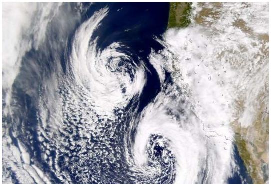
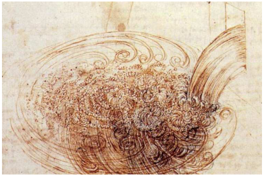
On the dynamics of free boundaries between incompressible fluids

Rafael Granero-Belinchón

In this talk I will present some new ideas and results for

1. incompressible fluids moving through a porous media
2. the modeling of the Rayleigh-Taylor instability for two-phase incompressible Euler equations





Euler equations (1757)

$$\underbrace{\rho}_{\text{mass}}\left(\underbrace{\partial_t u+(u\cdot\nabla)u}_{\text{acceleration}}\right)=\underbrace{-\nabla p-g\rho e_d^t}_{\text{force}}\ \text{Newton's Law}$$

$$\nabla\cdot u=0\ \text{incompressibility}$$

$$\partial_t\rho+u\cdot\nabla\rho=0\ \text{conservation of mass}$$

$$u:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d, d=2,3,$$

$$p:\mathbb{R}^d\times[0,T]\to\mathbb{R},$$

$$\rho:\mathbb{R}^d\times[0,T]\to[0,\infty)$$

and $g=9.8m/s^2$ (the acceleration due to gravity).

When the fluid moves through porous media, the *usual* momentum equation (Euler/Navier-Stokes) changes to describe the effect of the solid matrix.

Darcy's law (1856)

$$\frac{\mu}{\kappa}u = -\nabla p - g\rho e_d^t \text{ Darcy's law}$$

Figure : Water and air in a porous medium.

Darcy's law is similar to the Hele-Shaw problem modeling flow between two paralled plates separated by a small distance (b)

Hele-Shaw (1898)

$$\frac{12\mu}{b^2}u = -\nabla p - g e_2^t \quad \text{Hele-Shaw eq}$$

The Muskat (1931) or Hele-Shaw problem is then

Muskat problem

$$\frac{\mu}{\kappa}u = -\nabla p - g\rho e_d^t \quad \text{Darcy's law}$$

$$\nabla \cdot u = 0 \quad \text{incompressibility}$$

$$\partial_t \rho + u \cdot \nabla \rho = 0 \quad \text{conservation of mass}$$

The fundamental questions are

- do either *weak or strong solutions* exist for all time?
- are there mechanisms that create a *finite-time breakdown* of the solutions?

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In other words, we are interested in the dichotomy global existence vs. finite time singularity.

Singularities in the fluid bulk

Beale-Kato-Majda blow up criterion

If

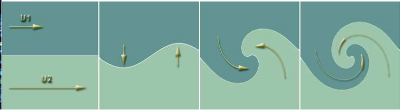
$$\int_0^t \max_x |\text{curl}\, u(x,s)| ds < \infty$$

then the velocity u and the pressure p remain smooth.

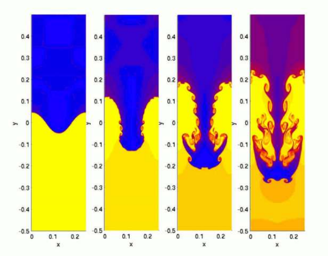
Singularities on surfaces of discontinuity (for instance, water waves)
Kelvin-Helmholtz instability (named after their works in 1871 and 1868, respectively)

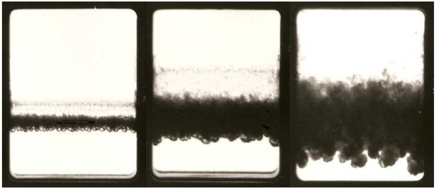




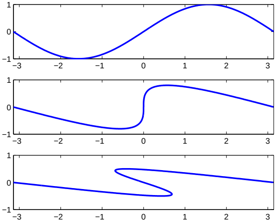


Rayleigh-Taylor instability (named after their works in 1883 and 1950, respectively)

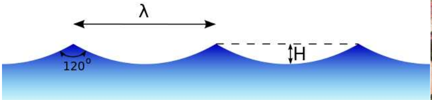


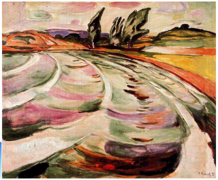


Turning waves & self-intersection



Curvature blow-up (for instance, for the so called *extreme Stokes waves* and the Stokes conjecture (1880))





Some of the main difficulties to understand the dychotomy global existence vs. finite time singularity are the following:

1

We need to study hyperbolic or coupled hyperbolic-parabolic free-boundary problems. The domains are themselves unknown in the problem.

2

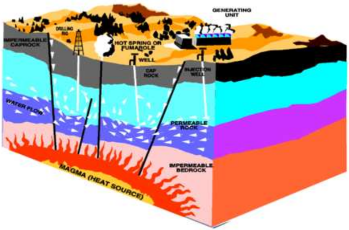
Not only can gradients of velocity, pressure become infinite, but the interface can form a *corner*, or self-intersect, etc. We have to study the geometric properties of the interface.

3

In particular, we need to develop a local theory for well-posedness for these free-boundary problems. We have to design *good* norms that capture the fundamental behavior of fluids coupled to interfaces.

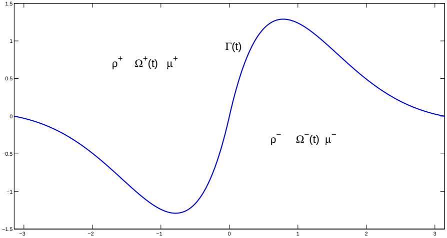
4

Besides the free boundary, real problems occur on bounded domains with arbitrary geometries. We have to take into account the topography and its effect in the dynamics of the interface.



The Muskat/Hele-Shaw problem:

Some notation



$$\Gamma(t) = \partial\Omega^+(t) \cap \partial\Omega^-(t) \text{ (free boundary)},$$

$$\Omega(t) = \Omega^-(t) \cup \Omega^+(t).$$

The 2D Muskat/Hele-Shaw problem

$$\frac{\mu}{\kappa}u = -\nabla p - g\rho e_2^t \ \ (x,y) \in \Omega(t)$$
$$\nabla \cdot u = 0 \ \ (x,y) \in \Omega(t)$$
$$\partial_t \rho + u \cdot \nabla \rho = 0 \ \ (x,y) \in \Omega(t)$$
$$\rho(0) = \rho^- \mathbf{1}_{\Omega_0^-} + \rho^+ \mathbf{1}_{\Omega_0^+} \text{ initial data}$$
$$\Gamma(0) = (x, f_0(x)) \text{ initial data}$$

Unknowns of the problem:

- the fluid velocity u
- the fluid pressure p
- the fluid domains $\Omega^+(t), \Omega^-(t)$. Equivalently, the interface $f(x,t)$

The 2D Muskat/Hele-Shaw problem

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This problem has been studied by many authors: D. Ambrose, R. Caflisch, P. Constantin, D. Córdoba, J. Escher, C. Fefferman, F. Gancedo, S. Howison, F. Lin, B. Matioc, R. Shvidkoy, R. Strain, V. Vicol, etc.

Most of the previous results used the following contour equation for the interface $(x, f(x,t))$,

$$\partial_t f(x,t) = \frac{\rho^- - \rho^+}{2\pi} p.v. \int_{\mathbb{R}} \frac{(\partial_x f(x,t) - \partial_x f(x-y,t))y}{y^2 + (f(x,t) - f(x-y,t))^2} dy.$$

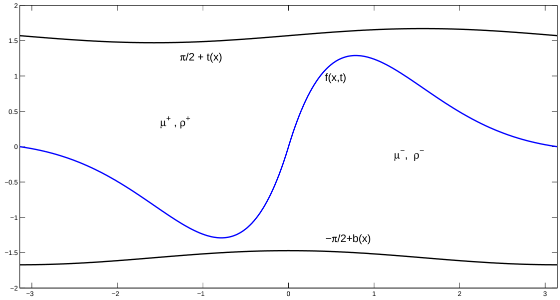
Some limitations of this formulation:

- This contour equation is valid only when the fluids fill the whole plane, *i.e.* they *do not* consider the bottom or top topography.
- The fluids have the same viscosities, *i.e.* $\mu^- = \mu^+$
- The well-posedness theory allowed for $f_0 \in H^3$ initial data, *i.e.* the curvature is *finite*.

There are some cases where we can reduce the previous drawbacks while still use a contour equation formulation:

- the case where the fluids fill a flat strip but have the same viscosities (Córdoba, RGB, Orive, *Comm. Math. Sci*).
- the case where the fluids have different viscosities but fill the plane (no top/bottom boundaries) (Córdoba, Córdoba, Gancedo, *Ann. of Math*).

We consider the following setting for the Muskat problem



We assume that $t(x), b(x), f(x,t) \in H^2$. Thus, they may have unbounded curvature.

Furthermore, we consider the Rayleigh-Taylor stable case where the fluids satisfy

$$RT(t) = (\nabla p^{+}(x, f(x,t)) - \nabla p^{-}(x, f(x,t))) \cdot (-\partial_x f(x,t), 1) > 0.$$

Note that in the case where $\mu^{+} = \mu^{-}$, we have that the Rayleigh-Taylor condition reduces to

$$RT(t) = -(\rho^{+} - \rho^{-}),$$

i.e. *good* fluid stratification.

Main contributions:

- Initial data with unbounded curvature
- Arbitrary bottom with unbounded curvature

Theorem: Two-phase problem (Cheng, RGB & Shkoller, *Adv. in Math*, 2016)

Assume that $\rho^- > \rho^+$ (RT stable), and

$$f_0 \in H^2, \ \|f_0\|_{H^{1.75}} + \|b\|_{H^2} + \|t\|_{H^2} \leq C_1$$

(for a small enough constant C_1). Then, there exists a unique solution for the Muskat problem

$$f(x,t) \in C([0,T(f_0)),H^2) \cap L^2([0,T(f_0)),H^{2.5}).$$

and this solution becomes C^∞ in $(0, T(f_0))$.

This result was the first that allowed for initial data and arbitrary bottom and top with unbounded curvature.

The new ideas are the following:

- By using a good *change of variables*, we fix the domain. free boundary PDE with constant coefficients $\Rightarrow$ fixed boundary PDE with variable coefficients.
- We designed an *energy* that involves both the interface and the velocity of the fluid in the bulk. We analyse the behavior of the interface and the fluid bulk together.
- We crucially use a new identity that links the curvature of the interface with the tangential discontinuity.

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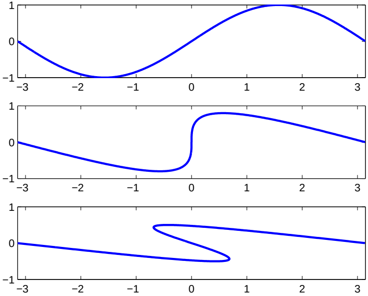
Let us study rigorously wave breaking for the Muskat problem

Figure : Jean-Désiré-Gustave Courbet, The Wave, 1870



Theorem: Turning/Recoiling (Gómez-Serrano & RGB)

There exists initial data $z_0(\alpha) = (z_1(\alpha), z_2(\alpha))$ such that:

- if the depth is infinite, the solution becomes a regular graph.
- if the depth is finite, the solution cannot be parametrized as a graph (turning singularity).

So Shakespeare was right when he wrote
"Smooth runs the water where the brook is deep".

This is a *computer assisted proof*. The ideas in this proof are the following:

- We reformulate the contour equation to allow for curves with vertical tangent vector. These curves cannot be parametrized as smooth graphs.
- We prove (forward and backward) local existence via an *ad hoc* Cauchy-Kovalevsky theorem.
- We construct a *suitable* initial data with vertical tangent vector.
- To ensure that the velocity points in the desired direction, we rigorously compute $\partial_\alpha u_1$ using interval arithmetics.

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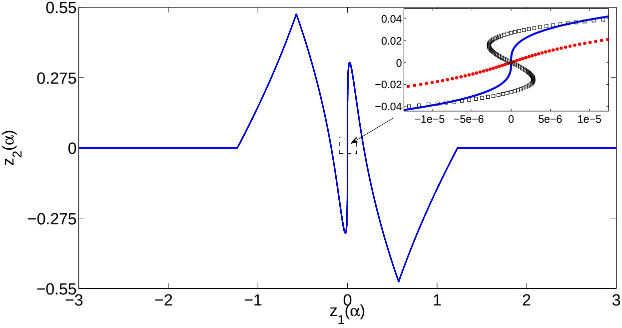
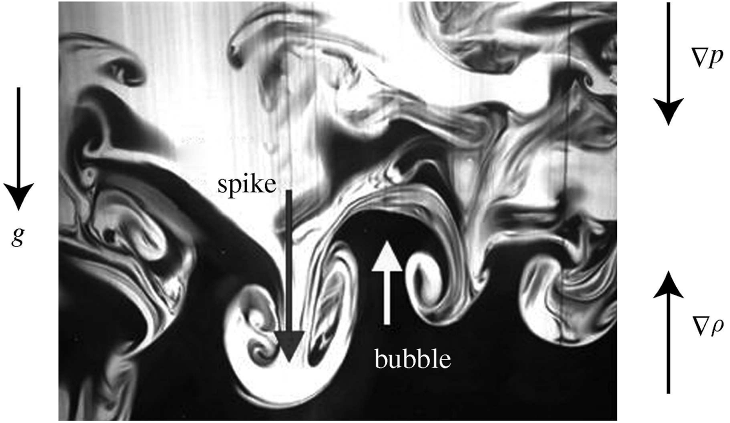


Figure : The curve. Inset: Close caption around zero, solid: initial condition, dotted: normal component of the velocity for the infinitely deep case, squared: normal component of the velocity for the finitely deep case. The normal components have been scaled by a factor 1/100.

The Rayleigh-Taylor instability & mixing:



$\nabla p \cdot \nabla \rho < 0$

The 2D Rayleigh-Taylor instability

$$\rho\left(\partial_t u + (u\cdot\nabla)u\right) = -\nabla p - g\rho e_2^t \quad \text{Euler}$$
$$\nabla\cdot u = 0 \quad \text{incompressibility}$$
$$\text{curl}\, u = 0 \quad \text{irrotationality}$$
$$\partial_t\rho + u\cdot\nabla\rho = 0 \quad \text{conservation of mass}$$
$$\rho(x,0) = \rho^{-}\mathbf{1}_{\Omega^{-}(0)} + \rho^{+}\mathbf{1}_{\Omega^{+}(0)} \quad \text{initial data (density)}$$
$$u(0) = u_0^{-}\mathbf{1}_{\Omega_0^{-}} + u_0^{+}\mathbf{1}_{\Omega_0^{+}} \quad \text{initial data}$$

We also have to attach an initial free boundary $\Gamma(0)$.
Unknowns:

- velocity u
- pressure p
- interface $\Gamma(t)$ the domains are unknowns

Two-phase Euler (in absence of surface tension) is

- ill-posed,
- highly unstable,
- computationally intractable.

RT mixing is crucial in many different physical problems and applications. However, with nearly 200 papers published yearly on RT mixing in scientific journals, our knowledge of this process is still limited.

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According to Youngs (1984), the development of a RT driven mixing zone can be described as a three step process:

- Initially an exponential growth of infinitesimal perturbations that correspond to linear stability analysis.
- After a short amount of time of exponential-in-time growth, the interface can be described by *bubbles* of the lighter fluid and *spikes* of the heavier fluid. In this second regime, exponential growth of the bubbles slows down and nonlinear terms in the equations of motion can no longer be ignored. This is called *saturation/bubble competition*.

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- Eventually, due to the growth of the bubbles, they merge and mix chaotically. This develops into a region of turbulent self-similar mixing whose half-width can be described by the following formula

$$\text{half-width of the turbulent mixing region} \approx \alpha\left(\frac{\rho^+-\rho^-}{\rho^++\rho^-}\right)gt^2.$$

α is a dimensionless but non-universal parameter.
How many physical parameters does it take to describe α? At least 6, 3 describing fluid parameters and 3 describing the initial conditions (Glimm *et. al.*).

One of the fundamental questions regarding RT mixing is
How can we estimate α?
Difficulties:

- Two-phase Euler is a highly unstable system $\Rightarrow$ Direct Numerical Simulations (DNS) extremely expensive.
- Two-phase Euler is ill-posed $\Rightarrow$ the analysis of the system extremely difficult.

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Current approaches:
Real experiments (Read & Youngs, Read, Smeeton & Youngs)

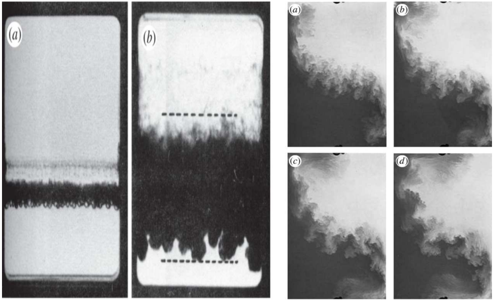

Figure : a) Rocket rig and b) Tilted rig experiments where the fluids are NaI solution ($\rho^- = 1.89g/cm^3$) and Hexane ($\rho^+ = 0.66g/cm^3$)

Current approaches:
Modal models (Goncharov, Rollin & Andrews).
Such models are large systems of nonlinear ODEs for expansions (Taylor, Fourier) of the interface and the velocity.
These models (Goncharov) predicts a constant velocity during the saturation/bubble competition regime.

Our main contributions are a couple of new mathematical models for two-fluid interface motion, subjected to the Rayleigh-Taylor (RT) instability in two-dimensional fluid flow.
The basis of our approach is very different to the prior modal modes and leads to two different nonlinear and nonlocal PDE's modeling RT instability.

The ideas are the following:

1. We write the 2D Euler system for (u, p, Γ) in the Birkhoff-Rott integral-kernel formulation (BR). New unknowns: the interface Γ and the tangential discontinuity $\varpi = (u^{-} - u^{+}) \cdot \tau$ ($\tau(t)$ denotes the tangent vector to $\Gamma(t)$).
2. We reduce the problem to a system of 2 (Γ is a graph) or 3 (Γ is not a graph) nonlinear and nonlocal PDE's.
3. We either
   - expand the integral kernel in the BR formulation assuming that the slope of the interface is small (*h-model*)
   - restrict the nonlocality present in the the original BR kernel (*z-model*)

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When the interface $\Gamma(t)$ is given by a graph $(x, h(x,t))$, we use an asymptotic expansion of the BR formulation and assume that $|\partial_x h| \ll 1$.

Example:

$$\frac{p.v.}{\pi}\int \varpi(y,t)\frac{(x-y)}{(x-y)^2+(h(x,t)-h(y,t))^2}dy$$

$$\approx \frac{p.v.}{\pi}\int \frac{\varpi(y,t)}{x-y}\left(1-\left(\frac{h(x,t)-h(y,t)}{x-y}\right)^2\right)dy$$

$$\approx H\varpi$$

where H denotes the Hilbert transform $\widehat{Hg}(k) = -i\frac{k}{|k|}\hat{g}(k)$.

This leads to the *h-model*.

$$h_{tt} = Ag\Lambda h - A\partial_x(Hh_t h_t)$$

where $\Lambda = H\partial_x$,

$$A = \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-} \text{ Atwood number}$$

The $h-$model predicts a constant velocity if certain stability condition is satisfied. The stability condition is similar to the saturation/bubble competition regime. This recovers Goncharov predictions.

When the interface $\Gamma(t)$ is given as a smooth curve $(z_1(x,t), z_2(x,t))$, we replace the BR kernel by a *localized* kernel. Example:

$$\frac{p.v.}{\pi}\int \varpi(y)\frac{(z(x,t)-z(y,t))^\perp}{|z(x,t)-z(y,t)|^2}dy \approx \frac{(\partial_\alpha z(\alpha,t))^\perp}{|\partial_\alpha z(\alpha,t)|^2}\frac{p.v.}{\pi}\int\frac{\varpi(y)}{x-y}dy$$

$$
\begin{aligned}
z_{tt} = {} & \Lambda\left[\frac{A}{|\partial_x z|^2} H\left(z_t \cdot (\partial_x z)^{\perp} H(z_t \cdot (\partial_x z)^{\perp})\right)\right]\frac{(\partial_x z)^{\perp}}{|\partial_x z|^2} \\
& + z_t \cdot (\partial_x z)^{\perp}\left(\frac{(\partial_x z_t)^{\perp}}{|\partial_x z|^2} - \frac{(\partial_x z)^{\perp} 2(\partial_x z \cdot \partial_x z_t)}{|\partial_x z|^4}\right) \\
& + Ag\Lambda z_2 \frac{(\partial_x z)^{\perp}}{|\partial_x z|^2}
\end{aligned}
$$

here the interface is parametrized as $z = (z_1, z_2)$.

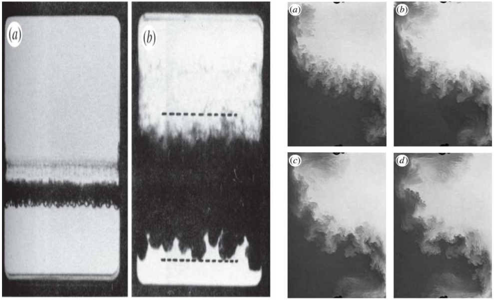
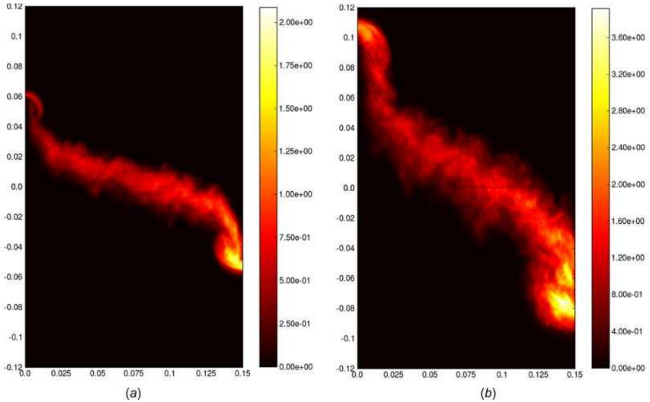


Figure : a) Rocket rig and b) Tilted rig experiments where the fluids are NaI solution ($\rho^- = 1.89g/cm^3$) and Hexane ($\rho^+ = 0.66g/cm^3$)


0.12
0.1
0.08
0.06
0.04
0.02
0.0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
0.0
0.025
0.05
0.075
0.1
0.125
0.15
2.00e+00
1.75e+00
1.50e+00
1.25e+00
1.00e+00
7.50e-01
5.00e-01
2.50e-01
0.00e+00
(a)
3.60e+00
3.20e+00
2.80e+00
2.40e+00
2.00e+00
1.60e+00
1.20e+00
8.00e-01
4.00e-01
0.00e+00
(b)

For the rocket rig:

- DNS suggests $\alpha \in [0.04, 0.05]$
- the empirical value is $\alpha \approx 0.063$

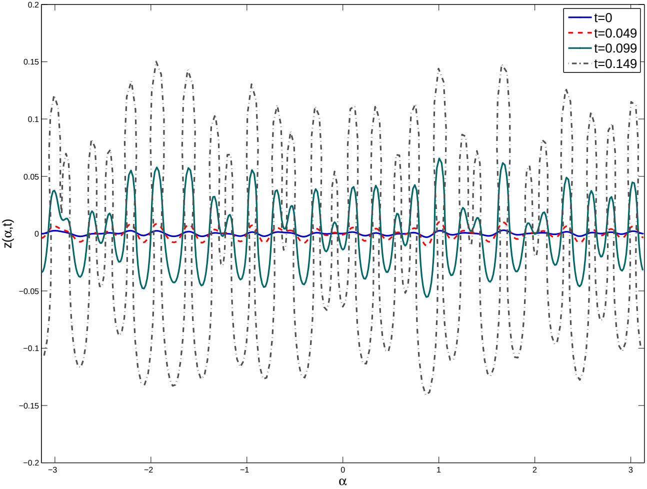
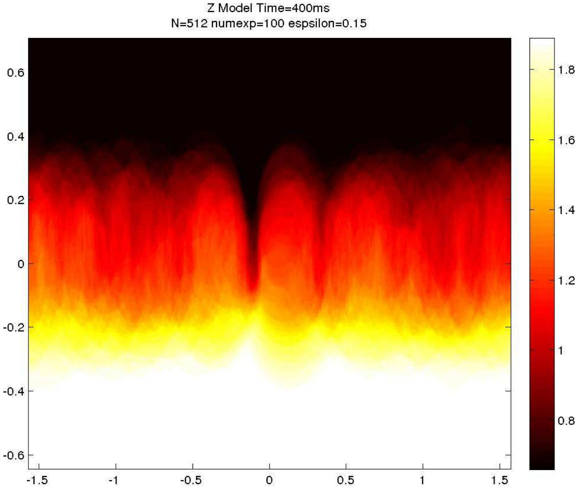


Figure : Interface position $z(\alpha, t_j)$ for $t_0 = 0$, $t_1 = 0.049$, $t_2 = 0.099$ and $t_3 = 0.149$.


Z Model Time=400ms
N=512 numexp=100 espsilon=0.15
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5
1.8
1.6
1.4
1.2
1
0.8

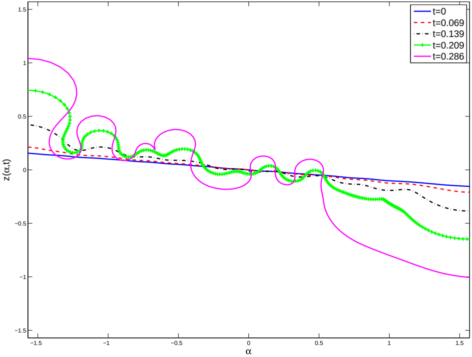


Figure : Interface position for $t_0 = 0$, $t_1 = 0.069$, $t_2 = 0.139$, $t_3 = 0.209$ and $t_4 = 0.286$.

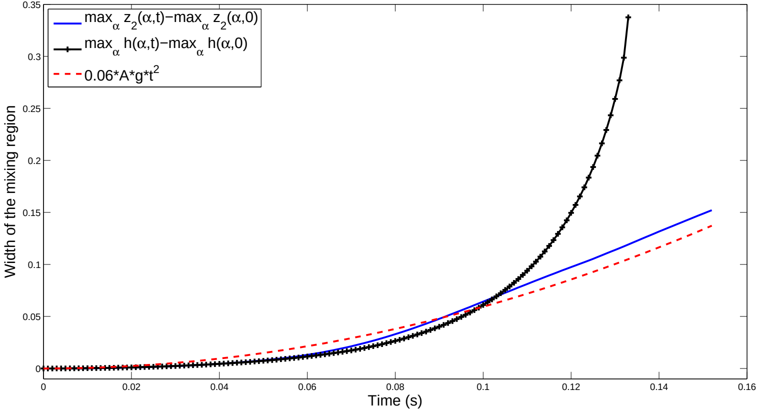


Figure : Comparison between $\max_x z_2(x,t) - \max_x z_2(x,0)$, $\max_x h(x,t) - \max_x h(x,0)$ and (**??**) with $\alpha = 0.06$.

Future work:

- Scenarios for finite time curvature blow up for interfaces
- Modeling of the RT instability
  - various levels of fidelity
  - inviscid regularizations
  - in 3D
  - compressibility effects
  - velocity fields with non-zero curl