The inhomogeneous Muskat problem

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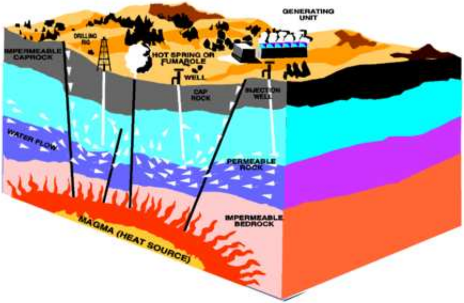
The Muskat problem

The question that we want to address is the evolution of an interface between two different fluids in a porous medium.

Figure : Water and air in a porous medium.



Thus, we have

The Muskat problem

$$
\begin{aligned}
\frac{\mu u}{\kappa} &= -\nabla p - (0,\rho), && \text{Balance of momentum}\\
\nabla \cdot u &= 0, && \text{Incompressibility}\\
\partial_t \rho + \nabla \cdot (u\rho) &= 0, && \text{Mass conservation.}
\end{aligned}
$$

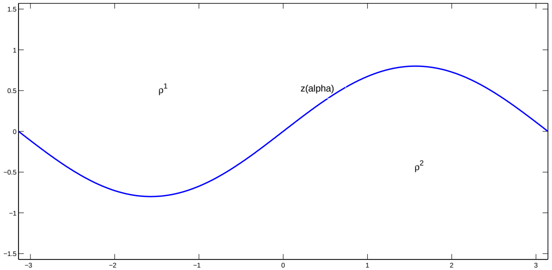
This system is equivalent to the standard (vertical) Hele-Shaw cell problem with gravity.

Part 1: The homogeneous Muskat problem

We need the definition of κ to close the system. There are some possible choices of $\kappa(x)$. For instance

Case 1: Infinitely deep & homogeneous

$$\kappa(x) = 1$$



We write

$$\rho^- - \rho^+ = 2\pi, \ \mu^+ = \mu^-.$$

Then, the equation in the infinitely deep case is

Case 1: Infinitely deep & homogeneous

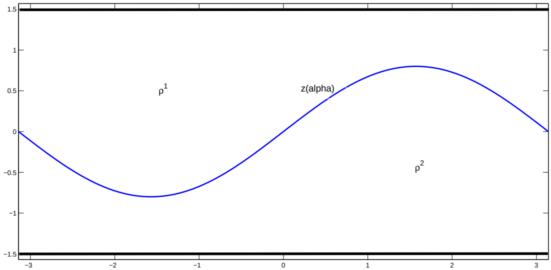
$$\partial_t f = \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x f(x) - \partial_x f(x-\eta))\eta}{\eta^2 + (f(x) - f(x-\eta))^2} d\eta.$$

The previous equation has been studied by many authors: D. Ambrose, R. Caflisch, A. Castro, P. Constantin, A. Córdoba, D. Córdoba, F. Deng, J. Escher, C. Fefferman, F. Gancedo, H. Kawarada, H. Koshigoe, Z. Lei, F. Lin, M. López-Fernández, A. Matioc, B. Matioc, T. Pernás-Castaño, L. Rodríguez-Piazza, R. Shvidkoy, R. Strain, V. Vicol etc.

In the case where we have parallel impervious walls, the permeability is

Case 2: Finitely deep

$$\kappa(x) = \mathbf{1}_{\{-\pi/2 \leq y \leq \pi/2\}}$$



The equation in the confined case is

Case 2: Finitely deep ($l = \pi/2$) & homogeneous

$$\partial_t f(x) = \frac{1}{2}\text{P.V.}\int_{\mathbb{R}} \frac{(\partial_x f(x) - \partial_x f(x-\eta))\sinh(\eta)}{\cosh(\eta) - \cos(f(x) - f(x-\eta))} + \frac{(\partial_x f(x) + \partial_x f(x-\eta)\sinh(\eta)}{\cosh(\eta) + \cos(f(x) + f(x-\eta))} d\eta.$$

The linearized equations are
Infinitely deep:

$$\partial_t g = -\frac{\rho^- - \rho^+}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{g(x) - g(x-\eta)}{\eta^2} d\eta,$$

Finitely deep:

$$\partial_t g = -\frac{\rho^- - \rho^+}{4l} \frac{\pi}{2l} \text{P.V.} \int_{\mathbb{R}} \frac{(g(x) - g(x-\eta)) \cosh\left(\frac{\pi}{2l}\eta\right)}{\sinh^2\left(\frac{\pi}{2l}\eta\right)} d\eta$$

We observe that this equation is well-posed if $\rho^- > \rho^+$.

This motivates the following definition

Rayleigh-Taylor condition

$$RT(x,t) = -\left[\nabla p^{-}(z(x,t)) - \nabla p^{+}(z(x,t))\right] \cdot \partial_x^{\perp} z(x,t).$$

From this point onwards, we assume that this condition holds.

In particular, in the case with the same viscosities, for a graph, this condition is satisfied if the denser fluid is above the lighter one.

Local existence in Sobolev spaces (homogeneous case):

1. Infinitely deep case with arbitrary H^3 initial data with the same viscosities $\mu^+ = \mu^-$ (D. Córdoba & F. Gancedo, *Comm. Math. Phys.* 2007)
2. Infinitely deep case with arbitrary H^3 initial data with arbitrary viscosities $\mu^+ \neq \mu^-$ (A. Córdoba, D. Córdoba & F. Gancedo, *Annals of Math.* 2011)
3. Finitely deep case with arbitrary H^3 initial data with the same viscosities $\mu^+ = \mu^-$ (D. Córdoba, RGB & R. Orive, *Comm. Math. Sciences* 2014)

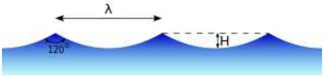
An interesting question is:

Can we prove local existence for initial data with unbounded curvature?

1. Arbitrary domain with H^2 initial data satisfying smallness restriction in $H^{1.5+}$ with arbitrary viscosities $\mu^+ \neq \mu^-$. In the case with viscosities $\mu^+ = 0, \mu^- > 0$ the result holds true for arbitrary H^2 initial data. (A. Cheng, RGB & S. Shkoller, *Adv. in Math. 2016*)
2. Infinitely deep case with arbitrary H^2 initial data with the same viscosities $\mu^+ = \mu^-$ (P. Constantin, F. Gancedo, R. Shvidkoy & V. Vicol, *Arxiv preprint* 2015)

These results close the problem at least in the cases with equal viscosities or $\mu^+ = 0$.

The problem of existence with infinite curvature is related with classical problems as the Stokes waves



Let us state rigorously our results

Theorem: Local existence of H^2 solutions (C. Cheng, RGB, S. Shkoller)

Let's consider $\mu^1, \mu^2 > 0$ (not necessarily equal!), $s > 0$ and $f_0 \in H^2$ in the stable regime such that

$$\|f_0\|_{H^{1.5+s}} \leq C,$$

for some universal constant C. Then there exists a unique solution

$$f(x,t) \in C([0, T(f_0)), H^2) \cap L^2([0, T(f_0)), H^{2.5}).$$

Here any domain geometry is allowed

Figure : $H^{1.5+s}$ norm by M. Rothko.

Figure : H^2 norm by Peridis.

Let's consider now $\mu^+ = 0 \neq \mu^- = 1$.

Theorem: Local existence of H^2 solutions (C. Cheng, RGB, S. Shkoller)

Let's consider $\mu^+ = 0, \mu^- > 0$, and $f_0 \in H^2$ in the stable regime. Then there exists a unique solution

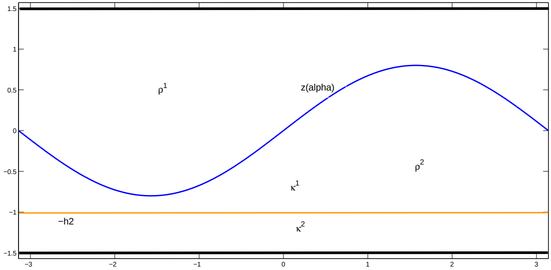
$$f(x,t) \in C([0,T(f_0)), H^2) \cap L^2([0,T(f_0)), H^{2.5}).$$

Furthermore, this solution becomes C^∞ instantly.

Again, here any domain geometry is allowed

Part 2: The inhomogeneous Muskat problem
What happens if the permeability takes more than 1 positive value?

Case 3: Finitely deep ($l = \pi/2$) & inhomogeneous



We write

$$\mathcal{K}=\frac{\kappa^1-\kappa^2}{\kappa^1+\kappa^2},\kappa^1=1 \text{ y } \rho^2-\rho^1=2\pi.$$

Case 3: Finitely deep ($l=\pi/2$) & inhomogeneous

$$\begin{aligned}\partial_t f(x)=&\frac{1}{2}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)-\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)-\cos(f(x)-f(\beta))}d\beta\\&+\frac{1}{2}\text{P.V.}\int_{\mathbb{R}}\frac{(\partial_x f(x)+\partial_x f(\beta))\sinh(x-\beta)}{\cosh(x-\beta)+\cos(f(x)+f(\beta))}d\beta\end{aligned}$$

$$\begin{aligned}&+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)(\sinh(x-\beta)+\partial_x f(x)\sin(f(x)+h_2))}{\cosh(x-\beta)-\cos(f(x)+h_2)}d\beta\\&+\frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}}\frac{\varpi_2(\beta)(-\sinh(x-\beta)+\partial_x f(x)\sin(f(x)-h_2))}{\cosh(x-\beta)+\cos(f(x)-h_2)}d\beta\end{aligned}$$

$$\varpi_2(x) = \mathcal{K}\left(\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \right.$$
$$\left. + \text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta\right)$$

$$+ \frac{\mathcal{K}^2}{\sqrt{2\pi}} G_{h_2,\mathcal{K}} * \left(\text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(h_2+f(\beta))}{\cosh(x-\beta)-\cos(h_2+f(\beta))}d\beta \right.$$
$$\left. - \text{P.V.}\int_{\mathbb{R}} \frac{\partial_x f(\beta)\sin(-h_2+f(\beta))}{\cosh(x-\beta)+\cos(-h_2+f(\beta))}d\beta\right).$$

Theorem: Well-posedness (L.Berselli, D.Córdoba & RGB)

If the Rayleigh-Taylor condition is satisfied, *i.e.* $\rho^2-\rho^1>0$, and the initial data $-h_2<f_0(x)=f(x,0)\in H^3_l(\mathbb{R})$, then there exists an unique classical solution $f\in C([0,T],H^k_l(\mathbb{R}))$ where $T=T(f_0)$. Furthermore, the solution verifies

$$\|f(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\frac{\|v\|^2_{L^2(\mathbb{R}\times(-h_2,\pi/2))}}{\kappa^1(\rho^2-\rho^1)}ds+\int_0^t\frac{\|v\|^2_{L^2(\mathbb{R}\times(-\pi/2,-h_2))}}{\kappa^2(\rho^2-\rho^1)}ds=\|f_0\|^2_{L^2(\mathbb{R})}.$$

Notice that there is not condition on the sign of $\mathcal{K}$. To better understand the role of $\mathcal{K}$ it is interesting to perform numerics. Let's see some videos.

However, what if the curve where the permeability changes is not flat?

Theorem: Local existence of H^3 solutions (T. Pernás-Castaño, Arxiv Preprint, 2016)

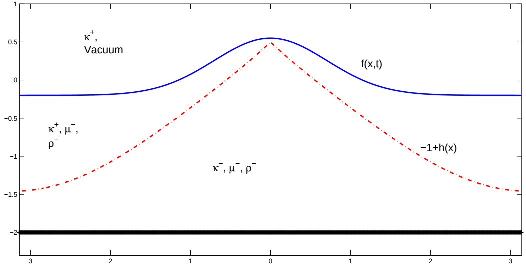
Let's consider $\mu^1 > 0, \mu^2 > 0$ (not necessarily equal), $h \in H^3$ and $f_0 \in H^3$ in the stable regime. Assume also that the domain is unbounded. Then there exists a unique solution

$$f(x,t) \in C([0, T(f_0)), H^3).$$

However, what if the initial data has unbounded curvature?

Case 4: finitely deep & inhomogeneous

$$\kappa(x,y) = \kappa^1 \mathbf{1}_{\{-1+h(x)<y\}} + \kappa^2 \mathbf{1}_{\{-2<y<-1+h(x)\}}$$



Theorem: Local existence of H^2 solutions (RGB & S. Shkoller, Preprint)

Let's consider $\mu^1 = 0, \mu^2 > 0$, $h \in H^{2.5}$ and $f_0 \in H^2$ in the stable regime. Then there exists a unique solution

$$f(x,t) \in C([0,T(f_0)),H^2) \cap L^2([0,T(f_0)),H^{2.5}).$$

Finite time singularities The wave could turn over

Figure : Jean-Désiré-Gustave Courbet, The Wave, 1870

Or, with a clearer picture,

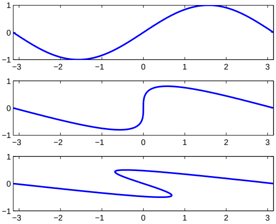


Figure : Possible turning

Theorem: Turning waves (J. Gómez-Serrano & RGB)

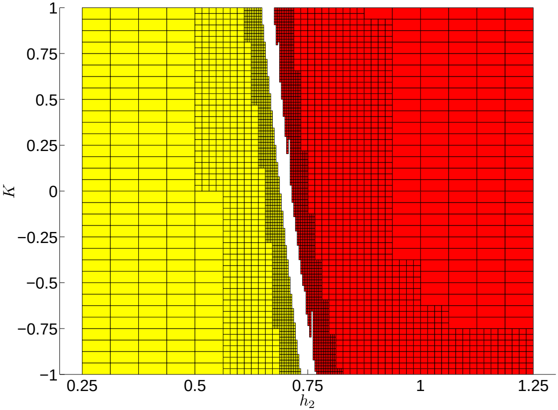


Figure : Yellow: not turning, red: turning.

Theorem: Turning waves (J. Gómez-Serrano & RGB)

There exists a C^1 curve $(h_2, \mathcal{K}(h_2))$, located in $[0.648, 0.77] \times (-1, 1)$, such that for every h_2 for which the curve is defined, for every $\mathcal{K} < \mathcal{K}(h_2)$ the curve does not turn and for every $\mathcal{K} > \mathcal{K}(h_2)$ the curve turns.

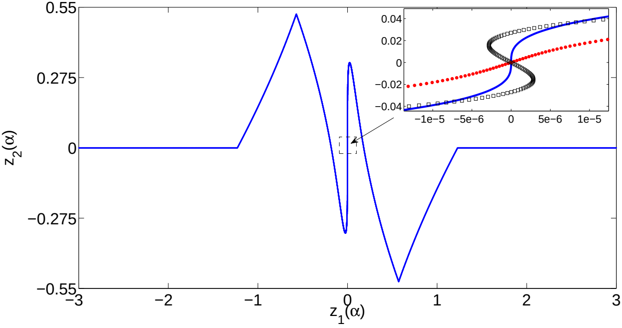


Figure : The curve. Inset: Close caption around zero, solid: initial condition, dotted: normal component of the velocity for the infinitely deep case, squared: normal component of the velocity for the finitely deep case. The normal components have been scaled by a factor 1/100.