Symbolic-Numeric Approaches for Intersection Problems in Computer Aided Geometric Design

CAGD@UC: The Computer Aided Geometric Design (CAGD) Group at the Universidad de Cantabria

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Abstract

Computing the intersection curve between two surfaces is a key problem in many areas such as the CAD/CAM treatment of complicated shapes, computer animation, NC machining, solid modelling, ... The main goal concerning the surface-to-surface intersection problem is to develop robust, accurate and fast algorithms solving such a problem, needing the least user intervention.

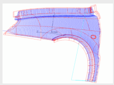
We present several solutions to different instances of the surface-to-surface intersection problem which are characterized either by involving real algebraic curves or surfaces presented implicitly, or by reducing the considered problem to determine the topology of a real algebraic plane curve implicitely defined.

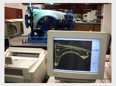
The concrete problems, whose solution is to be presented, are: computing the intersection curve between two surfaces (when presented parametrically or implicitly, when intersecting tangentially, when one of them is a ruled, revolution or canal surface, when they are quadratic or Bezier triangular patches or general quadrics, when sectioning offsets, when they are ellipsoids parametrically dependent, ...), computing the intersection points between two plane curves (when presented parametrically or implicitly, when they are conics parametrically dependent, ...), etc.

These algorithms have been implemented in Maple. The accomplished implementation involves the simultaneous use of symbolic and numerical techniques. Thus, when needed, implicitization is done symbolically, topology determination involves both symbolic and numerical methods, and final computations, concerning the searched output, are performed numerically.

Objectives

The long and winding road: from equations to real applications through algorithm design and software development.





The Surface-to-Surface Intersection Problem in CAGD:

Surfaces in CAGD are usually presented parametrically but the availability of their implicit equation can be very helpful in order to solve in a much more efficient and accurate way the surface-to-surface intersection problem.

General (and highly simplified) scheme for determining the intersection curve between two surfaces **J** and **L** parametrically presented:

$$\mathbf{J}(u,v) := \begin{cases} x(u,v) = \dfrac{f(u,v)}{q(u,v)} \\ y(u,v) = \dfrac{g(u,v)}{q(u,v)} \\ z(u,v) = \dfrac{h(u,v)}{q(u,v)} \end{cases} \qquad \mathbf{L}(s,t) := \begin{cases} x(s,t) = \dfrac{a(s,t)}{p(s,t)} \\ y(s,t) = \dfrac{b(s,t)}{p(s,t)} \\ z(s,t) = \dfrac{c(s,t)}{p(s,t)} \end{cases}$$

- To compute the implicit equation of **J**: **H**(x,y,z)=0.
- To substitute the parametrization of **L** into **H**(x,y,z):
  H(x(s,t),y(s,t),z(s,t))=**R**(s,t)=0.
- To determine the curve **R**(s,t)=0.
- To lift the curve **R**(s,t)=0 to the surface **L** by using its parametrization.

Questions considered and analyzed:

- Fast computation of implicit equations with a good numerical behavior.
- To solve accurately the equation R(s,t)=0 in high degrees.
- To extend this approach for dealing with offsets (in general, without a parametric representation).
- To use symbolic techniques to derive closed formulae of interest in CAGD: generic implicitation, conics and quadrics interference, etc.
- To provide a problem solving environment where to test symbolic and numerical techniques in CAGD.

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Methods

Generic Implicitization, Subresultant Implicitization, Approximate Implicitization, ...

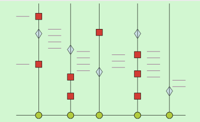
By using symbolic techniques (resultants, Grobner Basis, ...) the implicit equation of low degree parametric surfaces is precomputed and kept into a database.
This approach avoids the problems preventing the use of implicit equations in CAGD: floating point real numbers as coefficients in the parameterization and efficiency.

$$\begin{aligned} x(u,v) &= a_1v^2 + a_2v \\ y(u,v) &= b_1u^2 + b_2 \\ z(u,v) &= c_1uv + c_2u + c_3v \end{aligned}$$

Generic Implicit Equation

$$\begin{aligned} &c_1^4x^2y^2 + (2c_1^4b_2 - 2c_1^2b_1c_3^2)x^2y + (b_1^2c_3^4 - 2b_1c_3^2b_2c_1^2 + b_2^2c_1^4)x^2 + (2c_1^3a_2c_2 - 2c_1^2a_1c_2^2)xy^2 \\ &-2b_1a_1c_1^2xyz^2 + (2c_1^2b_1a_2c_3 - 8a_1c_2b_1c_1c_3)xyz + (-4c_1^2b_2a_1c_2^2 + 4c_1^3b_2a_2c_2 - 2b_1c_3^2a_2c_1c_2 \\ &-2b_1c_3^2a_1c_2^2)xy + (-2b_1^2a_1c_3^2 - 2b_1a_1b_2c_1^2)xz^2 + (-2b_1^2c_3^2a_2 + 2b_1a_2c_3b_2c_1^2 - 8b_2a_1c_2b_1c_1c_3)xz \\ &+(-2b_1c_3^2b_2a_1c_2^2 - 2b_1c_3^2b_2a_2c_1c_2 - 2b_2^2a_1c_1^2c_2^2 + 2b_2^2c_1^3a_2c_2)x + (a_1^2c_2^4 + a_2^2c_1^2c_2^2 - 2a_2c_1c_2^3a_1)y^2 \\ &+(2b_1a_1a_2c_1c_2 - 2b_1a_1^2c_2^2 - a_2^2c_1^2b_1)yz^2 - 2b_1a_2c_3a_1c_2^2yz + (2b_2a_1^2c_2^4 - 4a_2c_1c_2^3b_2a_1 \\ &+2a_2^2c_1^2c_2^2b_2 - a_2^2c_3^2c_2^2b_1)y + b_1^2a_1^2z^4 + 2b_1^2a_1a_2c_3z^3 + (b_1^2a_2^2c_3^2 - 2b_1a_1^2b_2c_2^2 + 2b_1a_1b_2a_2c_1c_2 \\ &-b_2a_2^2c_1^2b_1)z^2 - 2b_1a_2c_3b_2a_1c_2^2z + b_2^2a_1^2c_2^4 + b_2^2a_2^2c_1^2c_2^2 - 2b_2^2a_1c_2^3a_2c_1 - b_2a_2^2c_3^2c_2^2b_1 = 0 \end{aligned}$$

A seminumerical algorithm computing the topology of a real algebraic plane curve f(x,y)=0



$$f(x,y) = 0$$

General Position

$$D(x) = \mathrm{resultant}_y\left(f, \frac{\partial f}{\partial y}\right)$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m$$

$$H_j(x,y) = \mathbf{StHa}_j(f) = h_j(x)y^j + h_{j,j-1}(x)y^{j-1} + \ldots + h_{j,0}(x)$$

$$r_k(x) = -\frac{1}{k}\frac{h_{k,k-1}(x)}{h_k(x)}$$

$$\beta_j^{(k)} = r_k(\alpha_j^{(k)})$$

If for every $k \in \{1,\ldots,t\}$ and $j \in \{1,\ldots,s_k\}$ the following condition is verified:

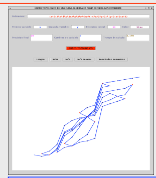
$$H_k(\alpha_j^{(k)}, y) = h_k(\alpha_j^{(k)})(y - \beta_j^{(k)})^k$$

then the curve $\mathcal{C}(f)$ is in generic position.

$$F_k(\alpha_j^{(k)}, \beta_j^{(k)}, y) = \frac{f(\alpha_j^{(k)}, y)}{(y - \beta_j^{(k)})^k}$$

$$f\left(\frac{\alpha_k + \alpha_{k+1}}{2}, y\right) = 0$$

Algorithm TOP



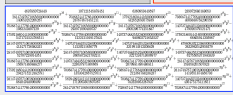
This is the kernel of any surface-to-surface intersection algorithm. The final shape is obtained through the numerical integration of:

$$f_x(x,y)\cdot x' + f_y(x,y)\cdot y' = 0$$

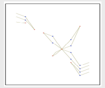
over the certified topology produced by the algorithm TOP.

The most algebraic version of TOP is based on subresultants and squarefree decompositions and the most numerical version of TOP replaces the symbolic tools by a generalized eigenvalue problem plus several singular value decomposition computations.

Maple Implementation: Maplets plus adaptative precision.



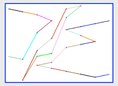
Example extracted from:
S. Becuwe, A. Cuyt: *On the Froissart phenomenon in multivariate homogeneous Pade approximation*. Adv. Comput. Math. 11, 21-40, 1999.



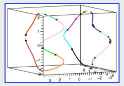
Intersection problems involving ruled, revolution and ringed surfaces

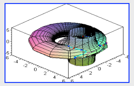
Revolution, ruled and ringed surfaces are very easy to implicitize and the general scheme can be easily applied here. Special attention must be paid to the detection of the so called geometric extraneous components.

$$S_1 := \left[-\frac{2\,(3+s^2)\,(-1+u^2)}{(1+s^2)\,(1+u^2)}, \frac{4\,(3+s^2)\,u}{(1+s^2)\,(1+u^2)}, \frac{4\,s}{1+s^2}\right] \qquad S_2 := \left[\frac{(2\,t^2+4)\,(1-v^2)}{(t^2+1)\,(1+v^2)}, 3+\frac{2\,(2\,t^2+4)\,v}{(t^2+1)\,(1+v^2)}, t\right]$$

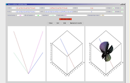


Topology in the v-t domain





Intersection curve together with the two surfaces

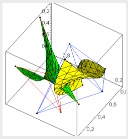


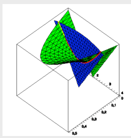
Maplet Interface

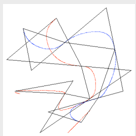
Computing intersections between patches

The using of algebraic and numerical tecniques has produced a new family of algorithms for computing, topologically certified, the intersection curve between:

- Bezier triangular patches
- Quadratic triangulatr patches
- Quadrics
- Cubic A-splines







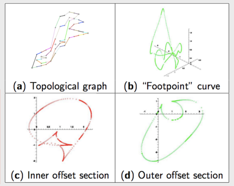
Computing offset sections

A fast and efficient method to compute the intersection curve between a plane Π and the offset $\mathcal{O}_d$ to distance d of a parametric surface $\mathcal{S}$. This is achieved by eliminating the parameters defining the plane Π into the system of algebraic equations characterizing the points in $\mathcal{O}_d$. This linear elimination produces a planar curve $\mathcal{C}_{st}$, implicitly defined in the parametric domain of $\mathcal{S}$, whose lifting to $\mathbb{R}^3$ provides the searched intersection curve between $\mathcal{S}$ and Π.

Computing the intersection curve between the plane *z=1* and offset to distance *1* of the surface:

$$\begin{aligned} x(s,t) &= st + 1 \\ y(s,t) &= st + s \\ z(s,t) &= s^2 + t^2 + s \end{aligned}$$

This method handles perfectly the usual numerical and geometrical difficulties arising when intersecting offsets, such as self-intersections or singular points, and avoids the computation of any approximation of the considered offset.



(a) Topological graph (b) "Footpoint" curve (c) Inner offset section (d) Outer offset section

Closed Formulae for the interference analysis of conics and quadrics

The obtained closed-formulae for conics and ellipsoids have been obtained by using Quantifier Elimination, Invariant Theory and Symbolic Computation. They provide the most efficient way for solving, both, the ellipses and ellipsoids moving problem.

Let $\mathcal{A} : X^TAX = 0$ and $\mathcal{B} : X^TBX = 0$ be two ellipses and

$$f(\lambda) = \det(\lambda A + B) = \lambda^3 + a\lambda^2 + b\lambda + c$$

their characteristic polynomial (once turned monic). $\mathcal{A}$ and $\mathcal{B}$ are separated iff

$$\begin{cases} a \geq 0 \\ a^2 - 3b > 0 \\ 3ac + ba^2 - 4b^2 < 0 \\ a^2b^2 - 27c^2 + 18cab - 4a^3c - 4b^3 > 0 \end{cases} \quad \text{or} \quad \begin{cases} a < 0 \\ a^2 - 3b > 0 \\ a^2b^2 - 27c^2 + 18cab - 4a^3c - 4b^3 > 0 \end{cases}$$

Let $\mathcal{A} : X^TAX = 0$ and $\mathcal{B} : X^TBX = 0$ be two ellipsoids and

$$f(\lambda) = \det(\lambda A + B) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$$

their characteristic polynomial (once turned monic). $\mathcal{A}$ and $\mathcal{B}$ are separated iff

$$\begin{array}{lll} p_1 > 0, p_2 > 0, p_3 > 0, q_2 < 0 & \text{or} & p_1 > 0, p_2 > 0, p_3 = 0, q_1 < 0, q_2 = 0 \quad \text{or} \\ p_1 > 0, p_2 > 0, p_3 = 0, q_2 < 0, q_3 > 0 & \text{or} & p_1 > 0, p_2 > 0, p_3 > 0, q_1 \neq 0, q_2 > 0, q_3 \neq 0 \quad \text{or} \\ p_1 > 0, p_2 > 0, p_3 > 0, q_1 \neq 0, q_2 = 0, q_3 \geq 0 & \text{or} & p_1 > 0, p_2 > 0, p_3 = 0, q_1 < 0, q_2 > 0, q_3 > 0 \end{array}$$

where

$$\begin{aligned} p_1 &= -8b + 3a^2, \\ p_2 &= -4b^3 + a^2b^2 + 16bd + 14cab - 6a^2d - 3ca^3 - 18c^2, \\ p_3 &= -27d^2a^4 - 4a^3c^3 + 18a^3dcb + a^2c^2b^2 + 144a^2bd^2 - 6a^2c^2d - 4a^2b^3d - 192ad^2c - 80ab^2cd \\ &\quad +18ac^3b - 27c^4 + 144c^2bd + 256d^3 - 128d^2b^2 + 16db^4 - 4b^3c^2, \\ q_1 &= a, \\ q_2 &= ba^2 + 3ac - 4b^2, \\ q_3 &= 4a^3c^2 - 3da^3b + 7dca^2 - b^2ca^2 + 12dab^2 - 18ac^2b + 16ad^2 + 27c^3 - 48dcb + 4b^3c. \end{aligned}$$

On Going Work

- Algorithms computing the implicit equation of (x=x(t),y=y(t)) by using the information in the subresultant sequence of x(t)-x and y(t)-y with respect to t.
- Faster algorithms for determining the topology of an arrangement of cubic and quartic implicit plane curves by using the geometry of their singularities.
- A new familiy of intersection algorithms for procedural curves and surfaces: they are easy to evaluate but their explicit algebraic description is not available or it is very costly to generate (for example, NURBS).
- Seminumerical computation of bisectors, silhouettes, medial axis, ... of curves and surfaces.
- Computation of the implicit ideal of a hypercircle.
- Topological analysis of offsets in terms of the distance.

Projects

Projects with the company CANDEMAT:

- *Nuevas prestaciones para el entorno CAD/CAM/CAE industrial para matriceria CSIS: medida de bordes, interrogación, suavizado y offseting.*
- *Integración de nuevas técnicas algebro-numéricas en Diseño Geométrico Asistido por Ordenador (CAGD): desarrollo de prototipos e implementación en un entorno CAD/CAM industrial.*

Projects funded by the European Union:

- *Real Algebraic and Analytic Geometry - RAAG* (HPRN-CT-2001-00271).
- *Intersection algorithms for geometry based IT-applications using approximate algebraic methods-GAIA II* (IST-2002-35512).

Projects funded by the Spanish Ministry of Education and Research:

- *Curvas y Superficies: Fundamentos, Algoritmos y Aplicaciones* (BFM2002-04402-C02-02).
- *Geometria Algebraica Real y Algoritmos para Curvas y Superficies* (MTM2005-08690-C02-02).

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