On two nonlocal equations showing chaotic behaviour

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Part 1: Overture

In this talk we consider the problem given by

$$\partial_t u = \mathcal{L}u + \partial_x(u\mathcal{T}u) + f(u), \tag{1}$$

where $\mathcal{L}$ is defined using a Fourier multiplier,

$$\widehat{\mathcal{L}u} = m(|\xi|)\hat{u},$$

$\mathcal{T}u$ is certain operator, $f(u)$ is some polynomial forcing. We also consider periodic boundary conditions.

The previous equation arises from a variety of applications:
1) Chemotaxis, *i.e.* the movement of cells along a chemical concentration gradient (joint work with J. Burczak): This is very interesting as an explanation of the formation of multicellular structures from the aggregation of unicellular bacteria. An example of this behaviour is the slime mold *Dictyostelium discoideum* who changes from a collection of unicellular amoebae into a multicellular slug and then into a fruiting body within its lifetime.

$$\mathcal{L} = -(-\partial_x^2)^{\alpha/2} = -\Lambda^{\alpha},\ 0 < \alpha \leq 2 \quad f(u) = ru(1-u),$$

$$\begin{aligned}\mathcal{T}u &= \Lambda^{\beta-1} H v(x,t) \\ &= \Lambda^{\beta-1} H \left(v_0 e^{(-\Lambda^{\beta}+1)t} + \int_0^t e^{-(\Lambda^{\beta}+1)(t-s)} u(s) ds \right),\ 0 < \beta \leq 2.\end{aligned}$$

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2) Fluid dynamics (joint work with J. K. Hunter): Stabilization of a Hadamard instability, with growth rate proportional to the absolute value of the wavenumber, by viscous diffusion (e.g., the Kelvin-Helmholtz instability for the Euler equations) and kinetic equations (e.g., the Vlasov equations, negative Landau damping of plasma waves), in which the growth rate of long waves is determined by a parameter with the dimensions of velocity.

$$\mathcal{L}=\Lambda^{\gamma}-\epsilon\Lambda^{\alpha},\ 0<\gamma<\alpha\leq 2,\ 0<\epsilon\ll 1,\quad f(u)=0,$$

$$\mathcal{T}u=-\frac{u}{2}.$$

This equation also appears as a model of

- crime
- electrolytes (Debye system)
- pattern formation of *Escherichia Coli*, *Salmonella Typhimurium*
- astrophysics (Zeldovich approximation, primordial universe)

GOAL:

To study the global in time existence of solutions and its behaviour.

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Part 2: Chemotaxis The problem 1) is equivalent to the system

$$\partial_t u = -\Lambda^\alpha u + \partial_x(u\Lambda^{\beta-1}Hv) + ru(1-u),$$

$$\partial_t v = -(\Lambda^\beta + 1)v + u$$

Notice that $\Lambda H = -\partial_x$

Global existence (Burczak, RGB):

Let $2 \geq \alpha > 1$, $r \geq 0$, $2 \geq \beta \geq \alpha/2$ and the initial data $(u_0, v_0) \in H^{3\alpha}(\mathbb{T}) \times H^{3\alpha+\beta/2}(\mathbb{T})$, then there exists a unique global in time solution (u_0, v_0) that enjoys

$$u \in C([0, T], H^{3\alpha}(\mathbb{T})) \quad \forall\ T < \infty,$$

$$v \in C([0, T], H^{3\alpha+\beta/2}(\mathbb{T})) \quad \forall\ T < \infty.$$

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A short digression...

Spoiler alert! In fact, in a joint work with J. Burczak, we study the system

$$\partial_t u = -\Lambda^\alpha u + \partial_x(u\Lambda^{\beta-1}Hv) + ru(1-u),$$

$$(1+\Lambda^\beta)v = u,$$

and proved global classical solutions for

$$1 - r < \alpha \leq 2.$$

If $r = 0$, we prove global classical solution in the limiting, critical case $\alpha = 1$.

Thus,

Open problem 1:

Assume that, $0 < r \ll 1$, $1 - c(r) < \alpha < 1$, for certain $c(r) > 0$. Is there a global strong solution for the doubly parabolic Keller-Segel system?

Open problem 2:

Assume that $\alpha = 1$ and $r = 0$. Is there a global strong solution for the doubly parabolic Keller-Segel system?

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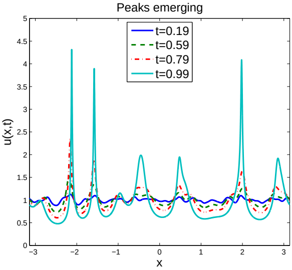
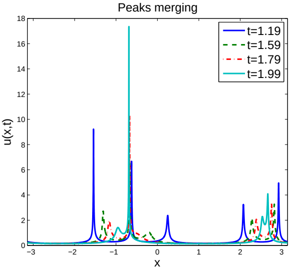
Let see how is the behaviour of these solutions:





The previous simulations suggests that the solution is bounded (in certain functional spaces) and develops spatio-temporal chaos.

Absorbing set (Burczak, RGB):

Let $2 \geq \beta \geq \alpha > 1$, $r > 0$ and the initial data $(u_0, v_0) \in H^{3\alpha}(\mathbb{T}) \times H^{3\alpha+\beta/2}(\mathbb{T})$ be given. Then there exist positive numbers T^*, $S(\cdot)$ such that

$$\|u(t+1)\|^2_{\dot{H}^{3\alpha/2}} \leq S(\dot{H}^{3\alpha/2}) \quad \forall\, t \geq T^*, 0 \leq 3\alpha.$$

In particular, it implies that there exists C, depending on the parameters present in the problem and on the initial data, such that

$$\max_{0\leq t<\infty} \{\|u(t)\|^2_{H^{3\alpha/2}} + \|v(t)\|^2_{H^{\beta+\alpha}}\} \leq C.$$

Attractor (Burczak, RGB):

Given $r > 0$, $2 \geq \beta \geq \alpha \geq 8/7$, the doubly parabolic Keller-Segel system has a maximal, connected, compact attractor in the space $H^{3\alpha}(\mathbb{T}) \times H^{3\alpha+\beta/2}(\mathbb{T})$.

The restriction $8/7 \leq \alpha$ is to get

$$3\alpha + \beta/2 \geq 3.5\alpha \geq 4.$$

This condition is, somehow, *an artifact*.

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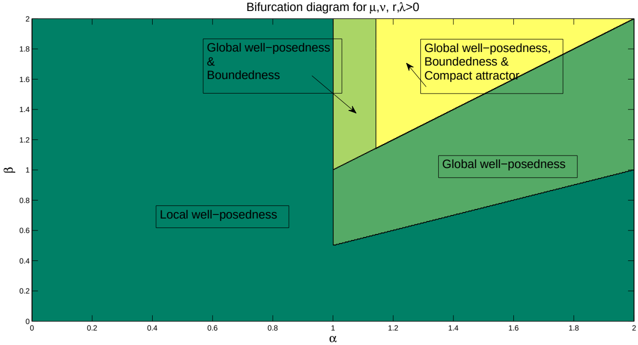

Peaks I (Burczak, RGB):

Let $2 \geq \alpha, \beta \geq 1$, $r > 0$ and the initial data $(u_0, v_0) \in H^3(\mathbb{T}) \times H^4(\mathbb{T})$ be given. Then, there exists $\mathcal{W}(u_0, v_0, \alpha, \beta)$ (with an explicit expression) such that, for any $0 < \tilde{T}/2 < t < \tilde{T}$,

$$\mathbb{T} = I^u \cup R^u = I^v \cup R^v,$$

where I^u, I^v are the union of at most $[\frac{4\pi}{\mathcal{W}}]$ open intervals in $\mathbb{T}$, and

- $|\partial_x u(x)| \leq 1,$ for all $x \in I^u$,
- $\#\{x \in R^u : \partial_x u(x) = 0\} \leq \frac{2}{\log 2} \frac{2\pi}{\mathcal{W}} \log\left(\frac{\sqrt{2}2\|u_0\|_{L^\infty(\mathbb{T})}}{\mathcal{W}}\right),$
- $|\partial_x v(x)| \leq 1,$ for all $x \in I^u$,
- $\#\{x \in R^v : \partial_x v(x) = 0\} \leq \frac{2}{\log 2} \frac{2\pi}{\mathcal{W}} \log\left(\frac{\sqrt{2}2\|v_0\|_{L^\infty(\mathbb{T})}}{\mathcal{W}}\right).$

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Using the explicit expression for $\mathcal{W}$ for $\min\{\alpha,\beta\}>1$, we obtain

Peaks II (Burczak, RGB):

Let $r>0$, $2\geq\beta\geq\alpha\geq 8/7$ and (u,v) be a solution in the attractor, then the number of peaks for u can be bounded as

$$\#\{\text{peaks for } u\}\leq\frac{12\pi\mathcal{K}_1}{\log 2}\log\left(6\sqrt{2}\mathcal{K}_1C_{SE}^2(\alpha)S(H^{\alpha/2})\right),$$

where $S(H^{\alpha/2})$ and $\mathcal{K}_1$ are explicit (with intricate expression).

Remark: The explicit expressions are available in the paper.

Part 3: Fluid dynamics The problem 2) is the nonlocal analog of the well-known Kuramoto-Sivashinsky equation

$$\partial_t u = (\Lambda^\gamma - \epsilon\Lambda^\alpha)u - u\partial_x u.$$

Some simulations to compare between the local and nonlocal effects:

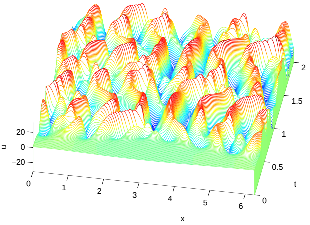

Figure : $\gamma = 2, \alpha = 4, \epsilon = 0.001$

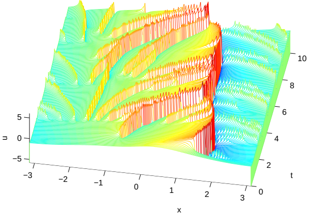

Figure : $\gamma = 1, \alpha = 2, \epsilon = 0.001$

Some simulations to compare between the nonlocal effects:

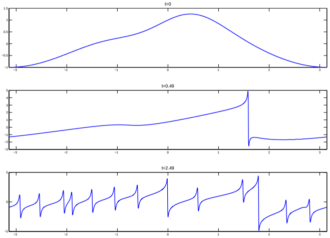

Figure : $\gamma = 1, \alpha = 2, \epsilon = 0.001$

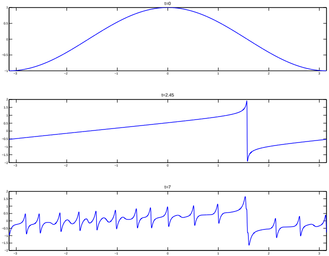

Figure : $\gamma = 1.45, \alpha = 1.5, \epsilon = 0.8$

Global existence (RGB, Hunter):

Suppose that $\epsilon > 0$, $0 \leq \gamma < \alpha \leq 2$. If

$$u_0 \in H^3,$$

then for every $0 < T < \infty$ the initial value problem has a unique classical solution

$$u(x,t) \in C([0,T], H^3).$$

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Absorbing set (RGB, Hunter):

Suppose that $u_0 \in H^3(\mathbb{T})$ has zero mean. Then for every $0 < s \leq 3$ the solution u satisfies

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} \leq C\left(s, \epsilon, \alpha, \gamma, \|u_0\|_{L^2}\right).$$

Attractor (RGB, Hunter):

The nonlocal analog of the Kuramoto-Sivashinsky has a maximal, connected, compact attractor in the space H^3.

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Oscillations (RGB, Hunter):

Let u be the solution for initial data $u_0 \in H^3$. Then, there exist $\tau > 0$ and $I, R \subset \mathbb{T}$, where I a union of at most $[4\pi/\tau]$ open intervals, such that $\mathbb{T} = I \cup R$ and the following estimates hold for $T/\sqrt{2} < t < T$ ($T = T(\alpha, \gamma, u_0)$ is explicit):

$$|\partial_x u(x,t)| \leq \|u_0\|_{L^\infty(\mathbb{T})} \qquad \text{for all } x \in I,$$

$$\text{card}\{x \in R : \partial_x u(x,t) = 0\} \leq \frac{4\pi}{\log 2} \frac{\log\left(\sqrt{2}/\tau\right)}{\tau}.$$

An explicit choice in the range $0 < \epsilon \ll 1$ verifies

$$\tau = \frac{1}{\sqrt{2}} \left(\frac{\log\left(\mathcal{E}/c + 1\right)}{3\mathcal{E}} \right),$$

with

$$\mathcal{E} = O\left(\left(\frac{\max\{1,\gamma\}\left(S+1\right)}{\epsilon\alpha} \right)^{\frac{\max\{1,\gamma\}}{\alpha - \max\{1,\gamma\}}} \right),$$

where S is the size of the absorbing set in H^3.

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Part 4: How to prove the number of peaks/oscillations

1. Prove the local/global existence of classical solution
2. Prove that the solution becomes complex analytic in a complex strip containing the real axis
3. Obtain careful estimates on the width of the complex strip
4. Use the relationship between the width of analyticity and the number of peaks/oscillation

The delicate part is the gain of regularity. To do that, one need to define the appropriate Hardy-Sobolev spaces and the *complexified* equation.

This method may be used in (basically) any drift-diffusion equation/system of equations.

J. Burczak and R. Granero-Belinchón.
On a generalized, doubly parabolic Keller-Segel system in one spatial dimension.
Submitted. Arxiv Preprint arXiv:1407.2793 [math.AP].

R. Granero-Belinchón and J. Hunter.
On a nonlocal analog of the kuramoto-sivashinsky equation.
Nonlinearity, 28(4):1103–1133, 2015.