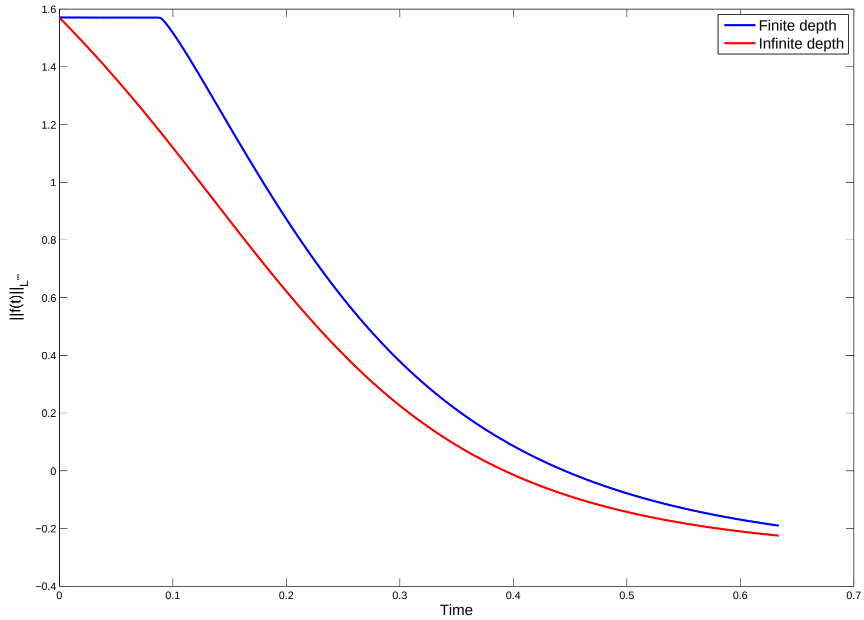

Figure 13.4: Evolution of $\|f^{\pi/2}(t)\|_{L^\infty}$ (blue) and $\|f^{\infty}(t)\|_{L^\infty}$ (red) and initial datum given by (13.4).

13.3 Homogeneous vs. Inhomogeneous porous medium

13.3.1 Turning waves

Numerical evidence 13.2. *There are curves, which are solutions of* (2.12) *or* (2.14)*, such that for every* $|\mathcal{K}|<1$ *and* $h_2=\pi/2$ *turn over.*

Let us consider first the periodic setting. Recall the fact that $h_2=\pi/2$ and let us define

$$z_1(\alpha)=\alpha-\sin(\alpha),\quad z_2(\alpha)=\frac{\sin(3\alpha)}{3}-\sin(\alpha)\left(e^{-(\alpha+2)^2}+e^{-(\alpha-2)^2}\right)\text{ for }\alpha\in\mathbb{T}. \tag{13.6}$$

Inserting this curve in (12.5), we obtain that for any possible $-1<\mathcal{K}<1$,

$$I_1(0)+|I_2(0)|<0.$$

In particular,

$$\partial_\alpha v_1^p(0)=I_1(0)+I_2(0)<I_1(0)+|I_2(0)|<0.$$

Let us introduce the algorithm we use. We need to compute

$$\partial_\alpha v_1^p(0)=\int_0^\pi\mathcal{I}_1+\int_0^\pi\mathcal{I}_2,$$

where $\mathcal{I}_i$ means the $i-$integral in (12.5). Recall that $\mathcal{I}_i$ is two times differentiable, so, we can use the sharp error bound for the trapezoidal rule. We denote dx the mesh size when we compute the

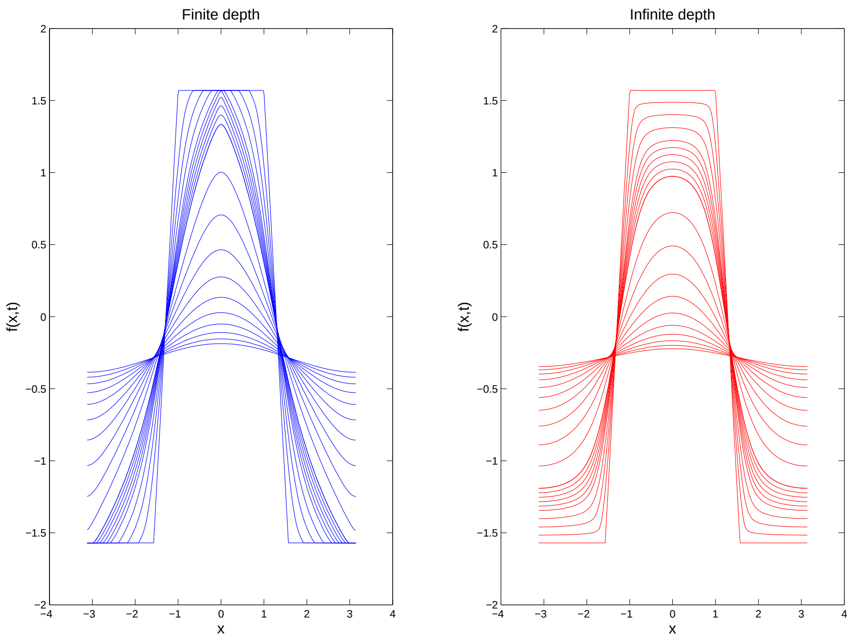

Figure 13.5: $f^{\pi/2}(x,t)$ (blue) and $f^{\infty}(x,t)$ (red) for the same times t_i and initial datum given by (13.4).

first integral. We approximate the integral of $\mathcal{I}_1$ using the trapezoidal rule between $(0.1,\pi)$. We neglect the integral in the interval $(0,0.1)$, paying with an error denoted by $|E^1_{PV}|=O(10^{-3})$. The trapezoidal rule gives us an error

$$|E^1_I|\leq\frac{dx^2(\pi-0.1)}{12}\|\partial^2_\alpha\mathcal{I}_1\|_{L^\infty}.$$

As we know the curve z, we can bound $\partial^2_\alpha\mathcal{I}_1$. We obtain,

$$|E^1_I|\leq dx^2\frac{(\pi-0.1)}{6}10^5.$$

We take $dx=10^{-7}$. Putting all together we obtain

$$|E^1|\leq|E^1_{PV}|+|E^1_I|+\leq 3O(10^{-3})=O(10^{-2}).$$

Then, we can ensure that

$$\partial_\alpha z_2(0)\int_0^\pi\frac{\partial_\alpha z_1(\beta)\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta))-\cos(z_1(\beta)))^2}d\beta\leq-0.7+|E^1|<-0.6. \tag{13.7}$$

We need to control analytically the error in the integral involving ϖ^p_2. This second integral has the error coming form the numerical integration, E^2_I and a new error coming from the fact that ϖ^p_2 is known with some error. We denote this new error as E^2_ϖ. Let us write $\tilde{dx}$ the mesh size for the second integral. Then, using the smoothness of $\mathcal{I}_2$, we have

$$|E^2_I|\leq\frac{\tilde{dx}^2}{16}\|\varpi^p_2\|_{C^2}\leq\frac{\tilde{dx}^2}{4}\cdot 50.$$

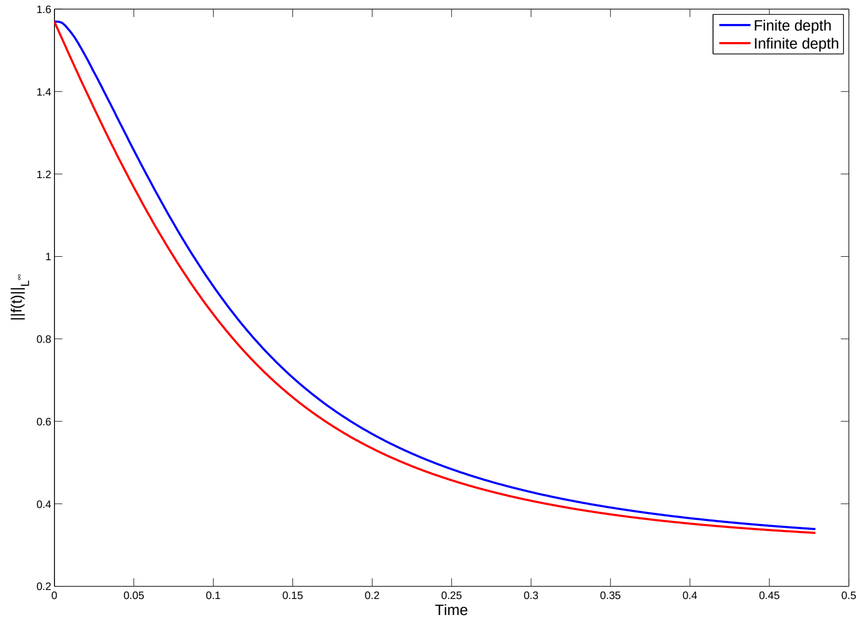

Figure 13.6: Evolution of $\|f^{\pi/2}(t)\|_{L^\infty}$ (blue) and $\|f^{\infty}(t)\|_{L^\infty}$ (red) and initial datum given by (13.5).

We take $\tilde{dx} = 10^{-4}$. It remains the error coming from ϖ_2^p. The second vorticity, ϖ_2^p, is given by the integral (12.6). We compute the integral (12.6) using the same mesh size as for I_2, $\tilde{dx}$. Thus, the errors are

$$|E^2_\varpi| \leq O(10^{-3}),$$

Putting all together we have

$$|E^2| \leq |E^2_I| + |E^2_\varpi| \leq O(10^{-2}),$$

and we conclude

$$\left|\frac{\partial_\alpha z_2(0)}{4\pi}\int_0^\pi \frac{(\varpi_2^p(\beta)+\varpi_2^p(-\beta))(-1+\cosh(h_2)\cos(\beta))}{(\cosh(h_2)-\cos(\beta))^2}d\beta\right| \leq 0.1 + |E^2| < 0.2. \qquad (13.8)$$

Now, using (13.7) and (13.8), we obtain $\partial_\alpha v_1^p(0) < 0$, and we are done with the periodic case.

We proceed with the flat at infinity case. We have to deal with the unboundedness of the domain so we define

$$\begin{aligned} z_1(\alpha) &= \alpha - \sin(\alpha)\exp(-\alpha^2/100) \\ z_2(\alpha) &= \frac{\sin(3\alpha)}{3} - \sin(\alpha)\left(e^{-(\alpha+2)^2} + e^{-(\alpha-2)^2}\right)\mathbf{1}_{\{|\alpha|<\pi\}}. \end{aligned} \qquad (13.9)$$

Inserting this curve in (12.3) we obtain that for any possible $-1 < \mathcal{K} < 1$,

$$I_1(0) + |I_2(0)| < 0.$$

Then, as before,

$$\partial_\alpha v_1(0) = I_1(0) + I_2(0) < I_1(0) + |I_2(0)| < 0.$$

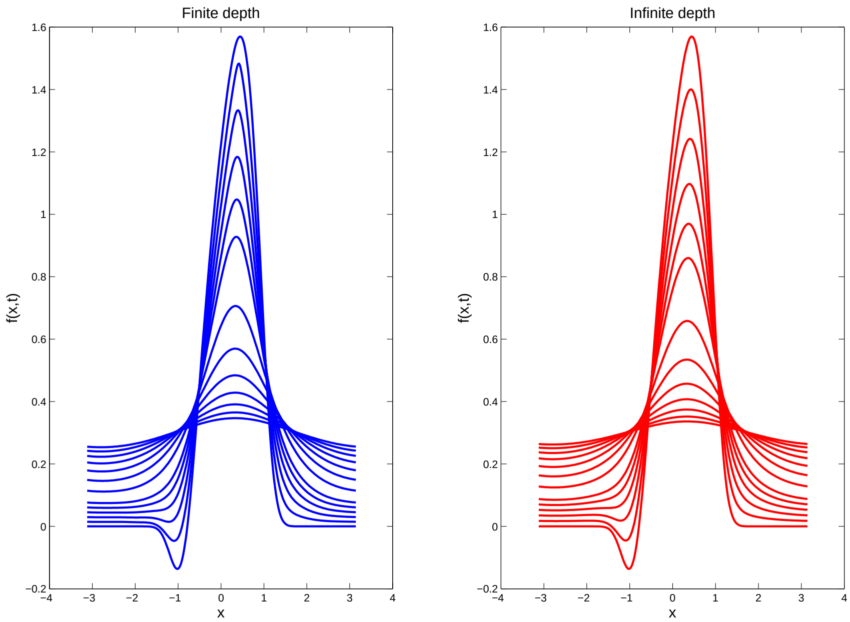

Figure 13.7: $f^{\pi/2}(x,t)$ (blue) and $f^{\infty}(x,t)$ (red) for the same times t_i and initial datum given by (13.5).

The function z_2 is Lipschitz, so the same for $\mathcal{I}_1$, where now $\mathcal{I}_i$ are the expressions in (12.3) and the second integral I_2 is over an unbounded interval. To avoid these problems we compute the numerical aproximation of

$$\int_{0.1}^{\pi-dx}\mathcal{I}_1+\int_0^{L_2}\mathcal{I}_2.$$

Recall that ϖ_2 is given by (12.4) and then, due to the definition of z_2, we can approximate it by an integral over $(0,\pi-\tilde{dx})$. The lack of analyticity of z_2 and the truncation of $I_2(0)$ introduces two new sources of error. We denote them by $E^1_{z_2}$ and $E^2_{\mathbb{R}}$. We take $dx=10^{-7}, \tilde{dx}=10^{-4}$ and $L_2=2\pi$. Using the bounds $z_1\leq\pi$, $\partial_\alpha z_1\leq 2$ and $z_2\leq h_2$ we obtain

$$|E^1_{z_2}|\leq\left|\int_{\pi-dx}^{\pi}\mathcal{I}_1\right|\leq dx\cdot 0.2\cdot 4\pi^2\leq 8\cdot 10^{-7}.$$

We have

$$|\varpi_2(\beta)|\leq 4\pi\frac{(h_2+\max_\gamma|z_2(\gamma)|)\max_\gamma|\partial_\alpha z_2(\gamma)|}{\min_\gamma(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}d\gamma\leq\frac{4\pi\cdot 3\cdot 2}{\min_\gamma(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}=C(\beta),$$

with $C(\beta)<C(L_2)$ for $\beta>L_2$. Using this inequality we get the desired bound for the second error as follows:

$$|E^2_{\mathbb{R}}|\leq\frac{|C(L_2)|}{\pi}\int_{L_2}^{\infty}\frac{\beta^2}{\left(\beta^2+\left(\frac{\pi}{2}\right)^2\right)^2}\leq\frac{4\pi\cdot 3\cdot 2}{10}\cdot 0.05<4\cdot 10^{-1}.$$

The other errors can be bounded as before, obtaining,

$$|E^1|\leq|E^1_{PV}|+|E^1_I|+|E^1_{z_2}|=O(10^{-2}),$$

$$|E^2| \leq |E^2_{\varpi}| + |E^2_I| + |E^2_{\mathbb{R}}| = 0.42.$$

We conclude

$$\partial_\alpha z_2(0) \cdot 4\text{P.V.} \int_0^\infty \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta \leq -0.7 + |E^1| < -0.6, \tag{13.10}$$

and

$$\left| -\frac{1}{2\pi} \text{P.V.} \int_0^\infty \frac{(\varpi_2(\beta) + \varpi_2(-\beta))\beta^2}{(\beta^2 + h_2^2)^2} d\beta \right| < 0.02 + |E^2| < 0.5. \tag{13.11}$$

Putting together (13.10) and (13.11) we conclude $\partial_\alpha v_1(0) < 0$.

In order to complete a rigorous enclosure of the integral, we are left with the bounding of the errors coming from the floating point representation and the computer operations and their propagation (see [33]). By using interval arithmetics we give a computer assisted proof of this result below.

13.3.2 Decay in C^1

In this section we perform numerical simulations to better understand the role of ϖ_2. We consider equation (2.14) where $\kappa^1 = 1$, $\rho^2 - \rho^1 = 4\pi$ and $h_2 = \pi/2$. For each initial datum we approximate the solution of (2.14) corresponding to different $\mathcal{K}$. Indeed, we take different κ^2 to get

$$\mathcal{K} = \frac{-999}{1001}, \frac{-1}{3}, 0, \frac{1}{3} \text{ and } \frac{999}{1001}.$$

We change slightly the notation and write $f^{\mathcal{K}}(x,t)$ for the solution corresponding to (2.14) for a certain value of $\mathcal{K}$. With this notation we computing $f^{\frac{-999}{1001}}(x,t)$,$f^{\frac{-1}{3}}(x,t)$, $f^0(x,t)$, $f^{\frac{1}{3}}(x,t)$ and $f^{\frac{999}{1001}}(x,t)$ with a common initial datum $f_0(x)$.

To perform the simulations we follow the ideas in Section 13.2.2 (see also [25]). In the simulations we take $N = 120$ and $dt = 10^{-3}$.

- Case 1: The first initial datum considered is

$$f_0(x) = -\left(\frac{\pi}{2} - 0.000001\right) e^{-x^{12}}. \tag{13.12}$$

  We obtain Figures 13.8 and 13.9.

- Case 2: The second initial datum is

$$f_0(x) = -\left(\frac{\pi}{2} - 0.000001\right) \cos(x^2). \tag{13.13}$$

  We obtain Figures 13.10 and 13.11.

- Case 3: We approximate the solution (see Figures 13.12 and 13.13) corresponding to the initial datum

$$f_0(x) = -\left(\frac{\pi}{2} - 0.000001\right) e^{-(x-2)^{12}} - \left(\frac{\pi}{2} - 0.000001\right) e^{-(x+2)^{12}} + e^{-x^2} \cos^2(x). \tag{13.14}$$

In these simulations we observe that $\|f\|_{C^1}$ decays but rather differently depending on $\mathcal{K}$. If $\mathcal{K} < 0$ the decay of $\|f\|_{L^\infty}$ is faster when compared with the case $\mathcal{K} = 0$. In the case where $\mathcal{K} > 0$ the term corresponding to ϖ_2 slows down the decay of $\|f\|_{L^\infty}$ but we observe still a decay. Particularly, we observe that if $\mathcal{K} \approx 1$ ($\kappa^2 \approx 0$) the decay is initially almost zero and then slowly increases.

When the evolution of $\|\partial_x f\|_{L^\infty}$ is considered the situation is reversed. Now the simulations corresponding to $\mathcal{K} > 0$ have the faster decay. With these result we can not define a *stable* regime for $\mathcal{K}$ in which the evolution would be *smoother*. Recall that we know that there is not any hypothesis on the sign or size of $\mathcal{K}$ to ensure the existence (see Theorem 11.2 and 11.3).

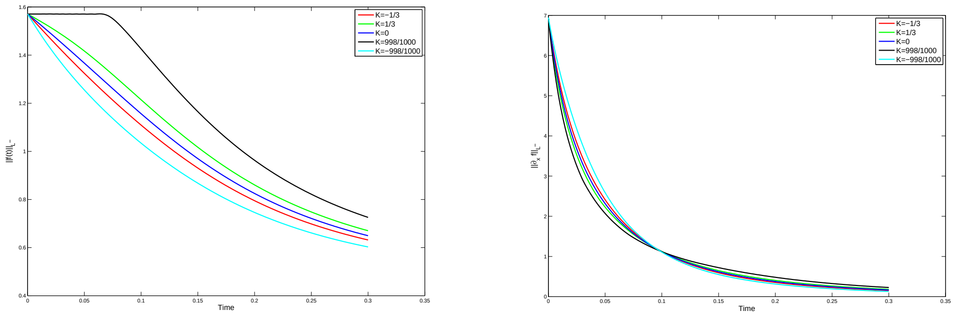

Figure 13.8: Evolution of $\|f^{\mathcal{K}}\|_{L^\infty}$ (left) and $\|\partial_x f^{\mathcal{K}}\|_{L^\infty}$ (right) with $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , 0 (blue) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) and initial datum given by (13.12).

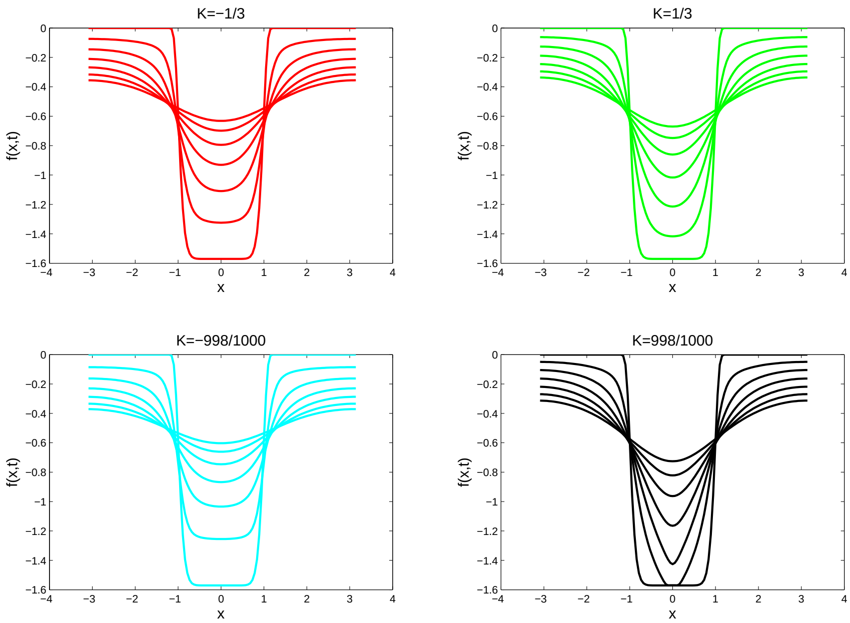

Figure 13.9: Evolution $f^{\mathcal{K}}(x,t)$ for $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) at the same times t_i and initial datum given by (13.12).

13.4 Computer assisted proofs

In this section we outline the analytical component Theorems 13.1, 13.2 and Corollary 13.1. For the details about the computation of the integrals using interval arithmetic and its background, the codes of the proof and some other technical issues see [32] and [33].

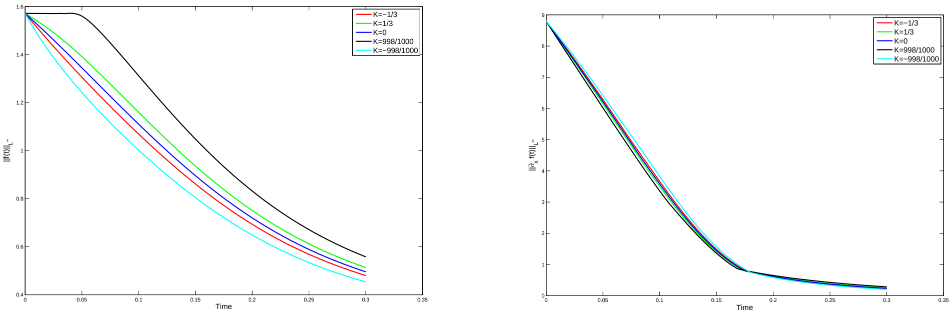

Figure 13.10: Evolution of $\|f^{\mathcal{K}}\|_{L^\infty}$ (left) and $\|\partial_x f^{\mathcal{K}}\|_{L^\infty}$ (right) with $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , 0 (blue) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) and initial datum given by (13.13).

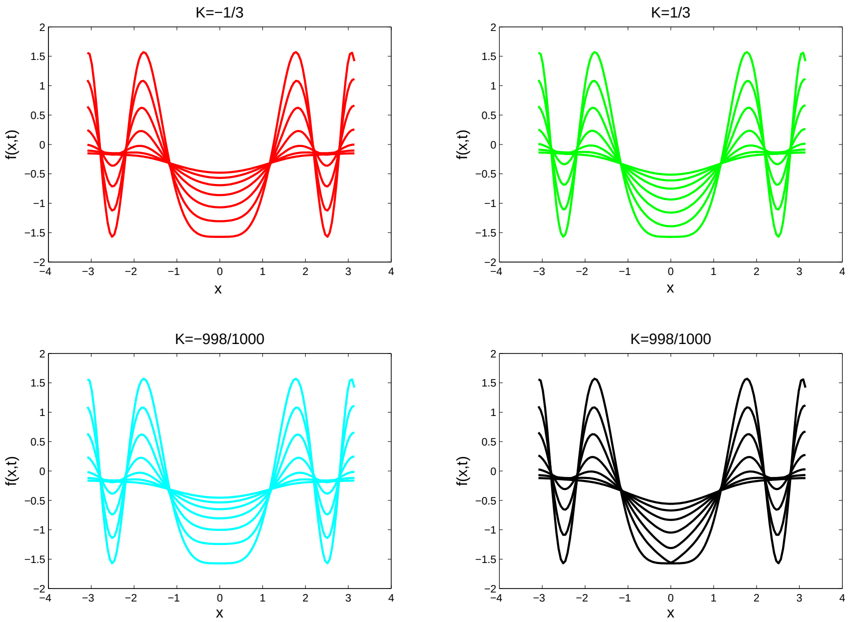

Figure 13.11: Evolution $f^{\mathcal{K}}(x,t)$ for $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) at the same times t_i and initial datum given by (13.13).

13.4.1 Finite vs. Infinite depth

In this section we prove that the boundaries make the Muskat problem more singular from the point of view of singularity formation. Equivalently, the boundaries decreases the diffusion rate (see [26]).

Proof of Theorem 13.1. We take $l = \pi/2$ and $\rho^2 - \rho^1 = 4\pi$. Then, after some lengthly computations (see (8.4) in Chapter 8), we get the following expression for the appropriate quantity in the confined

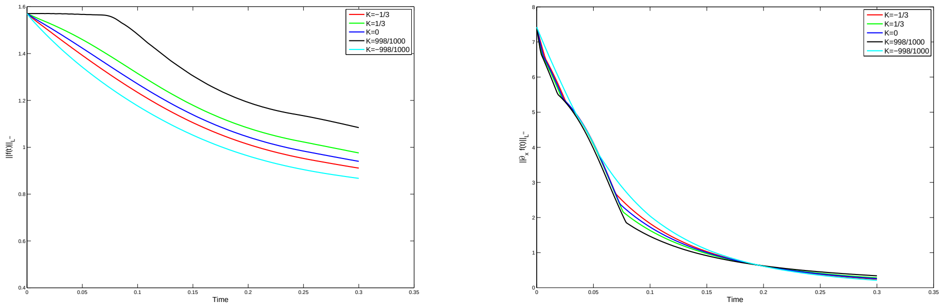

Figure 13.12: Evolution of $\|f^{\mathcal{K}}\|_{L^\infty}$ (left) and $\|\partial_x f^{\mathcal{K}}\|_{L^\infty}$ (right) with $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , 0 (blue) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) and initial datum given by (13.14).

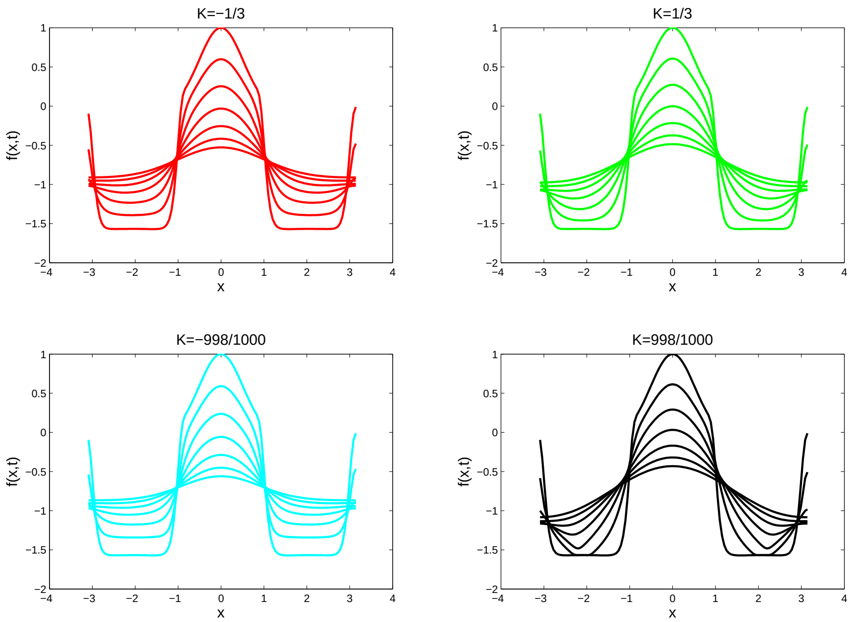

Figure 13.13: Evolution $f^{\mathcal{K}}(x,t)$ for $\mathcal{K} = \frac{-999}{1001}$ (cyan) , $\frac{-1}{3}$ (red) , $\frac{1}{3}$ (green) and $\frac{999}{1001}$ (black) at the same times t_i and initial datum given by (13.14).

case:

$$\frac{\partial_\alpha v_1(0)}{2} = \partial z_2(0) \int_0^\infty \partial_\alpha z_1(\eta) \sinh(z_1(\eta)) \sin(z_2(\eta)) \Bigg(\frac{1}{(\cosh(z_1(\eta)) - \cos(z_2(\eta)))^2} + \frac{1}{(\cosh(z_1(\eta)) + \cos(z_2(\eta)))^2} \Bigg) d\eta. \quad (13.15)$$

With the same ideas, for the infinitely deep case, we obtain the expression

$$\frac{\partial_\alpha v_1(0)}{2} = 4\partial z_2(0) \int_0^\infty \frac{\partial_\alpha z_1(\eta) z_1(\eta) z_2(\eta)}{(z_1(\eta))^2 + (z_2(\eta))^2)^2} d\eta. \quad (13.16)$$

We take z_0 as was defined in the Numerical Evidence 13.1, *i.e.*

$$z_1(\alpha) = \alpha - e^{-\alpha^2 k}\sin(\alpha),$$

with $k = 1e-4$ and

$$z_2(\alpha) = \begin{cases} \frac{1}{3}\sin(3\alpha) & \text{if } 0 \leq \alpha \leq \frac{\pi}{3}, \\ -\alpha + \frac{\pi}{3} & \text{if } \frac{\pi}{3} \leq \alpha < \frac{\pi}{2}, \\ \alpha - \frac{2\pi}{3} & \text{if } \frac{\pi}{2} \leq \alpha < \frac{2\pi}{3}, \\ 0 & \text{if } \frac{2\pi}{3} \leq \alpha. \end{cases}$$

For this curve we compute (13.15) and (13.16) using interval arithmetics (see [32, 33] for the details). We obtain that (13.15) is contained in the interval $(-0.013681974, -0.013681812)$ while (13.16) is contained in the interval $(0.021217219, 0.021217381)$. Then, we obtain a rigorous enclosure of the sign of the integral (8.4). We approximate the curve z by analytic functions and we obtain the forward and backward solvability applying Theorem 8.3 in Chapter 8. This concludes the proof. □

This Theorem also implies the following result

Proof of Corollary 13.1. Take the same curve as before and define the initial amplitude for the vorticity as $\partial_\alpha z_2(\alpha, 0)$. With these initial data we have a solution z of the water waves problem and the result follows (see [10]). □

13.4.2 Inhomogeneus Muskat problem

In this section we obtain that if $h_2 = O(1)$ there exist turning waves for any $\mathcal{K}$:

Proof of Theorem 13.2. Let us consider first the periodic setting (see equation (2.14)). Recall the fact that $h_2 = \pi/2$ and let us define, for $\alpha \in \mathbb{T}$,

$$\begin{aligned} z_1(\alpha) &= \alpha - \sin(\alpha) \\ z_2(\alpha) &= \frac{\sin(3\alpha)}{3} - \sin(\alpha)\left(e^{-(\alpha+2)^2} + e^{-(\alpha-2)^2}\right). \end{aligned} \tag{13.17}$$

While for the flat at infinity case (equation (2.12)) we define

$$\begin{aligned} z_1(\alpha) &= \alpha - \sin(\alpha)\exp(-\alpha^2/100) \\ z_2(\alpha) &= \frac{\sin(3\alpha)}{3} - \sin(\alpha)\left(e^{-(\alpha+2)^2} + e^{-(\alpha-2)^2}\right)\mathbf{1}_{\{|\alpha|<\pi\}}. \end{aligned} \tag{13.18}$$

For these curves we compute (12.3) and (12.5) using interval arithmetics. □

This result indicates that the different permeabilities can not prevent the singularity for any value of $\mathcal{K}$.

Index

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