New models for Euler flows with a free boundary: the Rayleigh-Taylor and Kelvin-Helmholtz instabilities

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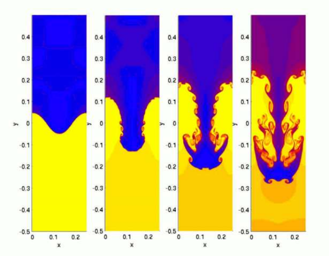
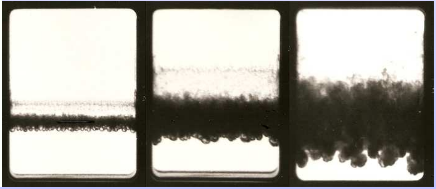
BML, June 2017

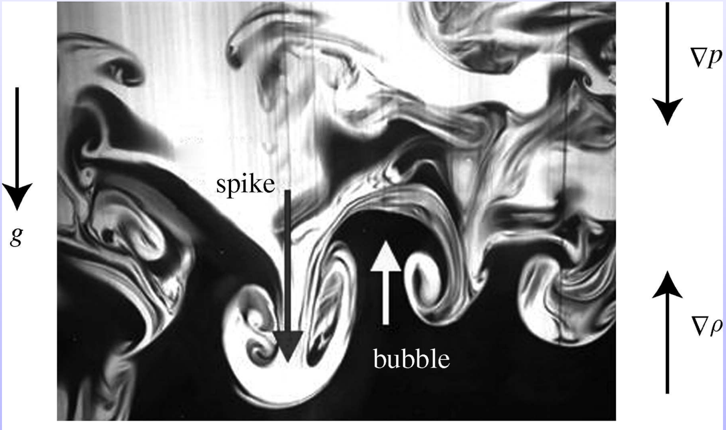
Rayleigh-Taylor instability

Rayleigh-Taylor instability: instability occuring when a layer of heavy fluid is supported by a layer of a light one (Rayleigh [1878] and Taylor [1950]).





RT occurs under gravity and, equivalently, under an acceleration of the fluid system in the direction toward the denser fluid. Whenever the pressure is higher in the lighter fluid, the differential acceleration causes the two fluids to mix.

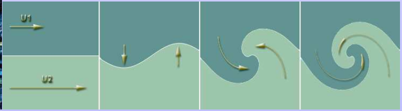


$$\nabla p \cdot \nabla \rho < 0$$

Kelvin-Helmholtz instability

Kelvin-Helmholtz instability: instability occuring when there is a velocity difference across the interface between two fluids (Lord Kelvin [1871] and von Helmholtz [1868])).





Goal of the talk

In this talk, I will describe two different strategies for deriving asymptotic models for RT interface growth and mixing.

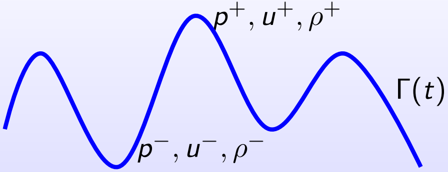
Euler equations [1757]

$$\underbrace{\rho}_{\text{mass}} \left(\underbrace{\partial_t u + (u \cdot \nabla) u}_{\text{acceleration}} \right) = \underbrace{-\nabla p - g\rho e_d^t}_{\text{force}} \text{ Newton's Law}$$

$$\nabla \cdot u = 0 \text{ incompressibility}$$

$$\nabla \times u = 0 \text{ irrotationality}$$

$$\partial_t \rho + u \cdot \nabla \rho = 0 \text{ conservation of mass}$$



The blue curve is an illustration of the interface $\Gamma(t)$, separating both fluids. The fluid on top of $\Gamma(t)$ has density ρ^+, pressure p^+ and velocity u^+, while the fluid on the bottom has density ρ^-, pressure p^- and velocity u^-.

The jump conditions are

$$[\![p]\!] = 0 \text{ (absence of surface tension)},$$

$$[\![u \cdot n]\!] = 0.$$

Define the Atwood number

$$A = \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-}.$$

Then, the linear equation when $\Gamma(t) = (x, h(x,t))$, $|h| \ll 1$, is

$$h_{tt} = Ag(-\partial_x^2)^{1/2} h.$$

- $A < 0$ (heavy fluid below)$\Rightarrow$ stable case it's a linear wave equation!
- $A > 0$ (heavy fluid on top)$\Rightarrow$ unstable case

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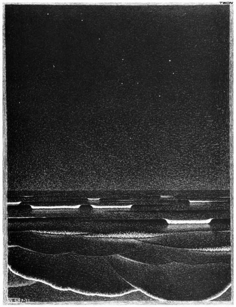
- ill-posed in Sobolev spaces,
- highly unstable,
- computationally intractable.

So, two-phase Euler is much more challenging than one-phase Euler and there is a great need for models that can answer the basic questions about interfaces, mixing, etc

Or, as M.C. Escher (in 1959 in a letter to his son George) put it

> *Those waves! Very soon I will try once more to draw something similar to the waves. But how can one suggest motion on a static plane? And how can you simplify something as complicated as a wave in the open sea, making it understandable?*



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- Initially an exponential growth of infinitesimal perturbations that correspond to linear stability analysis.

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- Initially an exponential growth of infinitesimal perturbations that correspond to linear stability analysis.
- After a short amount of time of exponential-in-time growth, the interface can be described by *bubbles* of the lighter fluid and *spikes* of the heavier fluid. In this second regime, exponential growth of the bubbles slows down and nonlinear terms in the equations of motion can no longer be ignored. This is called *saturation/bubble competition*.

- Eventually, due to the growth of the bubbles, they merge and mix chaotically. This develops into a region of turbulent self-similar mixing whose half-width can be described by the following formula

$$\text{half-width of the turbulent mixing region} \approx \alpha \left(\frac{\rho^+ - \rho^-}{\rho^+ + \rho^-} \right) gt^2.$$

α is a dimensionless but non-universal parameter.
How many physical parameters does it take to describe α? At least 6, 3 describing fluid parameters and 3 describing the initial conditions (Glimm *et. al.*).

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How can we estimate α?
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Difficulties:

- Two-phase Euler is a highly unstable system $\Rightarrow$ Direct Numerical Simulations (DNS) extremely expensive.
- Two-phase Euler is ill-posed $\Rightarrow$ the analysis of the system is extremely difficult.

Current approaches:
Real experiments (Read & Youngs, Read, Smeeton & Youngs)

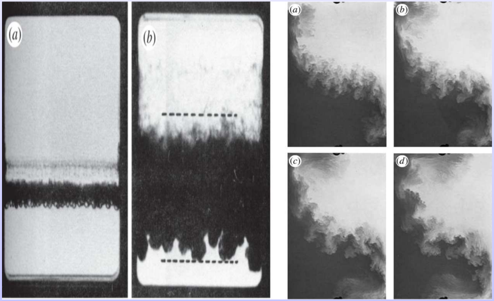

Figure: a) Rocket rig and b) Tilted rig experiments where the fluids are NaI solution ($\rho^{-} = 1.89g/cm^3$) and Hexane ($\rho^{+} = 0.66g/cm^3$)

Current approaches:

ODE models (large systems of nonlinear ODEs for expansions (Taylor, Fourier) of the interface and the velocity.):

B. Rollin and M. J. Andrews. On generating initial conditions for turbulence models: the case of Rayleigh-Taylor instability turbulent mixing. Journal of Turbulence,14(3):77–106, 2013

$$\frac{d^2\hat{h}_k}{dt^2} = Ag|k|\hat{h}_k + A|k|\sum_p\left(1-\frac{p\cdot k}{|p||k|}\right)\frac{d^2\hat{h}_p}{dt^2}\hat{h}_{k-p}$$

$$+A|k|\sum_p\left(\frac{1}{2}-\frac{p\cdot k}{|p||k|}-\frac{1}{2}\frac{p\cdot(k-p)}{|p||k-p|}\right)\frac{d\hat{h}_p}{dt}\frac{d\hat{h}_{k-p}}{dt},$$

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ODE models (large systems of nonlinear ODEs for expansions (Taylor,Fourier) of the interface and the velocity.):

V.N. Goncharov. Analytical model of nonlinear, single-mode, classical Rayleigh-Taylor instability at arbitrary Atwood numbers. Physical Review Letters, 88(13), 2002

The interface and potentials are assumed to be

$$h(t,x)=h_0(t)+h_2(t)x^2,\ \phi^+(t,x,y)=a_1(t)\cos(kx)e^{-k(y-h_0)},$$
$$\phi^-(t,x,y)=b_1(t)\cos(kx)e^{k(y-h_0)}+b_2(t)y.$$

Then,

$$\frac{dh_2}{dt}=-\frac{dh_0}{dt}\frac{k}{2}(k+6h_2)$$
$$\frac{d^2h_0}{dt^2}f_1(k,h_2)=-Agh_2-\left(\frac{dh_0}{dt}\right)^2 f_2(k,h_2),$$

Our main contributions are a couple of new mathematical models for two-fluid interface motion, subjected to the Rayleigh-Taylor (RT) instability in two-dimensional fluid flow.
The basis of our approach is very different to the prior ODE models and leads to two different nonlinear and nonlocal PDE's modeling RT instability.

The ideas are the following:

1. We write the 2D Euler system for (u, p, Γ) in the Birkhoff-Rott integral-kernel formulation (BR). New unknowns: the interface Γ and the tangential discontinuity $\varpi = (u^{-} - u^{+}) \cdot \tau$ ($\tau(t)$ denotes the tangent vector to $\Gamma(t)$).

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3. We either
   - expand the integral kernel in the BR formulation assuming that the slope of the interface is small (*h-model*)
   - restrict the nonlocality present in the the original BR kernel (*z-model*)

When the interface $\Gamma(t)$ is given by a graph $(x, h(x,t))$, we use an asymptotic expansion of the BR formulation and assume that $|\partial_x h| \ll 1$.

Example:

$$\frac{p.v.}{\pi}\int \varpi(y,t)\frac{(x-y)}{(x-y)^2+(h(x,t)-h(y,t))^2}dy$$

$$\approx \frac{p.v.}{\pi}\int \frac{\varpi(y,t)}{x-y}\left(1-\left(\frac{h(x,t)-h(y,t)}{x-y}\right)^2\right)dy$$

$$\approx H\varpi$$

where H denotes the Hilbert transform $\widehat{Hg}(k) = -i\frac{k}{|k|}\hat{g}(k)$.

This leads to the *h-model*.

$$h_t = \frac{1}{2} H\varpi$$

$$\varpi_t = 2Ag\partial_x h - \frac{A}{2}\Lambda\left(\varpi H\varpi\right),$$

where $\Lambda = H\partial_x = (-\partial_x^2)^{1/2}$. Or, equivalently,

$$h_{tt} = Ag\,\Lambda h - A\partial_x(Hh_t h_t)$$

When the interface $\Gamma(t)$ is given as a smooth curve $(z_1(x,t), z_2(x,t))$, we replace the BR kernel by a *localized* kernel.

$$\frac{p.v.}{\pi}\int \varpi(y)\frac{(z(x,t)-z(y,t))^{\perp}}{|z(x,t)-z(y,t)|^2}dy \approx \frac{(\partial_x z(x,t))^{\perp}}{|\partial_x z(x,t)|^2}\frac{p.v.}{\pi}\int \frac{\varpi(y)}{x-y}dy$$

Then, the $z-$model reads

$$z_t = \frac{1}{2} H\varpi \frac{(\partial_x z)^{\perp}}{|\partial_x z|^2},$$
$$\varpi_t = -\partial_x \left[\frac{A}{2} \frac{H(\varpi H\varpi)}{|\partial_x z|^2} - 2Agz_2 \right].$$

Equivalently,

$$z_{tt} = \Lambda \left[\frac{A}{|\partial_x z|^2} H\left(z_t \cdot (\partial_x z)^{\perp} H(z_t \cdot (\partial_x z)^{\perp}) \right) \right] \frac{(\partial_x z)^{\perp}}{|\partial_x z|^2}$$
$$+ z_t \cdot (\partial_x z)^{\perp} \left(\frac{(\partial_x z_t)^{\perp}}{|\partial_x z|^2} - \frac{(\partial_x z)^{\perp} 2(\partial_x z \cdot \partial_x z_t)}{|\partial_x z|^4} \right)$$
$$+ Ag\Lambda z_2 \frac{(\partial_x z)^{\perp}}{|\partial_x z|^2}$$

To estimate the error, define

$$m = \min |\partial_x z|, \; M = \max |\partial_x z|.$$

Then

$$|u(z(x),t) - u_{mod}(z(x),t)| \leq 2\frac{\|\varpi\|_{L^2}}{\pi}\left(\frac{1}{m^2}+2\right)^{0.5}\frac{\|\partial_x^2 z\|_{L^\infty}^{0.5}}{m^{0.5}},$$

where

$$u_{mod} = \frac{1}{2}H\varpi(x)\frac{(\partial_x z(x))^\perp}{|\partial_x z(x)|^2},$$

and u is the real Euler velocity.

Comparison with classical RT experiments:

- Growth of the mixing region: the Rocket rig and Tilted rig experiments

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- Growth of the mixing region: the Rocket rig and Tilted rig experiments
- Closed contours: Falling drops & Rising bubbles (z-model)

Rocket and Tilted rig experiments:

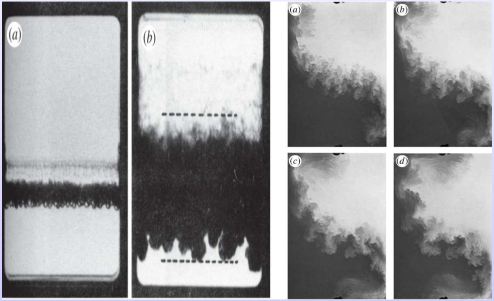
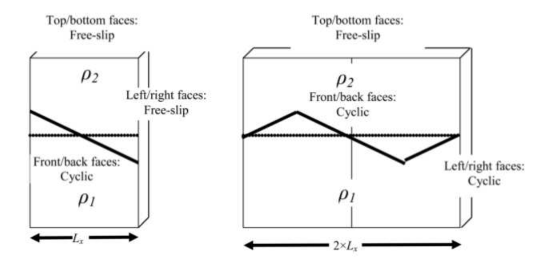


Figure: a) Rocket rig and b) Tilted rig experiments

Rocket rig experiment:

- Initial data: $\Gamma(0) = (x, h_0(x))$, h_0 random small initial data
- Time: the experiment runs for up to 70-73 ms
- Gravity points upwards
- Fluids: Heavy fluid (below) NaI solution ($\rho^- = 1.89g/cm^3$) and Light fluid (on top) Hexane ($\rho^+ = 0.66g/cm^3$)
- Size of the tank: 15 cm x 2.5 cm x 24 cm (so we have to rescale gravity)



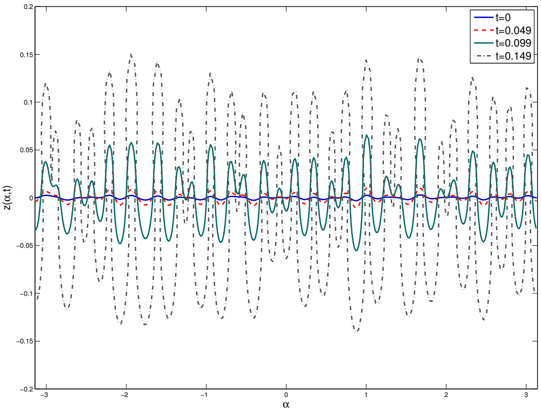
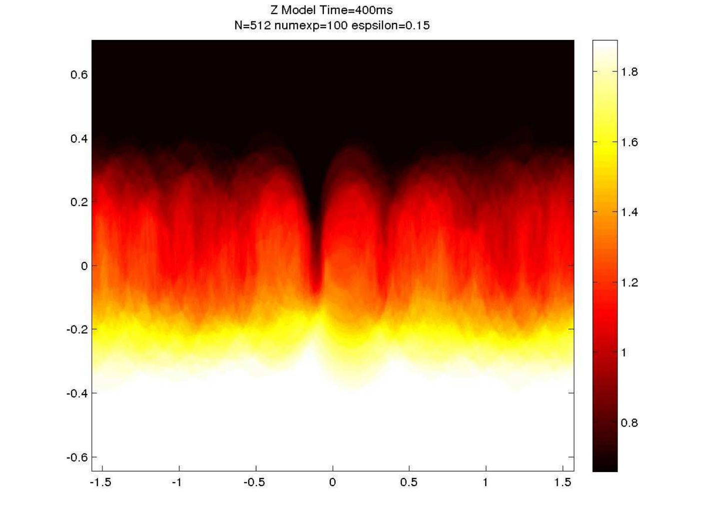


Figure: Interface position $z(\alpha, t_j)$ for $t_0 = 0,\ t_1 = 0.049,\ t_2 = 0.099$ and $t_3 = 0.149$.


Z Model Time=400ms
N=512 numexp=100 espsilon=0.15
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5
1.8
1.6
1.4
1.2
1
0.8

For the rocket rig:

- DNS suggests $\alpha \in [0.04, 0.05]$
- the empirical value is $\alpha \approx 0.063$

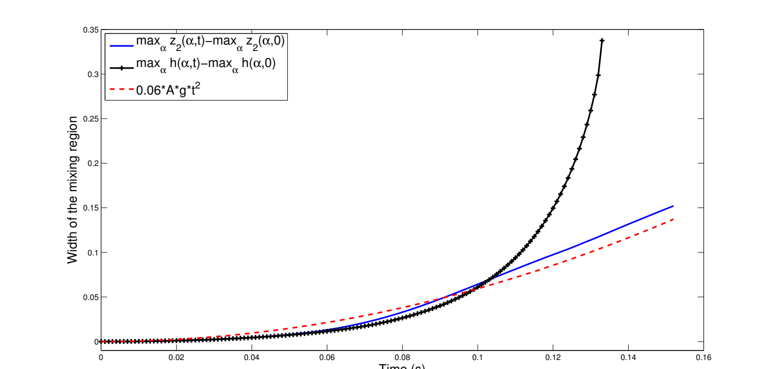
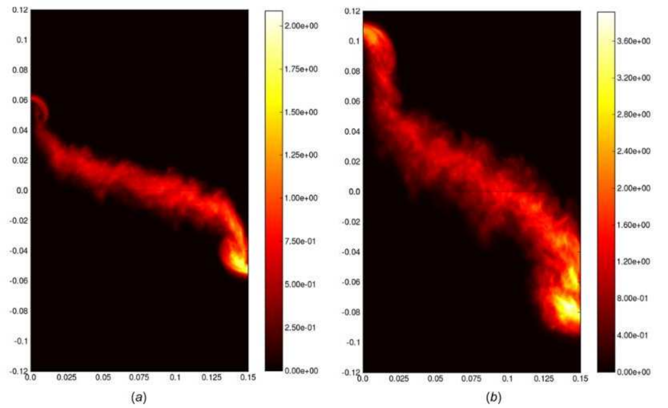


Figure: Comparison between $\max_x z_2(x,t) - \max_x z_2(x,0)$, $\max_x h(x,t) - \max_x h(x,0)$ and the theoretical prediction with $\alpha = 0.06$.

Tilted rig experiment:



(a) (b)

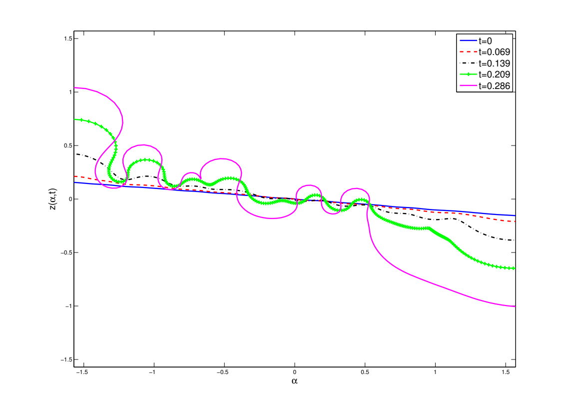


Figure: Interface position for $t_0 = 0$, $t_1 = 0.069$, $t_2 = 0.139$, $t_3 = 0.209$ and $t_4 = 0.286$.

Falling drops:

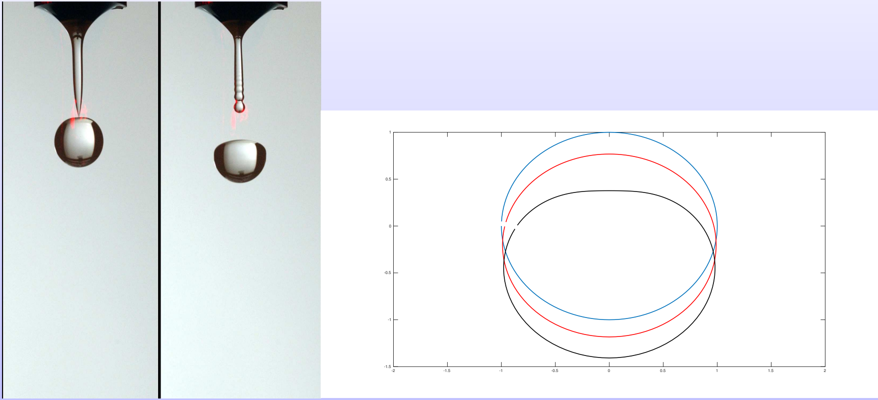


Figure: a) Real water drops, b) Simulation using the $z-$model

Rising plumes:

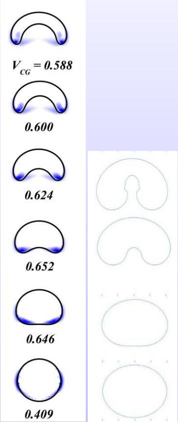


Figure: a) Plume as in Tripathi Sahu, Govindarajan (Nature,2014), b) Plume simulation using the $z-$model

Recovering the nonlinear saturation phenomenon:
The RT mixing zone has three different stages:

1. Initially, the growth of the mixing region is exponential in time (linear theory)
2. Then, there is a period where nonlinear saturation occurs (nonlinear effects). The growth slows down to merely linear in time.
3. After that the turbulent mixing stage is attained and the growth is quadratic in time.

Goncharov's ODE model correctly captured the linear growth in the tip of the bubble (i.e. assuming that gravity points downward and denoting by v the normal velocity of the bubble, where $Av > 0$).

Global well-posedness and asymptotic behavior for the h-model

Let $\rho^+, \rho^- > 0$, $g \neq 0$ be fixed constants such that $Ag < 0$, and let (h_0, h_1) denote the initial position and velocity, respectively, for the h-model. Setting $h_2 := h_{tt}(\cdot, 0) = Ag\Lambda h_0 - A\partial_x(Hh_1 h_1)$. Suppose that $(h_0, h_1) \in H^{2.5}(\mathbb{T}) \times H^2(\mathbb{T})$ is given such that

$$\lambda := \min_{x \in \mathbb{T}} A\, h_1(x) > 0, \text{ and } \|h_2\|_{0.5}^2 + \|h_1\|_1^2 < \left(\frac{-\bar{h}_1}{5}\right)^2 . \quad (1)$$

Then there exists a unique classical solution of the $h-$model satisfying

$$h \in C([0,T]; H^{2.5}),\ h_t \in C([0,T]; H^2) \cap L^2(0,T; H^{2.5})\ \forall\, 0 \le T < \infty.$$

Furthermore, as $t \to \infty$, the solution $h(\cdot, t)$ converges to the homogeneous solution $h^\infty = \bar{h}_0 + \bar{h}_1 t$;

$$\limsup_{t \to \infty} \|h(t) - h^\infty\|_{\dot{H}^1} + \|h_t(t) - h_t^\infty\|_{\dot{H}^1} = 0.$$

Let us emphasize that the condition

$$\|h_2\|_{0.5}^2 + \|h_1\|_1^2 < \left(\frac{-\bar{h}_1}{5}\right)^2$$

does not require small initial data; rather, we require the data to be sufficiently close to an arbitrarily large homogeneous state. For example, we can consider

$$h_0 = A + Be^{\alpha i} \quad \text{and} \quad h_1 = -1000 + \frac{e^{\alpha i}}{6}$$

for constants A and B. A simple computation using the explicit form of h_0 and h_1 shows that the condition is satisfied when $B \leq 110$ and any A.

Kelvin-Helmholtz instability:

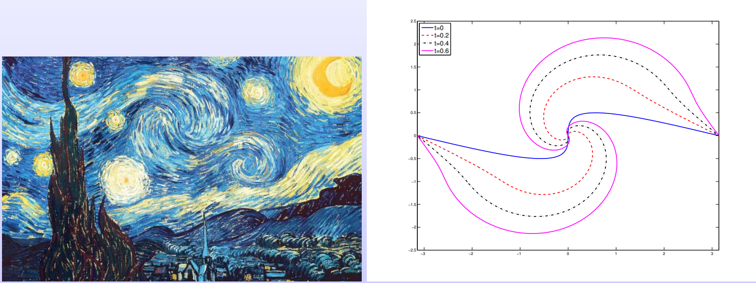


Figure: a)Kelvin-Helmholtz instability in clouds as drawn by Van Gogh. b) Simulation of the Kelvin-Helmholtz instability using the $z-$model

Closely related models:

1. W. Craig and C. Sulem. Numerical simulation of gravity waves. J. Comput. Phys., 108(1):73–83, 1993.
2. S. Liu and D.M. Ambrose. Sufficiently strong dispersion removes ill-posedness in truncated series models of water waves. Submitted, 2017

$$h_t = \frac{1}{2}H\varpi - \frac{1}{2}\partial_x\left(H(hH\varpi) + h\varpi\right)$$
$$\varpi_t = -2g\Lambda^p\partial_x h + \frac{1}{2}\Lambda\left(\varpi H\varpi\right),$$

- This model is obtained for gravity waves ($p = 0$), capillary waves ($p = 2$) or hydroelastic waves ($p = 4$) when the fluid above is replaced by vacuum.