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# Jacobian and polar curves of singular foliations

Nuria Corral

**Abstract** In this survey we describe results concerning topological properties of polar and jacobian curves of foliations. We also describe how invariants defined from polar or jacobian curves can be used to characterize generalized curve foliations or second type ones.

**Key words:** singular foliation, polar curve, jacobian curve, Camacho-Sad index, equisingularity, ramification, Newton polygon, analytic invariant

## Contents



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## 1.1 Introduction

Polar curves of a plane curve have been widely studied in order to better understand singularities of plane curves. Given a curve $C$ defined by $f(x,y)=0$ with $f\in\mathbb{C}\{x,y\}$ and a direction $[a:b]\in\mathbb{P}^1_{\mathbb{C}}$, the polar curve $\mathcal{P}_{[a:b]}$ is the curve defined by the equation
$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=0.$$
Thus polar curves are the elements of the pencil of curves defined by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (see [18, section 2.7] for more details about pencils of curves). All the curves $\mathcal{P}_{[a:b]}$ are equisingular except for a finite number of directions (see Proposition 8.5.1 and Remark 6.2.2 of [18]). Any of these equisingular polar curves $\mathcal{P}=\mathcal{P}^C$ is called a *generic polar curve* of $C$. It is well known that the topological type of a generic polar curve is not determined by the topological type of the curve $C$ as showed by Pham ([68]): consider $f_\lambda(x,y)=y^3-x^{11}+\lambda x^8 y$. All the curves $C_\lambda$ defined by $f_\lambda=0$ are equisingular, but the generic polar curve of $C_0$ is not equisingular to the generic polar curve of $C_\lambda$ with $\lambda\neq 0$. In particular, this example shows that the topological type of a generic polar curve of a plane curve depends on the analytic type of the curve.

However, there are some properties of the topological type of a generic polar curve that can be described from the equisingularity data of the curve $C$. These type of results are known as *decomposition theorems* of a generic polar curve (see for instance [65, 55, 40]). When $C$ is an irreducible curve with Puiseux pairs $\{(m_i,n_i)\}_{i=1}^g$, the decomposition theorem proved by M. Merle in [65] says that a generic polar curve $\mathcal{P}$ of $C$ has a decomposition
$$\mathcal{P}=\cup_{i=1}^g\mathcal{P}^i$$
such that the multiplicity at the origin of each curve $\mathcal{P}^i$ is given by $m_0(\mathcal{P}^i)=n_1\cdots n_{i-1}(n_i-1)$ and, for any irreducible component $\gamma$ of $\mathcal{P}^i$, the coincidence $C(\gamma,C)$ of $\gamma$ with $C$ is equal to $C(\gamma,C)=\beta_i/\beta_0$ where $\{\beta_0,\beta_1,\ldots,\beta_g\}$ are the characteristic exponents of $C$. In particular, the result of Merle implies that the topological type of the curve $C$ can be recovered from the set of polar quotients $Q(C)=\left\{\frac{(\gamma,C)_0}{m_0(\gamma)}\ :\ \gamma \text{ irreducible component of } \mathcal{P}\right\}$ together with the multiplicity $m_0(\mathcal{P})$ since we have that $m_0(\mathcal{P})=m_0(C)-1=\beta_0-1$ and $Q(C)=\left\{\bar\beta_1,\frac{\bar\beta_2}{n_1},\ldots,\frac{\bar\beta_g}{n_1\cdots n_{g-1}}\right\}$ where $\{\bar\beta_0,\bar\beta_1,\ldots,\bar\beta_g\}$ is the minimal system of generators of the semigroup $\Gamma_C$ of $C$. Polar quotients introduced by B. Teissier in [75] are also related to the $C^0$-*sufficiency degree* $\mathrm{Suff}(f)$ of the plane curve $f=0$. Recall that $\mathrm{Suff}(f)$ is the smallest integer $r$ such that the curve defined by the jet of order $r$, $j^r(f)$, has the same topological type as the curve $f=0$ (see [75, 76], [58, Sections 7, 8] or [18, Chapter 7] for more details). These results show the interest of studying polar curves to recover information about the curve $C$.

Polar curves can be thought as a particular case of the jacobian curve of a pair of curves when one of them is non-singular. If we consider $g(x,y)=ay-bx$, we have that the polar curve $\mathcal{P}_{[a:b]}$ is given by the jacobian determinant

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0.$$

More precisely, the *jacobian curve* of two germs of holomorphic functions $f, g \in \mathbb{C}\{x, y\}$ is the curve given by $J(f, g) = 0$ where $J(f, g) = f_x g_y - f_y g_x$. Jacobian curves of two plane curves $f = 0$ and $g = 0$ have been also considered to describe properties of the curves $f = 0$ and $g = 0$ (see [63, 56, 57, 19]) since the analytic type of the jacobian curve is an analytic invariant of the curves $f = 0$ and $g = 0$ (see [64]). Moreover, the topological type of the jacobian curve is not an invariant of the topological type of the pair of curves $f = 0$ and $g = 0$.

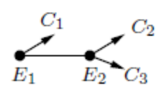
Note that this jacobian curve can be defined as the contact curve $df \wedge dg = 0$ between the hamiltonian foliations $df = 0$ and $dg = 0$. Hence, we can consider the more general case of two germs of foliations $\mathcal{F}$ and $\mathcal{G}$ defined by 1-forms $\omega = 0$ and $\eta = 0$ in $(\mathbb{C}^2, 0)$ and regard its contact curve, which is called the *jacobian curve* $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ of $\mathcal{F}$ and $\mathcal{G}$, given by

$$\omega \wedge \eta = 0.$$

A particular case of jacobian curves of foliations are *polar curves* of foliations: when the foliation $\mathcal{G}$ is non-singular, the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is a polar curve of the foliation $\mathcal{F}$. We have that polar curves of a foliation $\mathcal{F}$ are elements of the linear system defined by the ideal $\mathcal{I}_{\mathcal{F}}$ generated by the coefficients of the 1-form $\omega$. If $\nu = \nu_0(\mathcal{F})$ is the multiplicity of $\mathcal{F}$ and $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{x, y\}$, the curves given by the elements of $\mathcal{I}_{\mathcal{F}} \setminus \mathfrak{m}^{\nu+1}$ are polar curves of $\mathcal{F}$, and there is a Zariski open set of $\mathcal{I}_{\mathcal{F}} \setminus \mathfrak{m}^{\nu+1}$ such that all the curves defined by elements of this open set are equisingular. Moreover, if $\omega = A(x, y)dx + B(x, y)dy$ is a 1-form defining $\mathcal{F}$, a generic element of the curves of the pencil of curves $\{aA(x, y) + bB(x, y) = 0 \ : \ [a : b] \in \mathbb{P}^1_{\mathbb{C}}\}$ is equisingular to a generic element of the linear system defined by the ideal $\mathcal{I}_{\mathcal{F}}$. Hence, if we want to describe topological properties of generic polar curves, it is enough to consider a generic element of the above pencil of curves. We denote by $\mathcal{P}^{\mathcal{F}}$ one of these curves and we call it a *generic polar curve* of the foliation $\mathcal{F}$. Note that polar curves of plane curves can be thought as polar curves of hamiltonian foliations.

Hence, we will consider foliations with a fixed invariant curve (separatrix) $C$ and study the properties of the generic polar curves of these foliations. Note that generic polar curves of such foliations are not equisingular in general: consider for example the hamiltonian foliation $\mathcal{G}$ given by $df = 0$ and the foliation $\mathcal{F}$ given by $\omega = 11(-x^{10} + y^2x^6)dx + 5(y^4 - x^7y)dy$ with $f = y^5 - x^{11}$ (see [28] and Example 1.5.11). By the results about polar curves of plane curves, we know that there is a decomposition theorem for a generic polar curve of hamiltonian foliations. The description of this decomposition is given in terms of the topological data of the curve $C$; in particular, we can state it in terms of the dual graph of the reduction of singularities of $C$. Hence, we wonder if there is also a similar statement for generic polar curves of any foliation with $C$ as curve of separatrices. A first restriction will be to consider non dicritical foliations since it is clear that, if we consider the dicritical foliations $\mathcal{F}_{n,m}$ given by $mydx - nxdy = 0$ with $n, m \in \mathbb{N}$, $\gcd(n, m) = 1$, the topology of a generic polar curve $amy - bmx = 0$ does not depend on the topology of the curves $y^n - cx^m = 0$ which are invariant curves of $\mathcal{F}_{n,m}$. A second restriction will be to consider generalized curve foliations since in this case the reduction of singularities of $\mathcal{F}$ coincides with the minimal reduction of singularities of the curve $C$. If $\mathcal{F}$ is a non-dicritical generalized curve foliation with only an irreducible separatrix, P. Rouillé [71, 70] proved that the decomposition theorem given by Merle for hamiltonian foliations also holds for a generic polar curve of $\mathcal{F}$; he also defined polar quotients for the foliation $\mathcal{F}$ and showed that they are topological invariants of the foliation.

However, if we have a foliation whose curve of separatrices is not irreducible, we realize that, in general, for non-dicritical generalized curves, we can not give a decomposition as in the case of hamiltonian foliations [24]. Consider the foliation given by $\omega = (y^2+x^2y+x^4)dx+(y^2+x^3)dy$. This foliation has three non-singular irreducible separatrices $C_1$, $C_2$ and $C_3$. The dual graph of the reduction of singularities $\pi$ of $C$ is




The decomposition theorem in this situation implies that $\pi$ must be also a reduction of singularities of the generic polar curve (see [28]) but this is not true for the curve $a(y^2 + x^2y + x^4) + b(y^2 + x^3) = 0$.

In the case of foliations, the Camacho-Sad indices of the foliation play a determinant role in the topological behaviour of generic polar curves. The introduction of logarithmic models of generalized curve foliations [24, 26] allow us to codify the information which comes from the Camacho-Sad indices. We will show that there is a decomposition theorem for a generic polar curve of a singular foliation provided that the foliation is a non-dicritical generalized curve and avoids certain Camacho-Sad indices in its reduction of singularities. These results will be explained in Section 1.5. Properties of jacobian curves of foliations will be described in Section 1.6. Moreover, the point of view of foliations allows to recover the results for plane curves considering hamiltonian foliations which are a particular case of logarithmic foliations.

The study of polar curves of foliations have also been useful to study properties of foliations. In [14] a characterization of second type foliations and generalized curve foliations are given in terms of polar curves and also a description of the GSV-index. Section 1.3 is devoted to describe these properties and we also include some new results concerning jacobian curves. Recently, the jacobian curve of two foliations has been used in the study of the Zariski invariant of plane curves which are separatrices corresponding to a dicritical component of one of the foliations (see [44]).

The aim of this survey is to describe the results obtained by the author about polar and jacobian curves of foliations, explaining the main tools used in their proofs as well as relate the above statements with the ones known for polar curves or jacobian curves of plane curves. In the last section we review some invariants concerning analytic classification of irreducible plane curves relating them with the results of previous sections.

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## 1.2 Generalized curve foliations and logarithmic models

In this section we will introduce some basic notions concerning the theory of germs of singular holomorphic foliations in $(\mathbb{C}^2, 0)$. The reader can refer for instance to [10, 60, 23, 61] for a more detailed introduction to the subject.

A germ of singular holomorphic foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ is defined by $\omega = 0$, where $\omega = A(x, y)dx + B(x, y)dy$ is a 1-form and we will assume that $\gcd(A, B) = 1$. The origin is a *singular point* if $A(0) = B(0) = 0$. The *algebraic multiplicity* $\nu_0(\mathcal{F})$ is the minimum of the orders $\nu_0(A)$, $\nu_0(B)$ at the origin of the coefficients of the 1-form $\omega$. Hence, the origin is a singular point of $\mathcal{F}$ if $\nu_0(\mathcal{F}) \geq 1$.

We say that the origin is a *simple singularity* of $\mathcal{F}$ if there are local coordinates $(x, y)$ in $(\mathbb{C}^2, 0)$ such that $\mathcal{F}$ is defined by a 1-form

$$y(\lambda + a(x, y))dx - x(\mu + b(x, y))dy = 0 \tag{1.1}$$

with $a(0) = b(0) = 0$, $\mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}_{>0}$. If $\lambda = 0$, the singularity is called a *saddle-node*.

Consider the blow-up of the origin $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ and denote $E_1 = \pi_1^{-1}(0)$ the exceptional divisor. The blow-up $\pi_1$ is *non-dicritical* if $E_1$ is invariant by the strict transform $\pi_1^*\mathcal{F}$ of $\mathcal{F}$; otherwise, the exceptional divisor $E_1$ is generically transversal to $\pi_1^*\mathcal{F}$ and we say that the blow-up $\pi_1$ is *dicritical*. We also say that the divisor $E_1$ is dicritical or non-dicritical according to $\pi_1$ being or not dicritical.

We say that a morphism $\pi : X \to (\mathbb{C}^2, 0)$, composition of a finite number of punctual blow-ups, is a *reduction of singularities* of $\mathcal{F}$ if the strict transform $\pi^*\mathcal{F}$ of $\mathcal{F}$ verifies that

- each irreducible component of the exceptional divisor $\pi^{-1}(0)$ is either invariant by $\pi^*\mathcal{F}$ or transversal to $\pi^*\mathcal{F}$;
- all the singular points of $\pi^*\mathcal{F}$ are simple and do not belong to a dicritical component of the exceptional divisor.

The existence of a reduction of singularities for foliations in $(\mathbb{C}^2, 0)$ is a consequence of Seidenberg's Desingularization Theorem [73]. Moreover, there is a minimal morphism $\pi$ such that any other reduction of singularities of $\mathcal{F}$ factorizes through the minimal one. If all the irreducible components of the exceptional divisor are invariant by $\pi^*\mathcal{F}$ we say that the foliation $\mathcal{F}$ is *non-dicritical*; otherwise, $\mathcal{F}$ is called a *dicritical foliation*.

A *separatrix* $C$ of $\mathcal{F}$ is a germ of invariant irreducible curve, that is, if $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ is a parametrization of $C$, then $\gamma^*\omega \equiv 0$ where $\omega$ is a 1-form defining $\mathcal{F}$. If the curve $C$ is given by $f(x, y) = 0$, with $f \in \mathbb{C}\{x, y\}$, we have also that $C$ is a separatrix of $\mathcal{F}$ if and only if

$$\omega \wedge df = f h \, dx \wedge dy \tag{1.2}$$

where $h \in \mathbb{C}\{x, y\}$. We denote $\mathrm{Sep}(\mathcal{F})$ the set of separatrices of $\mathcal{F}$. When $\mathcal{F}$ is a non-dicritical foliation, we will denote by $S_{\mathcal{F}}$ the curve of separatrices of $\mathcal{F}$, that is, $S_{\mathcal{F}} = \bigcup_{S \in \mathrm{Sep}(\mathcal{F})} S$.

Observe that, if we have a foliation with a simple singularity given by the expression in (1.1), with $\lambda\mu \neq 0$, then $x = 0$ and $y = 0$ are separatrices of the foliation. The notion of separatrix can be extended considering also formal curves: in Section 1.3 we will also consider formal separatrices and $S_{\mathcal{F}}$ may be a formal curve. If a foliation has a saddle-node singularity, it has a convergent separatrix (called *strong*) and a formal separatrix (called *weak*) that can be divergent.

In [9], C. Camacho and P. Sad proved the existence of at least one (convergent) separatrix for any foliation in $(\mathbb{C}^2, 0)$. Note that, in dimension two, a foliation $\mathcal{F}$ is dicritical if and only if $\mathcal{F}$ has infinitely many separatrices.

We have that the minimal reduction $\pi$ of singularities of a foliation $\mathcal{F}$ gives a reduction of singularities of the curve of separatrices of $\mathcal{F}$ but in general the converse is not true. In [8], C. Camacho, A. Lins Neto and P. Sad introduce the notion of generalized curve foliations, a class of foliations for which the minimal reduction of singularities of the curve of separatrices gives also the minimal reduction of singularities of the foliation. More precisely, a non-dicritical foliation $\mathcal{F}$ is called a *generalized curve foliation* if there are not saddle-node singularities in the reduction of singularities of $\mathcal{F}$ . Dicritical generalized curve foliations are also studied in [8].

We will use properties of generalized curve foliations to study polar curves and jacobian curves of these type of foliations. But, in next section, we will also consider a wide class of foliations, called second type foliations. Recall that a foliation $\mathcal{F}$ is a *second type foliation* if all the singularities of saddle-node type which appear in its reduction of singularities are well oriented with respect to the exceptional divisor, this means that the saddle-node singularities are not corners of the exceptional divisor and that, for the trace type saddle-node singularities, the exceptional divisor is the strong separatrix (see [62]). Hence formal separatrices can appear for second type foliations.

Next result gives some properties of generalized curve and second type foliations:

**Theorem 1.2.1** *[8, 62] Let $\mathcal{F}$ be a non-dicritical foliation in $(\mathbb{C}^2, 0)$ and consider $\mathcal{G}_f$ the hamiltonian foliation defined by $df = 0$, where $f$ is a reduced equation of the curve $S_{\mathcal{F}}$ of separatrices of $\mathcal{F}$. Let $\pi : M \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\mathcal{F}$. Then we have that*

*(i) $\pi$ is also a reduction of singularities of $S_{\mathcal{F}}$. Moreover, $\pi$ is the minimal reduction of singularities of $S_{\mathcal{F}}$ if and only if $\mathcal{F}$ is of second type;*
*(ii) $\nu_0(\mathcal{F}) \geq \nu_0(\mathcal{G}_f)$, and equality holds if and only if $\mathcal{F}$ is a second type foliation;*
*(iii) $\mu_0(\mathcal{F}) \geq \mu_0(\mathcal{G}_f)$, where $\mu_0(\mathcal{F})$ denotes de Milnor number of $\mathcal{F}$. Equality holds if and only if $\mathcal{F}$ is a generalized curve foliation.*

Recall that the *Milnor number* $\mu_0(\mathcal{F})$ is defined by

$$\mu_0(\mathcal{F}) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(A,B)} = (A,B)_0$$

where $(A,B)_0$ denotes the intersection multiplicity of $A$ and $B$ at the origin.

We will denote $\mathbb{G}$ the space of non-dicritical generalized curve foliations in $(\mathbb{C}^2,0)$ and by $\mathbb{G}_C$ the generalized curve foliations such that for any foliation $\mathcal{F} \in \mathbb{G}_C$ the curve of separatrices of $\mathcal{F}$ is exactly the curve $C$, that is, $C = S_{\mathcal{F}} = \bigcup_{S \in \mathrm{Sep}(\mathcal{F})} S$.

In order to introduce the notion of logarithmic model we need to define the Camacho-Sad index. This index was introduced by C. Camacho and P. Sad in the seminal article [9] in order to prove the existence of separatrices for any singular foliation in $(\mathbb{C}^2,0)$. We will see that Camacho-Sad indices play an important role in the description of the topology both of the polar and jacobian curves of foliations (see Sections 1.5 and 1.6).

Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2,0)$ and assume that $S = (y=0)$ is a non-singular invariant curve of $\mathcal{F}$. Hence, a 1-form defining $\mathcal{F}$ can be written as $ya(x,y)dx + b(x,y)dy$. The *Camacho-Sad index of $\mathcal{F}$* relative to *$S$ at the origin* is given by

$$\mathcal{I}_0(\mathcal{F},S) = -\mathrm{Res}_0 \frac{a(x,0)}{b(x,0)},$$

where $\mathrm{Res}_0$ denote the residue at 0 of $\frac{a(x,0)}{b(x,0)}$. A generalization of the Camacho-Sad index of a foliation relative to a singular curve was given by A. Lins Neto in [59, p. 199] using expression (1.7). We give this generalization in Section 1.2.2.

Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2,0)$ defined by a 1-form $\omega = 0$ with $\omega = A(x,y)dx + B(x,y)dy$. Consider the blow-up of the origin $\pi_1 : X_1 \to (\mathbb{C}^2,0)$ and let $E_1 = \pi_1^{-1}(0)$ be the exceptional divisor. If we denote $\nu = \nu_0(\mathcal{F})$, we can write $A(x,y) = \sum_{i \geq \nu} A_i(x,y)$ and $B(x,y) = \sum_{i \geq \nu} B_i(x,y)$ with $A_i, B_i$ homogeneous polynomials of degree $i$ or zero, and $(A_\nu, B_\nu) \neq (0,0)$. The exceptional divisor $E_1$ is invariant by the strict transform of the foliation $\pi_1^*\mathcal{F}$ provided that $\pi_1$ is not dicritical. Note that this is equivalent to assume that $xA_\nu(x,y) + yB_\nu(x,y) \not\equiv 0$. In this case, we can compute the Camacho-Sad index of the strict transform $\pi_1^*\mathcal{F}$ relative to $E_1$ at the singular points of $\pi_1^*\mathcal{F}$ and, by the properties of the Camacho-Sad index, we have that ([9, Proposition 2.2])

$$\sum_{P \in E_1} \mathcal{I}_P(\pi_1^*\mathcal{F}, E_1) = -1.$$

Recall that the singular points of $\pi_1^*\mathcal{F}$ at $E_1$ are determined by the polynomial $xA_\nu(x,y)+yB_\nu(x,y)$ and that the *tangent cone* of $\mathcal{F}$ is the set of lines defined by

$$xA_\nu(x,y)+yB_\nu(x,y)=0. \tag{1.3}$$

This homogeneous polynomial can be factorized as $xA_\nu(x,y)+yB_\nu(x,y) = k\prod_{i=1}^{r}(\alpha_i x+\beta_i y)^{m_i}$ with $k\in\mathbb{C}\setminus\{0\}$ and the lines $\alpha_i x+\beta_i y=0$ are in bijection with the singular points of $\pi_1^*\mathcal{F}$ in $E_1$ (see also [10, Proposition 4.7]).

*Remark 1.2.2* Note that all the Camacho-Sad indices $\mathcal{I}_P(\pi_1^*\mathcal{F},E_1)$, for any $P\in E_1$, are determined by the jet $j^\nu(\omega)=A_\nu(x,y)dx+B_\nu(x,y)dy$.

Let $(x_1,y_1)$ be coordinates in the first chart of the blow-up $\pi_1$ such that $\pi_1(x_1,y_1)=(x_1,x_1y_1)$ and $E_1=(x_1=0)$. The strict transform of the foliation $\pi_1^*\mathcal{F}$ in this chart is given by $\omega_1=0$ where

$$\omega_1=(A_\nu(1,y_1)+y_1B_\nu(1,y_1)+x_1(\cdots))dx_1+x_1(B_\nu(1,y_1)+x_1(\cdots))dy_1.$$

We can assume that all the singularities $\{P_i\}_{i=1}^r$ of $\pi_1^*\mathcal{F}$ are in the first chart of the blow-up and then they are given by $P_i=(0,d_i)$ where $A_\nu(1,y)+yB_\nu(1,y) = k\prod_{i=1}^{r}(y-d_i)^{m_i}$ with $k\in\mathbb{C}\setminus\{0\}$. Hence, the Camacho-Sad index $\mathcal{I}_{P_i}(\pi_1^*\mathcal{F},E_1)$ is given by

$$\mathcal{I}_{P_i}(\pi_1^*\mathcal{F},E_1)=-\mathrm{Res}_{y=d_i}\frac{B_\nu(1,y)}{A_\nu(1,y)+yB_\nu(1,y)}=-\mathrm{Res}_{y=d_i}\frac{B_\nu(1,y)}{k\prod_{i=1}^{r}(y-d_i)^{m_i}}.$$

### 1.2.1 Logarithmic models

A particular case of generalized curve foliations are logarithmic foliations. Recall that a foliation is *logarithmic* if it is given by

$$f_1\cdots f_r\sum_{i=1}^{r}\lambda_i\frac{df_i}{f_i}=0 \tag{1.4}$$

where $f_i\in\mathbb{C}\{x,y\}$ and $\lambda_i\in\mathbb{C}$, $\lambda_i\neq 0$. Note that the curves defined by $f_i=0$ are separatrices of the logarithmic foliation defined by the 1-form above.

Logarithmic foliations are generalized curve foliations (see [70]) but they can be dicritical: for instance the foliation given by $xy(\frac{dx}{x}-\frac{dy}{y})=0$. A logarithmic foliation defined by a 1-form as in (1.4) is called *resonant* if $\sum_{i=1}^{r}n_i\lambda_i=0$ with $n_i$ non-negative integers not all zero. Non-resonant logarithmic foliations are non-dicritical (see [70, Proposition 2.0.20]) but the converse is not true: consider the logarithmic foliation associated to $f_1=y$, $f_2=y-x^2$, $f_3=y-x^3$ and $\lambda_1=1,\lambda_2=-\frac{1}{6}+i,\lambda_3=-\frac{1}{6}-i$.

We denote by $\mathcal{L}_{\lambda,f}$ the logarithmic foliation given by the 1-form in (1.4) where $f=f_1\cdots f_r$ and $\lambda=(\lambda_1,\ldots,\lambda_r)\in\mathbb{C}^r$. Note that, if $C$ is the curve given by $f=0$, we have that $\mathcal{L}_{\lambda,f}\in\mathbb{G}_C$ if $\lambda$ avoids certain resonances.

*Remark 1.2.3* Note that the resonances that must be avoided in order to have that $\mathcal{L}_{\lambda,f}$ is a non-dicritical foliation depend on the reduction of singularities of the curve $C$. Let us explain the type of resonances which appear.

Let $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ be the blow-up of the origin. Take coordinates $(x, y)$ in $(\mathbb{C}^2, 0)$ such that $x = 0$ is not in the tangent cone of $C$ and denote $m_i = m_0(C_i)$ the multiplicity at the origin of the curve $C_i$ with $C_i = (f_i = 0)$ (see Section 1.4 for the precise definition of these notions). Let $(x_1, y_1)$ be coordinates in the first chart of the blow-up such that $\pi_1(x_1, y_1) = (x_1, x_1 y_1)$ and the exceptional divisor $E_1 = \pi_1^{-1}(0)$ is given by $E_1 = (x_1 = 0)$. Then the strict transform of $\mathcal{L}_{\lambda,f}$ by $\pi_1$ is given by

$$x_1 \tilde{f}_1 \cdots \tilde{f}_r \left( (\sum_{i=1}^{r} \lambda_i m_i) \frac{dx_1}{x_1} + \sum_{i=1}^{r} \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i} \right) = 0 \tag{1.5}$$

where $f_i(x_1, x_1 y_1) = x_1^{m_i} \tilde{f}_i(x_1, y_1)$, that is, $\tilde{f}_i = 0$ is an equation of the strict transform of $C_i$ by $\pi_1$. From Expression (1.5) we get that, if $\sum_{i=1}^{r} \lambda_i m_i \neq 0$, then $E_1$ is an invariant curve of the strict transform of $\mathcal{L}_{\lambda,f}$ by $\pi_1$ and hence $\pi_1$ is non-dicritical for the foliation $\mathcal{L}_{\lambda,f}$.

We say that $\kappa_{E_1}(\mathcal{L}_{\lambda,f}) = \sum_{i=1}^{r} \lambda_i m_i$ is the *residue of the logarithmic foliation* $\mathcal{L}_{\lambda,f}$ *along the divisor* $E_1$. In Expression (3.2) of [31] the explicit expression of the residues along the divisors in the reduction of singularities of $C$ are given when $C$ has non-singular irreducible components (the reader can refer to the works of E. Paul concerning logarithmic foliations for more details, see for instance [66, 67]). Hence, we obtain a finite number of resonances to be avoided by performing the blow-ups needed to obtain a reduction of singularities of the curve $C$.

Let us give the notion of logarithmic model introduced in [25] (see also [24, 26]):

**Definition 1.2.4** Given a generalized curve foliation $\mathcal{F}$, we say that a logarithmic foliation $\mathcal{L}$ is a *logarithmic model* of $\mathcal{F}$ if both foliations have the same curve of separatrices and the Camacho-Sad indices of $\mathcal{F}$ and $\mathcal{L}$ coincide along the reduction of singularities.

If we have a foliation $\mathcal{F}$ with a simple singularity at $(\mathbb{C}^2, 0)$, the foliation is given by a 1-form

$$(\lambda + \cdots) y dx - (\mu + \cdots) x dy, \quad \lambda\mu \neq 0, \quad \lambda/\mu \notin \mathbb{Q}_{\geq 0}. \tag{1.6}$$

The quotient $\lambda/\mu$ is the Camacho-Sad index of the foliation with respect to $y = 0$ and it also determines the coefficient of the linear part of the holonomy. A logarithmic foliation in $(\mathbb{C}^2, 0)$, having holonomy with the same linear part as $\mathcal{F}$, is given by

$$xy \left( \lambda \frac{dx}{x} - \mu \frac{dy}{y} \right) = 0.$$

In this way, we can approach a germ of generalized curve foliation with a simple singularity in $(\mathbb{C}^2, 0)$ by a logarithmic foliation. In general, we have that

**Theorem 1.2.5** *[25, 26] Each non-dicritical generalized curve foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ has a logarithmic model. Moreover, the logarithmic model of $\mathcal{F}$ is unique once a reduced equation of the separatrices is fixed.*

Hence the logarithmic model of $\mathcal{F}$ can be considered as a foliation which gives "the linear part of the holonomy" of $\mathcal{F}$.

*Remark 1.2.6* Consider a foliation with a simple singularity (which is not a saddle-node) in $(\mathbb{C}^2, 0)$ given by a 1-form as in (1.6). The realization $h \in \text{Diff}(\mathbb{C}, 0)$ of the holonomy of the invariant curve $y = 0$ is the diffeomorphism

$$h(y) = e^{2\pi i\mu/\lambda} y + y^2 g(y).$$

The holonomy of an invariant variety of a foliation is an interesting invariant of the germ of singular foliation. For instance, a foliation $\mathcal{F}$ with a simple singularity with $\lambda\mu \neq 0$ is linearizable if and only if the holonomy of an invariant variety is linearizable (see [61, Theorem 2]). The reader can refer to [61, 23, 60] for more details concerning the holonomy of an invariant variety of a foliation.

Moreover, Theorem 1.2.5 allows to associate to each non-dicritical generalized curve $\mathcal{F}$ an exponent vector $\lambda(\mathcal{F}) = \lambda$ where $\mathcal{L}_{\lambda,f}$ is the logarithmic model of $\mathcal{F}$. Note that $\lambda(\mathcal{F})$ is unique as element in $\mathbb{P}^{r-1}_{\mathbb{C}}$.

The existence of logarithmic models for non-dicritical foliations, without saddle-node singularities, of codimension one in $(\mathbb{C}^n, 0)$ has been proved in [13]. There are some works studying problems related with the existence of logarithmic models for dicritical foliations in $(\mathbb{C}^2, 0)$ ([11, 12]). The existence of logarithmic models in the two dimensional real case has been proved in [34].

Although a generic polar curve $\mathcal{P}^{\mathcal{F}}$ of a foliation $\mathcal{F}$ and the one $\mathcal{P}^{\mathcal{L}}$ of its logarithmic model $\mathcal{L}$ are not equisingular in general (consider the foliations with $y^5 - x^{11} = 0$ as separatrix given in the introduction), the study of properties shared by $\mathcal{P}^{\mathcal{F}}$ and $\mathcal{P}^{\mathcal{L}}$ is essential to describe properties of $\mathcal{P}^{\mathcal{F}}$ thanks to the properties shared by the Newton polygons of $\mathcal{F}$ and $\mathcal{L}$ (see [26, 28] and Section 1.5). Logarithmic models are also crucial in the study of jacobian curves of foliations (see [31] and Section 1.6).

### 1.2.2 Camacho-Sad index relative to singular separatrices

We include here the generalization of the Camacho-Sad index of a foliation relative to a singular separatrix. This definition uses the decomposition of the 1-form defining the foliation given in Expression (1.7). This expression will be also used in the definition of the GSV-index in Section 1.3.

Let $C$ be a curve in $(\mathbb{C}^2, 0)$ with $f = 0$ a reduced equation of $C$ and let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2, 0)$ defined by a 1-form $\omega$. The curve $C$ is a a separatrix of the foliation $\mathcal{F}$ if and only if there exist $g, k \in \mathbb{C}\{x, y\}$, with $k$ and $f$ relatively prime, such that

$$g\omega = k df + f\theta \tag{1.7}$$

where $\theta$ is a holomorphic 1-form with either $\theta \wedge df \neq 0$ or $\theta = 0$. The proof of the characterization of $C$ being a separatrix of $\mathcal{F}$ with expression (1.7) was given in [59,

p. 198] when $f$ is irreducible and in [74, Lemma 1.1] in the general case. Let us explain the idea of the proof of this characterization. Note that if (1.7) holds, then

$$g\omega \wedge df = f\theta \wedge df = fh_1 dx \wedge dy$$

with $h_1 \neq 0$ if $\theta \neq 0$, and hence we have an expression as in (1.2) and $f = 0$ is a separatrix of $\mathcal{F}$. Conversely, assume that $f = 0$ is a separatrix of $\mathcal{F}$ and that Equation (1.2) holds. We can assume that $(f = 0) \neq (x = 0)$ and we write $\omega = A(x, y)dx + B(x, y)dy$. Let us show that, if we put $g = \frac{\partial f}{\partial y}$, $k = B$ and $\theta = hdx$, then (1.7) holds. We have that

$$g\omega = \frac{\partial f}{\partial y}(Adx + Bdy),$$
$$kdf + f\theta = B\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) + fhdx.$$

Since $A\frac{\partial f}{\partial y} - B\frac{\partial f}{\partial x} = fh$ by (1.2), then $g\omega = kdf + f\theta$ as wanted.

Note that the decomposition given in (1.7) is not unique. A. Lins Neto generalized in [59] (see also [5, 74]) the definition of Camacho-Sad index of a foliation $\mathcal{F}$ relative to a singular separatrix $C$ using expression (1.7). With the notations above we have that

$$\mathcal{I}_0(\mathcal{F}, C) = -\frac{1}{2\pi i}\int_{\partial C}\frac{1}{k}\theta$$

where $\partial C = C \cap S^3_\varepsilon$, with $S^3_\varepsilon$ is a small sphere centered at $0 \in \mathbb{C}^2$ oriented as the boundary of $C \cap B^4_\varepsilon$ for a ball $B^4_\varepsilon$ such that $S^3_\varepsilon = \partial B^4_\varepsilon$ (see [5]). Let us give an example.

*Example 1.2.7* Consider a logarithmic foliation $\mathcal{L}_{\lambda,f}$ given by the 1-form

$$\omega_{\lambda,f} = f_1 \cdots f_r \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i}.$$

Note that if we take the separatrix $C_1 = (f_1 = 0)$, we can write $\omega_{\lambda,f}$ as

$$\omega_{\lambda,f} = \lambda_1 f_2 \cdots f_r df_1 + f_1\theta$$

with $\theta = f_2 \cdots f_r(\sum_{i=2}^{r}\lambda_i\frac{df_i}{f_i})$. Hence, if we compute the Camacho-Sad index of $\mathcal{L}_{\lambda,f}$ relative to $C_1$ we obtain

$$\mathcal{I}_0(\mathcal{L}_{\lambda,f}, C_1) = -\frac{1}{2\pi i}\int_{\partial C_1}\frac{1}{\lambda_1 f_2 \cdots f_r}\theta = -\frac{1}{2\pi i}\sum_{i=2}^{r}\frac{\lambda_i}{\lambda_1}\int_{\partial C_1}\frac{df_i}{f_i}.$$

If $\gamma_1(t)$ is a Puiseux parametrization of the curve $C_1$, then

$$(C_1, C_i)_0 = \mathrm{ord}_t(f_i(\gamma_1(t))) = \frac{1}{2\pi i}\int\frac{df_i(\gamma_1(t))}{f_i(\gamma(t))} = \frac{1}{2\pi i}\int_{\partial C_1}\frac{df_i}{f_i}.$$

Hence, we conclude

$$\mathcal{I}_0(\mathcal{L}_{\lambda,f}, C_1) = -\sum_{i=2}^{r} \frac{\lambda_i}{\lambda_1}(C_1, C_i)_0$$

which extends to logarithmic foliations the computations given in [59, p. 201] (see also [70, p. 53]). This expression was also used in [34] to prove the existence of logarithmic models in the real case.

## 1.3 Polar and jacobian intersection multiplicities

As we mention in the introduction, polar curves of foliations can also be used to characterize properties of foliations. We recall here some results that can be found in [14] for polar curves of foliations and we include new similar properties that can be obtained for jacobian curves of foliations (see also [31, Appendix B]).

Let $\mathcal{F}$ be a singular foliation in $(\mathbb{C}^2, 0)$ and $\mathcal{P}^{\mathcal{F}}$ be a generic polar curve of $\mathcal{F}$. If $\mathcal{F}$ is a non-dicritical generalized curve foliation with $C$ as curve of separatrices, we proved (see [26, Proposition 3.7]) that the multiplicity of intersection $(\mathcal{P}^{\mathcal{F}}, C)_0$ can be computed in terms of local invariants of the foliation $\mathcal{F}$, that is,

$$(\mathcal{P}^{\mathcal{F}}, C)_0 = \mu_0(\mathcal{F}) + \nu_0(\mathcal{F}).$$

If besides we use the properties of generalized curves foliations given in Theorem 1.2.1, we get that $(\mathcal{P}^{\mathcal{F}}, C)_0 = \mu_0(C) + \nu_0(C) - 1$. Moreover, let us explain that the multiplicity of intersection of $\mathcal{P}^{\mathcal{F}}$ and $C$ can be used to characterize second type foliations.

In this section, we will denote $S_{\mathcal{F}}$ the formal curve whose irreducible components are all invariant curves of a non-dicritical foliation $\mathcal{F}$. Hence, we have that

**Proposition 1.3.1** *[14, Proposition 2] Given a non-dicritical foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$, we have that*

$$(\mathcal{P}^{\mathcal{F}}, S_{\mathcal{F}})_0 \leq \mu_0(\mathcal{F}) + \nu_0(\mathcal{F})$$

*and equality holds if and only if $\mathcal{F}$ is a second type foliation.*

Let us recall the notion of $C$-polar excess $\Delta(\mathcal{F}, C)$ introduced in [14, Definition 2] (this notion was extended to the dicritical case in [43]). Let $\mathcal{F}$ be a singular foliation in $(\mathbb{C}^2, 0)$ and $C$ be a formal invariant curve of $\mathcal{F}$. The *$C$-polar excess* $\Delta_0(\mathcal{F}, C)$ is given by

$$\Delta_0(\mathcal{F}, C) = (\mathcal{P}^{\mathcal{F}}, C)_0 - (\mathcal{P}^{\mathcal{G}}, C)_0 = (\mathcal{P}^{\mathcal{F}}, C)_0 - \mu_0(C) - \nu_0(C) + 1$$

where $\mathcal{G}$ is a generalized curve foliation with $C = S_{\mathcal{G}}$. We have that (see [14, Corollary 3])

$$\Delta_0(\mathcal{F}, C) \geq 0$$

for any non-dicritical foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ and any curve $C \subset S_{\mathcal{F}}$. Moreover, the polar excess can be used to characterize generalized curve foliations

**Corollary 1.3.2** *[14, Corollary 4] A non-dicritical foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ is a generalized curve foliation if and only if*

$$\Delta_0(\mathcal{F}, S_{\mathcal{F}}) = 0.$$

In [5], M. Brunella described generalized curve foliations using GSV-index (introduced by X. Gómez-Mont, J. Seade and A. Verjovsky in [45]). He proved that $GSV(\mathcal{F}, C) \geq 0$ for any invariant curve $C$ of a non-dicritical foliation $\mathcal{F}$ and $GSV(\mathcal{F}, S_{\mathcal{F}}) = 0$ if $\mathcal{F}$ is a generalized curve foliation. In fact, property $GSV(\mathcal{F}, S_{\mathcal{F}}) = 0$ characterizes non-dicritical generalized curve foliations (see also [22, Theorem 3.3]). Let us recall the definition of GSV-index and give the relationship with the $C$-polar excess.

Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2, 0)$ defined by a 1-form $\omega = 0$ with $\omega = A(x, y)dx + B(x, y)dy$ and consider a separatrix $C$ of the foliation $\mathcal{F}$ with $f = 0$ a reduced equation of $C$. In [45, Section 3], an algebraic definition of the GSV-index is given. In the case of dimension two, with the notations above, we get that the *GSV-index of $\mathcal{F}$ with respect to $C$ at the origin* is given by

$$GSV_0(\mathcal{F}, C) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(A, B, f)} - \dim_{\mathbb{C}} \frac{(f) \cap (A, B)}{(fA, fB)}$$

where the parentheses represent ideals in the local ring $\mathbb{C}\{x, y\}$ generated by the terms inside the parentheses. Moreover, in [45, Section 4], the authors showed that the index is a topological invariant.

M. Brunella in [5] proved that the GSV-index can be calculated as follows. Let $C$ be a separatrix of a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ and $f = 0$ a reduced equation of $C$. If $\omega$ is a 1-form defining $\mathcal{F}$, there exist $g, k \in \mathbb{C}\{x, y\}$ such $g\omega = kdf + f\theta$ with $\theta$ a holomorphic 1-form as explained in Section 1.2.2 and Equation (1.7). The GSV-index of $\mathcal{F}$ with respect to $C$ at the origin is given by

$$GSV_0(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \frac{g}{k} d\left(\frac{k}{g}\right)$$

where $\partial C = C \cap S^3_\varepsilon$, with $S^3_\varepsilon$ is a small sphere centered at $0 \in \mathbb{C}^2$ oriented as the boundary of $C \cap B^4_\varepsilon$ for a ball $B^4_\varepsilon$ such that $S^3_\varepsilon = \partial B^4_\varepsilon$ (see [5]). Note that if $C$ is an irreducible curve and $\gamma(t)$ is a Puiseux parametrization of $C$, we have that

$$GSV_0(\mathcal{F}, C) = \operatorname{ord}_t((k/g) \circ \gamma).$$

*Example 1.3.3* [5, p. 538] Let us compute the GSV-index for the foliation $\mathcal{F}$ given by $\omega = 0$ with

$$\omega = y^{p+1}dx - x(1 + \lambda y^p)dy, \qquad p \geq 1,\ \lambda \in \mathbb{C}$$

(the formal normal form of a saddle-node singularity). If $C_1 = (x = 0)$ and $C_2 = (y = 0)$ we have that

$$CSV_0(\mathcal{F}, C_1) = p + 1; \qquad CSV_0(\mathcal{F}, C_2) = 1.$$

The computation of the Camacho-Sad indices gives

$$\mathcal{I}_0(\mathcal{F}, C_1) = -\mathrm{Res}_{y=0} \frac{-(1 + \lambda y^p)}{y^{p+1}} = \lambda; \qquad \mathcal{I}_0(\mathcal{F}, C_2) = 0.$$

*Remark 1.3.4* Note that $GSV_0(\mathcal{F}, C)$ can be negative (see [6, p. 24]). Consider the dicritical foliation $\mathcal{F}$ given by $\omega = 3ydx - 2xdy$ and the separatrix $C = (f = 0)$ with $f = y^2 - x^3$. If we take $g = 2y$, $k = -2x$ and $\theta = 6dx$ we obtain the expression $g\omega = kdf + f\theta$ as in (1.7). If we consider $\gamma(t) = (t^2, t^3)$, then $GSV_0(\mathcal{F}, C) = \mathrm{ord}_t \left(\frac{-2x}{2y}(\gamma(t))\right) = \mathrm{ord}_t (\frac{-2t^2}{2t^3}) = -1$. However, if we consider the separatrix $S = (x = 0)$, then $GSV_0(\mathcal{F}, S) = \mathrm{ord}_t (3t) = 1$.

In [5], M. Brunella introduces the notion of *non-dicritical separatrices*, that is, separatrices whose reduction of singularities does not meet a dicritical component. Remark that this notion is different to the one of *isolated separatrices*, which are the separatrices of $\mathcal{F}$ whose strict transform by the reduction of singularities $\pi$ of $\mathcal{F}$ cut the exceptional divisor $\pi^{-1}(0)$ in a non-dicritical component (see [32, Remark 1]). In [5, Proposition 6], it is proved that if $C$ is a non-dicritical separatrix of $\mathcal{F}$, then $GSV_0(\mathcal{F}, C) \geq 0$. In particular, the GSV-index is non-negative for non-dicritical foliations.

The GSV-index is not additive on the separatrices (see [5, p. 532]), that is, if $C_1$ and $C_2$ are separatrices of $\mathcal{F}$ and $C = C_1 \cup C_2$, then

$$GSV_0(\mathcal{F}, C_1 \cup C_2) = GSV_0(\mathcal{F}, C_1) + GSV_0(\mathcal{F}, C_2) - 2(C_1, C_2)_0. \tag{1.8}$$

(note that the Camacho-Sad index has a similar behaviour [5, 74]).

In [14, Proposition 4] we prove that, if $\mathcal{F}$ is a non-dicritical foliation with $S_{\mathcal{F}}$ as curve of separatrices and $C \subset S_{\mathcal{F}}$ is a curve union of convergent separatrices of $\mathcal{F}$, then we have that

$$GSV_0(\mathcal{F}, C) = \Delta_0(\mathcal{F}, C).$$

In particular, this formula gives a way to generalize the definition of the GSV-index to formal invariant curves and also gives another interpretation of the non-negativity of the GSV-index for non-dicritical foliations.

Now we will prove some new results which relate local invariants of the foliations with jacobian curves. In the rest of the section, the foliations $\mathcal{F}$ and $\mathcal{G}$ may be dicritical ones unless otherwise stated. Next lemma shows that a description of the GSV-index can also be obtained if we consider jacobian curves of foliations. Note that we recover the above result ([14, Proposition 4]) when the foliation $\mathcal{F}$ in next lemma is non-singular.

**Lemma 1.3.5** *Consider an irreducible curve $C = (f = 0)$ in $(\mathbb{C}^2, 0)$. Let $\mathcal{G}$ be a foliation such that the irreducible components of $C$ are separatrices of $\mathcal{G}$ and* *consider the hamiltonian foliation $\mathcal{G}_f$ given by $df = 0$. Then, for any foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$, we have that*

$$GSV_0(\mathcal{G}, C) = (\mathcal{J}_{\mathcal{F},\mathcal{G}}, C)_0 - (\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C)_0.$$

***Proof*** Assume that $C$ is irreducible and let $\gamma(t) = (x(t), y(t))$ be a Puiseux parametrization of $C$. The general case follows from Property (1.8) of GSV-index and Lemma 1.3.6 below.

If $\mathcal{F}$ is defined by $\omega = 0$ with $\omega = Adx + Bdy$ and $\mathcal{G}$ is defined by $\eta = 0$ with $\eta = Pdx + Qdy$, we have that

$$(\mathcal{J}_{\mathcal{F},\mathcal{G}}, C)_0 = \operatorname{ord}_t(A(\gamma(t))Q(\gamma(t)) - B(\gamma(t))P(\gamma(t))).$$

From Equation (1.7) applied to $\eta$ we get that

$$P = \frac{k}{g} f_x + \frac{f}{g}\tilde{P}; \qquad Q = \frac{k}{g} f_y + \frac{f}{g}\tilde{Q}$$

and hence

$$\begin{aligned}(\mathcal{J}_{\mathcal{F},\mathcal{G}}, C)_0 &= \operatorname{ord}_t(A(\gamma(t)) f_y(\gamma(t)) - B(\gamma(t)) f_x(\gamma(t))) + \operatorname{ord}_t((k/g)\circ\gamma(t)) \\ &= (\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C)_0 + GSV_0(\mathcal{G}, C)\end{aligned}$$

as wanted. ◻

Let us prove the following lemma which was used in the previous proof.

**Lemma 1.3.6** *Let $\mathcal{G}$ be a foliation in $(\mathbb{C}^2, 0)$. Consider $C_1 = (f_1 = 0)$ and $C_2 = (f_2 = 0)$ two irreducible separatrices of $\mathcal{G}$. For any foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$, we have that*

$$\begin{aligned}(\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_1 \cup C_2)_0 - (\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C_1 \cup C_2)_0 = (\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_1)_0 - (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_1}}, C_1)_0 \\ + (\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_2)_0 - (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_2}}, C_2)_0 - 2(C_1, C_2)_0\end{aligned}$$

*where $f = f_1 f_2$.*

***Proof*** Since $(\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_1 \cup C_2)_0 = (\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_1)_0 + (\mathcal{J}_{\mathcal{F},\mathcal{G}}, C_2)_0$, we only need to show that

$$(\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C_1 \cup C_2)_0 = (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_1}}, C_1)_0 + (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_2}}, C_2)_0 + 2(C_1, C_2)_0.$$

If the foliation $\mathcal{F}$ is defined by $\omega = 0$ with $\omega = A(x, y)dx + B(x, y)dy$, then the curve $\mathcal{J}_{\mathcal{F},\mathcal{G}_f}$ is given by

$$f_2\left(A\frac{\partial f_1}{\partial y} - B\frac{\partial f_1}{\partial x}\right) + f_1\left(A\frac{\partial f_2}{\partial y} - B\frac{\partial f_2}{\partial x}\right) = 0.$$

Thus, if $\gamma_1(t)$ is a Puiseux parametrization of the curve $C_1$, we have that

$$
\begin{aligned}
(\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C_1)_0 &= \mathrm{ord}_t(f_2(\gamma_1(t))) + \mathrm{ord}_t\left(A(\gamma_1(t))\frac{\partial f_1}{\partial y}(\gamma_1(t)) - B(\gamma_1(t))\frac{\partial f_1}{\partial x}(\gamma_1(t))\right) \\
&= (C_1, C_2)_0 + (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_1}}, C_1)_0.
\end{aligned}
$$

A similar computation gives $(\mathcal{J}_{\mathcal{F},\mathcal{G}_f}, C_2)_0 = (C_1, C_2)_0 + (\mathcal{J}_{\mathcal{F},\mathcal{G}_{f_2}}, C_2)_0$ and this ends the proof. □

From Proposition B.1 in [31] we have that, if $\mathcal{F}$ and $\mathcal{G}$ are two foliations (that can be dicritical) in $(\mathbb{C}^2, 0)$ without common separatrices and $C$ is an irreducible separatrix of $\mathcal{G}$, then

$$
(\mathcal{J}_{\mathcal{F},\mathcal{G}}, C)_0 = \mu_0(\mathcal{G}, C) + \tau_0(\mathcal{F}, C)_0 \tag{1.9}
$$

where $\mu_0(\mathcal{G}, C)$ denotes the Milnor number of $\mathcal{G}$ along $C$ and $\tau_0(\mathcal{F}, C)_0$ is the tangency order of $\mathcal{F}$ with $C$. Note that, if $\mathcal{F}$ is a non-singular foliation, the result given in (1.9) generalizes Proposition 1 in [14] for a dicritical foliation $\mathcal{G}$. Let us recall the definition of the invariants which appear in Expression (1.9).

Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2, 0)$ defined by a 1-form $\omega = 0$ with $\omega = A(x, y)dx + B(x, y)dy$. Let $S$ be a formal curve at $(\mathbb{C}^2, 0)$ with a primitive parametrization $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $\gamma(t) = (x(t), y(t))$. If $S$ is a separatrix of $\mathcal{F}$, the *Milnor number* $\mu_0(\mathcal{F}, S)$ *of* $\mathcal{F}$ *along* $S$ is given by

$$
\mu_0(\mathcal{F}, S) = \begin{cases} \mathrm{ord}_t(B(\gamma(t))) - \mathrm{ord}_t(x(t)) + 1, & \text{if } x(t) \neq 0 \\ \mathrm{ord}_t(A(\gamma(t))) - \mathrm{ord}_t(y(t)) + 1, & \text{if } y(t) \neq 0. \end{cases}
$$

This number is also called *multiplicity of the vector field* $\mathbf{v}$ *along* $S$, see [8, p. 152-153], where $\mathbf{v}$ is the vector field $\mathbf{v} = -B(x, y)\frac{\partial}{\partial x} + A(x, y)\frac{\partial}{\partial y}$ also defines de foliation $\mathcal{F}$.

*Example 1.3.7* Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2, 0)$ with a simple singularity given by a 1-form as in Equation (1.1) with $\lambda\mu \neq 0$. If $S_1 = (x = 0)$ and $S_2 = (y = 0)$ are the two transversal separatrices of $\mathcal{F}$, then

$$
\mu_0(\mathcal{F}, S_1) = \mu_0(\mathcal{F}, S_2) = 1.
$$

Assume now that $\mathcal{F}$ has a saddle-node singularity in $(\mathbb{C}^2, 0)$ given by $\omega = y^{p+1}dx - x(1 + \lambda y^p)dy$, with $p \geq 1,\ \lambda \in \mathbb{C}$, then

$$
\mu_0(\mathcal{F}, S_1) = p + 1, \qquad \mu_0(\mathcal{F}, S_2) = 1.
$$

If $S$ is not a separatrix of $\mathcal{F}$, the *tangency order* $\tau_0(\mathcal{F}, S)$ of $\mathcal{F}$ with $S$ (at the origin) is given by $\tau_0(\mathcal{F}, S) = \mathrm{ord}_t(\alpha(t))$ where $\gamma^*\omega = \alpha(t)dt$ with $\alpha(t) = A(\gamma(t))\dot{x}(t) + B(\gamma(t))\dot{y}(t)$ (see [8, p. 167] when the curve $C$ is non-singular or [14] for the general case).

*Example 1.3.8* Let $\mathcal{F}$ be the foliation given by $\omega = (y^3+y^2-xy)dx-(2xy^2+xy-x^2)dy$ (Suzuki's example). Consider the curve $S = (y - x = 0)$ which is not an invariant curve of $\mathcal{F}$. A parametrization of $S$ is given by $\gamma(t) = (t, t)$ and hence $\tau_0(\mathcal{F}, S) = \mathrm{ord}_t(-t^3) = 3$.

Let $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ be the blow-up of the origin and take $(x_1, y_1)$ coordinates in the first chart such that $\pi_1(x_1, y_1) = (x_1, x_1 y_1)$. Then the strict transform $\pi_1^*\mathcal{F}$ of $\mathcal{F}$ by $\pi_1$ is given by $\omega_1 = 0$ with

$$\omega_1 = \frac{\pi_1^*\omega}{x_1^3} = -(y_1^3 dx_1 + (y_1 - 1 + 2x_1 y_1^2) dy_1)$$

in the first chart of the blow-up. Note that $\nu_0(\mathcal{F}) = 2$ and that $\pi_1$ is a dicritical blow-up for $\mathcal{F}$. Hence, we can compute the tangency order of $\pi_1^*\mathcal{F}$ with $E_1 = \pi_1^{-1}(0)$ at the point $P \in E_1$ given by $P = (0, 1)$ in coordinates $(x_1, y_1)$ and we get that $\tau_P(\pi_1^*\mathcal{F}, E_1) = 1$.

*Example 1.3.9* Consider a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ and let $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ be the blow-up of the origin. Let $\pi_1^*\mathcal{F}$ denote the strict transform by $\pi_1$ of the foliation $\mathcal{F}$ and $E_1 = \pi_1^{-1}(0)$ be the exceptional divisor. If the blow-up $\pi_1$ is a dicritical blow-up for $\mathcal{F}$, then we have that (see [8, Lemma 3])

$$\nu_0(\mathcal{F}) - 1 = \sum_{P \in E_1} \tau_P(\pi_1^*\mathcal{F}, E_1).$$

In [71, Lemma 3.7], P. Rouillé prove that if $\mathcal{F}$ is a non-dicritical generalized curve foliation and $\mathcal{G}_f$ is the hamiltonian foliation defined by $df = 0$ with $S_{\mathcal{F}} = (f = 0)$, then we have that

$$\tau_0(\mathcal{F}, S) = \tau_0(\mathcal{G}_f, S) = (S_{\mathcal{F}}, S)_0$$

for any irreducible curve $S$ which is not a separatrix of $\mathcal{F}$. Moreover, the tangency order can also be used to characterize second type foliations (see [14]):

**Lemma 1.3.10** *[14, Corollary 1] Consider a non-dicritical foliation $\mathcal{F}$ with $S_{\mathcal{F}} = (f = 0)$ and let $C$ be an irreducible curve which is not a separatrix of $\mathcal{F}$. Then*

$$\tau_0(\mathcal{F}, C) \geq \tau_0(\mathcal{G}_f, C) = (S_{\mathcal{F}}, C)_0 - 1 \tag{1.10}$$

*and the equality holds if and only if $\mathcal{F}$ is a second type foliation.*

The difference $\tau_0(\mathcal{F}, C) - \tau_0(\mathcal{G}_f, C)$ is determined explicitly, including the case of dicritical foliations, in [7, Lemma 4.2].

Finally, as a consequence of Equation (1.9) and Lemma 1.3.5 we obtain

**Corollary 1.3.11** *Let $C = (f = 0)$ be an irreducible curve in $(\mathbb{C}^2, 0)$, $\mathcal{G}$ be a foliation such $C$ is a separatrix of $\mathcal{G}$ and $\mathcal{G}_f$ be the hamiltonian foliation given by $df = 0$. Then*

$$GSV_0(\mathcal{G}, C) = \mu_0(\mathcal{G}, C) - \mu_0(\mathcal{G}_f, C).$$

*Consequently, if $\mathcal{G}$ is non-dicritical and $S_{\mathcal{G}} = C$ irreducible, we have that $\mathcal{G}$ is a generalized curve foliation if and only if $\mu_0(\mathcal{G}, C) = \mu_0(\mathcal{G}_f, C)$.*

In [29], we have proved a particular case of the result above thanks to the properties shared by the Newton polygons of non-dicritical generalized curve foliations with the same curve of separatrices. More precisely, we prove the following result:

**Lemma 1.3.12** *[29, Lemma 1] Consider a non-dicritical generalized curve foliation $\mathcal{F}$ with $C = (f = 0)$ as curve of separatrices and let $\mathcal{G}_f$ be the hamiltonian foliation given by $df = 0$. Let $\pi : (M, P) \to (\mathbb{C}^2, 0)$ be any morphism composition of a finite number of punctual blow-ups, $D = \pi^{-1}(0)$ be the exceptional divisor and $E$ be an irreducible component of $D$ with $P \in E$. Then*

$$\mu_P(\pi^*\mathcal{F}, E) = \mu_P(\pi^*\mathcal{G}_f, E).$$

## 1.4 Equisingularity data of a plane curve

Before explaining the results concerning the topological properties of polar and jacobian curves of singular foliations in Sections 1.5 and 1.6, we need to explain how to describe the equisingularity data of a plane curve.

The main classifications of complex analytic plane curve singularities are topological or analytic classifications. Two germs of plane curves $C_1$ and $C_2$ in $(\mathbb{C}^2, 0)$ are *topologically equivalent* (resp. *analytically equivalent*) if and only if there exists an homeomorphism $\varphi : U \to V$ (resp. analytic isomorphism) between open neighbourhoods $U$ and $V$ of the origin, where the curves $C_1$ and $C_2$ are defined, such that $\varphi(C_1 \cap U) = C_2 \cap V$. From the works of Zariski, it is known that the topological classification of curves is equivalent to the equisingular one: two plane curves are *equisingular* if they have the same minimal reduction of singularities. In this section we will describe topological invariants of a germ of plane curve in $(\mathbb{C}^2, 0)$. The description of analytic invariants of plane curves is more intricate; we will consider it in Section 1.7.

There are many equivalent ways of describing topological invariants associated to a plane curve $C$ in $(\mathbb{C}^2, 0)$. For instance, in [39, 40] the equisingularity data of the curve $C$ is described in terms of the Eggers diagram of the curve $C$, defined in [36]. In [18, 3], the authors consider the Enriques diagram (introduced in [37]) which keeps records of the proximity relations among the centers of blowing-up in the resolution of singularities of the curve. Since in the study of singularities of foliations in $(\mathbb{C}^2, 0)$, the reduction of singularities is a key tool, we will describe the equisingularity data of $C$ in terms of the dual graph $G(C)$ of the minimal reduction of singularities of $C$ (a more detailed description of the relationship between the equisingularity data of $C$ and the dual graph $G(C)$ can be found in [28, 31]). To deepen in the study of the invariants of the singularities of a plane curve and its reduction of singularities, the reader can consult for instance [79, 4, 18, 78, 77] or the recent survey [42] which gives an introduction to the use of toric and tropical geometry in the analysis of plane curve singularities.

### 1.4.1 Equisingularity data of an irreducible curve

Let us introduce some definitions concerning the description of the equisingularity data of a germ of irreducible plane curve, which will be also called a *branch* (for a more detailed description the reader can refer for instance to [18, 78, 47]). Let $C$ be a germ of irreducible curve in $(\mathbb{C}^2, 0)$ given by $f = 0$ with $f \in \mathbb{C}\{x, y\}$. We can write $f(x, y) = \sum_{i\geq n} f_i(x, y)$ with $f_i(x, y)$ homogeneous polynomials or zero, and $f_n(x, y) \not\equiv 0$. We say that $m_0(C) = n$ is the multiplicity of the curve $C$ at the origin. The *tangent cone* of the curve is equal to the set of lines given by the linear factors of $f_n(x, y) = 0$. In the rest of the section we will assume that $x = 0$ is not tangent to the curve (that is, $x = 0$ is not one of the lines in the tangent cone of $C$). The Newton-Puiseux Theorem shows that there exists a convergent fractionary power series

$$y(x) = \sum_{i\geq n} a_i x^{i/n}$$

such that $f(x, y(x)) = 0$. We say that $y(x)$ is a *Puiseux series* of $C$. Note that, if $y(x) = \sum_{i\geq n} a_i x^{i/n}$ is a Puiseux series of $C$ and $\varepsilon$ is an $n$-root of the unity, then $\sum_{i\geq n} a_i \varepsilon^i x^{i/n}$ is also a Puiseux series of $C$. If we denote by $y_1(x) = y(x), y_2(x), \ldots, y_n(x)$ all the Puiseux series of $C$ obtained from $y(x)$ by the action of each of the $n$-roots of the unity as we have shown, we have that

$$f(x, y) = u(x, y) \prod_{i=1}^{n} (y - y_i(x))$$

where $u(x, y)$ is a unit in $\mathbb{C}\{x, y\}$.

Given a Puiseux series $y(x) = \sum_{i\geq n} a_i x^{i/n}$ of an irreducible curve, we can consider a parametrization $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $\gamma(t) = (x(t), y(t))$ with

$$\begin{cases} x(t) = t^n \\ y(t) = \sum_{i\geq n} a_i t^i \end{cases}$$

such that $f(\gamma(t)) \equiv 0$. Let us introduce some of the invariants that codify the equisingularity data of the irreducible curve $C$.

The characteristic exponents of the curve $C$ are defined as follows (see [82, Section II.3]):

$$\beta_0 = n,$$
$$\beta_1 = \min\{i \;:\; a_i \neq 0, \;\; i \not\equiv 0 \mod (n)\}.$$

If $\beta_1$ does not exists, the $y(x)$ is a power series in $x$, and the curve $C$ is non-singular (this means that $n = 1$). Otherwise, let us consider $e_1 = \gcd(n, \beta_1)$ and define

$$\begin{aligned}\beta_2 &= \min\{i \ : \ a_i \neq 0, \ \ i \not\equiv 0 \mod (e_1)\},\\ e_2 &= \gcd(e_1, \beta_2).\end{aligned}$$

We have that $e_2 < e_1$. We repeat the above procedure and we define

$$\begin{aligned}e_k &= \gcd(e_{k-1}, \beta_k) = \gcd(\beta_0, \beta_1, \ldots, \beta_k),\\ \beta_{k+1} &= \min\{i \ : \ a_i \neq 0, \ \ i \not\equiv 0 \mod (e_k)\}.\end{aligned}$$

Since the sequence $e_0 = n > e_1 > e_2 > \cdots > e_k > \cdots$ is strictly decreasing, there exists $g$ such that $e_g = 1$. The sequence of positive integers $\{\beta_0, \beta_1, \ldots, \beta_g\}$ are called *characteristic exponents* of the curve $C$.

Note that $\beta_i$ is the first exponent which appears in the series which does not belong to the additive group generated by the precedent $\beta_j$. Hence the Puiseux series $y(x)$ can be written as

$$\begin{aligned}y(x) = &\sum_{\substack{i\in(n)\\ i<\beta_1}} a_i x^{i/n} + a_{\beta_1} x^{\beta_1/n} + \sum_{\substack{i\in(e_1)\\ \beta_1<i<\beta_2}} a_i x^{i/n} + a_{\beta_2} x^{\beta_2/n} + \cdots +\\ &+ \sum_{\substack{i\in(e_{g-1})\\ \beta_{g-1}<i<\beta_g}} a_i x^{i/n} + a_{\beta_g} x^{\beta_g/n} + \sum_{i>\beta_g} a_i x^{i/n}\end{aligned}$$

where the coefficients $a_{\beta_i}$, $i = 1, 2, \ldots, g$, are non-zero. The parametrization can be written as

$$\begin{cases} x(t) = t^n\\ y(t) = a_n t^n + a_{2n} t^{2n} + \cdots + a_{kn} t^{kn} + a_{\beta_1} t^{\beta_1} + a_{\beta_1+e_1} t^{\beta_1+e_1} + \cdots + a_{\beta_1+k_1e_1} t^{\beta_1+k_1e_1}\\ \qquad + a_{\beta_2} t^{\beta_2} + a_{\beta_2+e_2} t^{\beta_2+e_2} + \cdots + a_{\beta_q} t^{\beta_q} + a_{\beta_q+e_q} t^{\beta_q+e_q} + \cdots + a_{\beta_g} t^{\beta_g} +\\ \qquad + a_{\beta_g+1} t^{\beta_g+1} + \cdots \end{cases}$$

For $1 \leq i \leq g$, we can also define the integers $n_i$ and $m_i$ as

$$n_i = \frac{e_{i-1}}{e_i}, \quad \beta_i = m_i e_i \quad \text{with } \gcd(n_i, m_i) = 1.$$

We have that

$$m_0(C) = n = n_1 \cdots n_g, \qquad \frac{\beta_i}{n} = \frac{m_i}{n_1 \cdots n_i}.$$

The set $\{(m_i, n_i)\}_{i=1}^{g}$ are called *Puiseux pairs* of the curve $C$. Note that data of the Puiseux pairs is equivalent to the data of the characteristics exponents. Hence, we can write

$$\begin{aligned}y(x) = &\ a_n x + a_{2n} x^2 + \cdots + a_{kn} x^k + a_{\beta_1} x^{\frac{m_1}{n_1}} + a_{\beta_1+e_1} x^{\frac{m_1+1}{n_1}} + \cdots + a_{\beta_1+k_1e_1} x^{\frac{m_1+k_1}{n_1}}\\ &+ a_{\beta_2} x^{\frac{m_2}{n_1n_2}} + a_{\beta_2+e_2} x^{\frac{m_2+1}{n_1n_2}} + \cdots + a_{\beta_q} x^{\frac{m_q}{n_1n_2\cdots n_q}} + \cdots + a_{\beta_g} x^{\frac{m_g}{n_1n_2\cdots n_g}}\\ &+ a_{\beta_g+1} x^{\frac{m_g+1}{n_1n_2\cdots n_g}} + \cdots\end{aligned}$$

The characteristic exponents determine the Puiseux pairs of an irreducible plane curve and conversely. Moreover, two irreducible plane curves are equisingular if and only if they have the same characteristic exponents (see [81, Theorem 2.1]).

Let $C$ be an irreducible curve. The *semigroup* $\Gamma_C$ *of the curve* $C$ is the subset of $\mathbb{Z}_{\geq 0}$ given by

$$\begin{aligned}\Gamma_C &= \{(D,C)_0 \ : \ D \text{ is a germ of plane curve in } (\mathbb{C}^2,0)\}\\ &= \{\operatorname{ord}_t(g(\gamma(t))) \ : \ g \in \mathbb{C}\{x,y\}\}.\end{aligned}$$

Since the intersection multiplicity is additive and $0 \in \Gamma_C$, the set $\Gamma_C$ is a semigroup. Moreover, there exists an integer $c > 0$ such that any non-negative integer greater than or equal to $c$ is contained in $\Gamma_C$ but $c-1 \notin \Gamma_C$ (see [82, Theorem 1.1]). The number $c$ is called the *conductor* of the semigroup $\Gamma_C$. In a more algebraic way, the ideal $(t^c)$ is contained in the image of the morphism

$$\begin{aligned}\gamma^\sharp : \mathbb{C}\{x,y\} &\to \mathbb{C}\{t\}\\ g &\mapsto g(\gamma(t))\end{aligned}$$

but $t^{c-1}$ is not contained in the image of $\gamma^\sharp$ (see [82, Proposition 1.2]). For instance, if $C$ is an irreducible curve with only one Puiseux pair $\{(m,n)\}$, the semigroup of $C$ is given by $\Gamma_C = \{an + bm \ : \ a,b \in \mathbb{Z}_{\geq 0}\}$ and the conductor is equal to $c = (n-1)(m-1)$.

There exists a unique minimal finite set of integers $\{\bar\beta_0, \bar\beta_1, \ldots, \bar\beta_g\}$ which generates the semigroup $\Gamma_C$, that is, $\Gamma_C = \{t_0\bar\beta_0 + t_1\bar\beta_1 + \cdots + t_g\bar\beta_g \ : \ t_i \in \mathbb{Z}_{\geq 0}\}$ (see [82, Theorem 3.9]). Moreover, the generators of the semigroup are determined from the characteristic exponents as follows:

$$\begin{aligned}\bar\beta_0 &= \beta_0 = n,\\ \bar\beta_i &= n_{i-1}\bar\beta_{i-1} + \beta_i - \beta_{i-1} \qquad \text{for } i = 1,2,\ldots,g\end{aligned}$$

where we put $n_0 = 1$.

In particular, we obtain that two irreducible curves $C_1$ and $C_2$ are topologically equivalent if and only if $\Gamma_{C_1} = \Gamma_{C_2}$.

Note that if $\delta$ is a curve such that $(C,\delta)_0 = \bar\beta_i$ for some $0 \leq i \leq g$, then $\delta$ is an irreducible curve since, if $\delta = \delta_1 \cup \delta_2$, we have

$$(C,\delta)_0 = (C,\delta_1)_0 + (C,\delta_2)_0$$

with $(C,\delta_1)_0, (C,\delta_2)_0 \in \Gamma_C \setminus \{0\}$ in contradiction with the fact that $\{\bar\beta_0, \bar\beta_1, \ldots, \bar\beta_g\}$ is a minimal set of generators of $\Gamma_C$.

For any $1 \leq k \leq g$, a curve $C_k$ with $m_0(C_k) = n_0 n_1 \cdots n_{k-1} = \beta_0/e_{k-1}$ and $(C,C_k)_0 = \bar\beta_k$ (or equivalently the coincidence is given by $\mathcal{C}(C,C_k) = \beta_k/\beta_0$, see definition in Section 1.4.2 and Remark 1.4.3) is called a *k-semiroot* of $C$. The minimal set of generators of the semigroup $\Gamma_{C_k}$ is the set $\left\{\frac{\bar\beta_0}{e_{k-1}}, \frac{\bar\beta_1}{e_{k-1}}, \ldots, \frac{\bar\beta_{k-1}}{e_{k-1}}\right\}$ and the characteristic exponents of $C_k$ are given by $\left\{\frac{\beta_0}{e_{k-1}}, \frac{\beta_1}{e_{k-1}}, \ldots, \frac{\beta_{k-1}}{e_{k-1}}\right\}$ (see [69] or

[33] for more properties of semiroots of an irreducible plane curve). A particular set of semiroots of $C$ is given by the *characteristic approximate roots* introduced by Abhyankar and Moh in [1] (see also [46]).

Note that semiroots will also be useful to study analytic invariants of an irreducible plane curve as we will explain in Section 1.7.

### 1.4.2 Equisingularity data of a curve with several branches

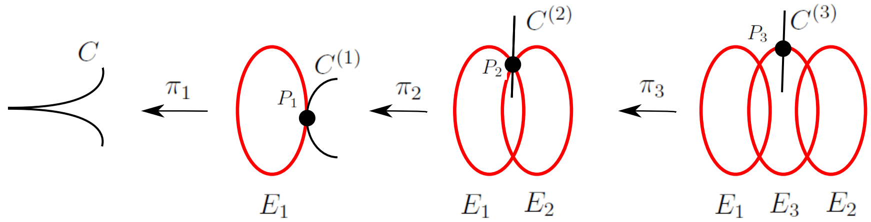
In general, if $C$ is any reduced curve, we can write $C = \cup_{i=1}^{r} C_i$ with $C_i$ irreducible for $i = 1, \ldots, r$. An irreducible curve $C_i$ of $C$ will also be called a *branch* of $C$.

Two germs of plane curves $C = \cup_{i=1}^{r} C_i$ and $D = \cup_{j=1}^{s} D_j$ in $(\mathbb{C}^2, 0)$ are *equisingular* if $r = s$ and there exists a bijection $\Psi : \{C_i\}_{i=1}^{r} \longrightarrow \{D_i\}_{i=1}^{r}$ with $\Psi(C_i) = D_i$ and such that the curves $C_i$ and $D_i$ are equisingular branches and $(C_i, C_j)_0 = (D_i, D_j)_0$ for all $i, j \in \{1, 2, \ldots, r\}$.

In the rest of the section we will consider a curve $C = \cup_{i=1}^{r} C_i$ and, for each irreducible component $C_i$ of $C$, we denote $n^i = m_0(C_i)$ the multiplicity of $C_i$ at the origin, $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ the characteristic exponents of $C_i$ and $\{(m_l^i, n_l^i)\}_{i=1}^{g_i}$ its Puiseux pairs.

Let $\pi_C : X_C \longrightarrow (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $C$, that is, $\pi_C$ is a morphism composition of a finite sequence of punctual blow-ups such that the strict transform of the curve $C$ by $\pi_C$ is a non-singular curve which has normal crossing with the exceptional divisor $\pi_C^{-1}(0)$. When $C$ is an irreducible curve, the idea of the proof of the existence of a reduction of singularities of a curve $C$ is based on the fact that, after a finite number of punctual blow-ups, the multiplicity of the strict transform of $C$ decreases and hence, a non-singular curve is obtained after a finite number of punctual blow-ups. In general, it is necessary to do additional blow-ups to assure that strict transform of the curve $C$ has normal crossings with the exceptional divisor (the details can be found for instance in [78, Theorem 3.3.1] for the irreducible case and [78, Theorem 3.4.4] for the general case).

*Example 1.4.1* Consider the curve $C = (y^2 - x^3 = 0)$. The strict transform $C^{(1)} = \pi_1^* C$ of $C$ by the blow-up of the origin $\pi_1 : X_1 \longrightarrow (\mathbb{C}^2, 0)$ is the non-singular curve $y_1^2 - x_1 = 0$ given in coordinates $(x_1, y_1)$ in the first chart of the blow-up. However, the curve $C^{(1)}$ is tangent to the exceptional divisor $E_1 = \pi_1^{-1}(0)$ at the origin $P_1$ of the first chart. Hence, to obtain a reduction of singularities with normal crossings we need to perform a new blow-up $\pi_2 : X_2 \longrightarrow (X_1, P_1)$ with center at the point $P_1$. Now the strict transform $C^{(2)}$ of $C$ goes through the intersection point $P_2$ of the two divisors $E_1$ and $E_2 = \pi_2^{-1}(P_1)$. Finally, we perform a new blow-up $\pi_3 : X_3 \longrightarrow (X_2, P_2)$ with center at $P_2$ and we get that $\pi_C = \pi_1 \circ \pi_2 \circ \pi_3$ is the minimal reduction of singularities of $C$.



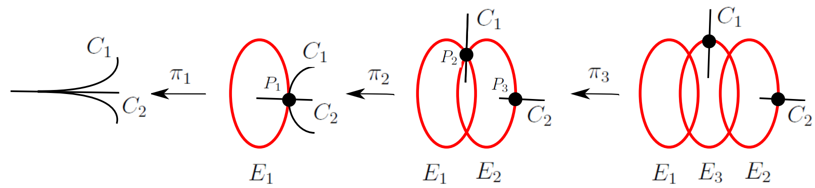

We can construct a graph associated to the minimal reduction of singularities $\pi_C$ of $C$ which also codifies the information concerning the equisingularity data of the curve $C$. The *dual graph* $G(C)$ is the graph whose vertices correspond to the irreducible components $E$ of the exceptional divisor $\pi_C^{-1}(0)$. Two vertices are joined by an edge when the corresponding divisors intersect. Each irreducible component $C_i$ of $C$ is represented by an arrow attached to the vertex which corresponds to the only component $E$ of the exceptional divisor which intersects the strict transform of $C_i$ by $\pi_C$. The reader can refer to [39, Section 1.4.3] for a detailed description of the construction of the dual graph of a curve from the equisingularity data given by the characteristic exponents.

Now we introduce some notations concerning the dual graph and the equisingularity data of the curve $C$ (for more details the reader can refer to [28, Appendix A] or [31, Section 2.3, Appendix A.1]). Given a vertex $E$ of $G(C)$, the valence of a divisor $E$ in the dual graph $G(C)$ is equal to the number of arrows and edges attached to $E$ in $G(C)$ (all the incoming and outgoing edges and arrows are counted). Denote by $E_1$ the component of $\pi_C^{-1}(0)$ which appears after the blow-up of the origin. We associate to $E$ a number $b_E$ defined by: $b_E + 1$ is the valence of $E$ if $E \neq E_1$ and $b_{E_1}$ is the valence of $E_1$.

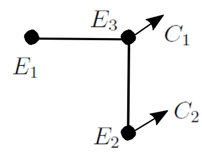
A divisor $E$ of $G(C)$ is called a *bifurcation divisor* of $G(C)$ if $b_E \geq 2$ and a *terminal divisor* if $b_E = 0$. A *dead arc* is a path which joins a bifurcation divisor with a terminal divisor without going through other bifurcation divisors. We denote $B(C)$ the set of bifurcation divisors of $G(C)$.

The *geodesic* of a divisor $E$ is the path which joins the first divisor $E_1$ with the divisor $E$. The geodesic of an irreducible component $C_i$ is the geodesic of the divisor which meets the strict transform of the curve $C_i$.

*Example 1.4.2* Consider the curve $C = C_1 \cup C_2$ with $C_1 = (y^2 - x^3 = 0)$ and $C_2 = (y = 0)$. The minimal reduction of singularities $\pi_C$ of $C$ is composed by three punctual blow-ups $\pi_C = \pi_1 \circ \pi_2 \circ \pi_3$. Let $\pi_1 : X_1 \longrightarrow (\mathbb{C}^2, 0)$ be the blow-up of the origin and $E_1 = \pi_1^{-1}(0)$. The strict transform $\pi_1^* C$ cuts $E_1$ in a point $P_1$, which correspond to the origin of the first chart of the blow-up. Let $\pi_2 : X_2 \longrightarrow (X_1, P_1)$ be the blow-up with center at $P_1$ and $E_2 = \pi_2^{-1}(P_1)$. Then the strict transform $\pi_2^*(\pi_1^* C)$ cuts $E_2$ in two points $P_2$ and $P_3$. Note that, although the two irreducible components of $\pi_2^*(\pi_1^* C)$ are non-singular curves, we need to blow-up with center at point $P_2$ to have normal crossings with the exceptional divisor.

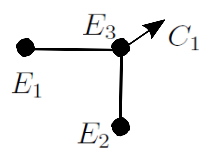



The dual graph $G(C)$ of the curve $C$ is the following one




We have that $b_{E_1} = 1$, $b_{E_2} = 1$ and $b_{E_3} = 2$. Hence, $E_3$ is a bifurcation divisor of $G(C)$.

The morphism $\pi_1 \circ \pi_2 \circ \pi_3$ is also the minimal reduction of singularities of the curve $C_1$ as shown in Example 1.4.1. In this case, the dual graph $G(C_1)$ is given by




and we have that $b_{E_1} = 1$, $b_{E_2} = 0$ and $b_{E_3} = 2$. Then, $E_3$ is a bifurcation divisor and $E_2$ is a terminal divisor of $G(C_1)$.

Given two irreducible curves $\gamma$ and $\xi$, the *coincidence* $C(\gamma, \xi)$ is defined as

$$C(\gamma,\xi) = \sup_{\substack{1\leq i\leq m_0(\gamma)\\ 1\leq j\leq m_0(\xi)}} \left\{ \mathrm{ord}_t(y_i^{\gamma}(x) - y_j^{\xi}(x)) \right\}$$

where $\{y_i^{\gamma}(x)\}_{i=1}^{m_0(\gamma)}$, $\{y_j^{\xi}(x)\}_{j=1}^{m_0(\xi)}$ are the Puiseux series of $\gamma$ and $\xi$ respectively.

*Remark 1.4.3* (see [65, Proposition 2.4]) Note that the coincidence $C(\gamma, \xi)$ determines the intersection multiplicity $(\gamma, \xi)_0$ and viceversa since we have that

$$C(\gamma,\xi) = \frac{\alpha}{m_0(\gamma)} \quad \text{if, and only if,} \quad \frac{(\gamma,\xi)_0}{m_0(\delta)} = \frac{\overline{\beta}_q}{n_1\cdots n_{q-1}} + \frac{\alpha-\beta_q}{n_1\cdots n_q}$$

where $\{\beta_0, \beta_1, \ldots, \beta_g\}$ are the characteristic exponents of $\gamma$ and $\alpha$ is a rational number with $\beta_q \leq \alpha < \beta_{q+1}$ ($\beta_{g+1} = \infty$), $\{(m_i, n_i)\}_{i=1}^{g}$ are the Puiseux pairs of $\gamma$ ($n_0 = 1$) and $\{\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g\}$ is the minimal system of generators of the semigroup $\Gamma_\gamma$ of $\gamma$.

Given an irreducible component $E$ of $\pi_C^{-1}(0)$, a *curvette* $\tilde{\gamma}$ of $E$ is a non-singular curve transversal to $E$ at a non-singular point of $\pi_C^{-1}(0)$. The projection $\gamma = \pi_C(\tilde{\gamma})$ is a germ of plane curve in $(\mathbb{C}^2, 0)$ and, by abuse of notation, we say that $\gamma$ is an *$E$-curvette*. All the $E$-curvettes have the same multiplicity at the origin that will be denoted by $m(E)$ and we denote by $\upsilon(E)$ the coincidence $C(\gamma_E, \gamma'_E)$ between two different $E$-curvettes $\gamma_E, \gamma'_E$ which cut $E$ in different points.

There is a partial order in the set of vertices of $G(C)$ given by $E < E'$ if the geodesic of $E'$ goes through $E$. Hence, if $E < E'$ we have that $\upsilon(E) < \upsilon(E')$.

*Example 1.4.4* In any of the dual graphs given in Example 1.4.2, we have that $\upsilon(E_1) = 1$, $\upsilon(E_2) = 2$, $\upsilon(E_3) = \frac{3}{2}$ and $m(E_1) = m(E_2) = 1$, $m(E_3) = 2$.

Given an irreducible component $E$ of $\pi_C^{-1}(0)$, we denote by $\pi_E : X_E \to (\mathbb{C}^2, 0)$ the *morphism reduction* of $\pi_C$ to $E$ (see [28, Appendix A]), that is, the morphism which verifies that

- the morphism $\pi_C$ factorizes as $\pi_C = \pi_E \circ \pi'_E$ where $\pi_E$ and $\pi'_E$ are composition of punctual blow-ups;
- the divisor $E$ is the strict transform by $\pi'_E$ of an irreducible component $E_{red}$ of $\pi_E^{-1}(0)$ and $E_{red} \subset X_E$ is the only component of $\pi_E^{-1}(0)$ with self-intersection equal to $-1$.

More precisely, we have

$$
\begin{tikzcd}
X_C \arrow[r, "\pi'_E"] \arrow[rr, bend left, "\pi_C"] & X_E \arrow[r, "\pi_E"] & (\mathbb{C}^2, 0) \\
E \arrow[r] & E_{red} &
\end{tikzcd}
$$

*Remark 1.4.5* Let $E$ be an irreducible component of $\pi_C^{-1}(0)$. Note that, if $\tilde{\gamma}_E$ is a curvette of $E$, then $\pi'_E(\tilde{\gamma}_E)$ is a curvette of $E_{red} \subset X_E$ and we have that curve $\gamma_E = \pi_C(\tilde{\gamma}_E) = \pi_E(\pi'_E(\tilde{\gamma}_E))$ is an $E$-curvette, with $\gamma_E$ a curve in $(\mathbb{C}^2, 0)$. It is clear that $m(E) = m(E_{red}) = m_0(\gamma_E)$ and $\upsilon(E) = \upsilon(E_{red})$.

We will denote by $\pi_E^* C$ the strict transform of $C$ by the morphism $\pi_E$. The points $\pi_E^* C \cap E_{red}$ are called *infinitely near points* of $C$ in $E$.

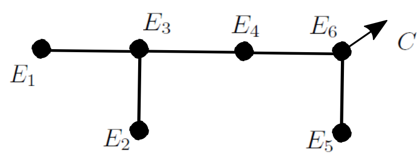
*Remark 1.4.6* The number of infinitely near points of $\pi_E^* C$ in $E_{red}$ is equal to $b_E$ when $C$ is a curve with only non-singular irreducible components.

Consider any $E$-curvette $\gamma_E$ of a divisor $E$ of $G(C)$. Let $\{\beta_0^E, \beta_1^E, \ldots, \beta_{g(E)}^E\}$ be the characteristic exponents of $\gamma_E$ and $\{(m_1^E, n_1^E), (m_2^E, n_2^E), \ldots, (m_{g(E)}^E, n_{g(E)}^E)\}$ be the Puiseux pairs of $\gamma_E$. Note that $m(E) = m_0(\gamma_E) = \beta_0^E$. If $\sigma : M_{\gamma_E} \to (\mathbb{C}^2, 0)$ is the minimal reduction of singularities of $\gamma_E$, then $\pi_E$ factorizes by $\sigma$. We have two possibilities for the value $\upsilon(E)$:

1. either $E$ corresponds to the last divisor which appears in $\sigma$ and hence $\upsilon(E) = \beta^E_{g(E)}/\beta^E_0$. We say that $E$ is a *Puiseux divisor* for $C$ and we put $n_E = n_{g(E)}$, $k_E = g(E) - 1$;
2. or $E$ appears in $\pi_E$ performing $q \geq 1$ punctual blow-ups after $\sigma$ and we have that $\upsilon(E) = (\beta^E_{g(E)} + q)/\beta^E_0$. We put $n_E = 1$ and $k_E = g(E)$. In this case, if we also have that $E$ is a bifurcation divisor, we say that $E$ is a *contact divisor*.

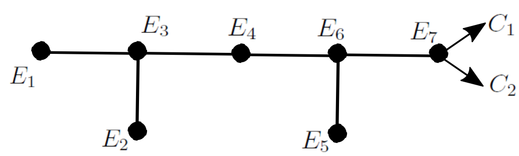
We define $\underline{n}_E = m(E)/n_E = n^E_1 \cdots n^E_{k_E}$. Note that if $E$ belongs to a dead arc whose terminal divisor is $F$ we have that $m(F) = \underline{n}_E$.

*Example 1.4.7* Let us explain with some examples the two possibilities above. Consider a curve $C$ with characteristic exponents $\{4, 6, 9\}$ whose dual graph is given by




If $\gamma$ is an $E_3$-curvette, then the minimal reduction of singularities of $\gamma$ coincides with $\pi_{E_3}$ and hence $E_3$ is a Puiseux divisor for $C$. Note that $m(E_3) = 2$ and $\upsilon(E_3) = \frac{3}{2}$. However, if $\delta$ is an $E_4$-curvette, then its minimal reduction of singularities is $\pi_{E_3}$ instead of $\pi_{E_4}$ (that is, we are in case 2.). Hence $E_4$ is not a Puiseux divisor for $C$. Note that $m(E_4) = 2$, $\upsilon(E_4) = 2$ and $\delta$ is a curve with one Puiseux pair $\{(3, 2)\}$. The bifurcation divisors of $G(C)$ are $E_3$ and $E_6$. The divisors $E_2$ and $E_5$ are terminal divisors with $m(E_2) = 1$ and $m(E_5) = 2$.

Consider now a curve $C = C_1 \cup C_2$ where $C_1$ and $C_2$ are two equisingular curves with characteristic exponents $\{4, 6, 9\}$ and $C(C_1, C_2) = 3$. Then the dual graph $G(C)$ is given by




The bifurcation divisors of $G(C)$ are $E_3$, $E_6$ and $E_7$. The divisors $E_3$ and $E_6$ are Puiseux divisors for $G(C)$ while $E_7$ is a contact divisor.

### 1.4.3 Ramification

Let us explain some properties concerning the behaviour of a plane curve under a ramification. For a detailed description the reader can refer to [26, 28, 31].

Consider a plane curve $C = \cup_{i=1}^r C_i$ in $(\mathbb{C}^2, 0)$. If $x = 0$ is not tangent to $C$, the ramification $\rho : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $\rho(u, v) = (u^n, v)$ is a $C$-*transversal* *ramification*. In order to have that $\widetilde{C} = \rho^{-1}C$ has only non-singular irreducible components, we need that $n \equiv 0 \mod (n^1, n^2, \dots, n^r)$ where $n^i = m_0(C_i)$. Each curve $\widetilde{C}_i = \rho^{-1}C_i$ has exactly $n^i$ irreducible components, $\{\sigma^i_j\}_{j=1}^{n^i}$, which are in bijection with the Puiseux series of $C_i$. Hence, the number of irreducible components of $\widetilde{C}$ is equal to $m_0(C) = n^1 + \cdots + n^r$. In [28] it is proved that the equisingularity data of $C$ can be recovered from the curve $\rho^{-1}C$.

Working with the ramification of a curve allows to deal with the information provided by the parametrizations of the curve as it was done in [55, 56, 57] where the authors consider the *tree model*. The relationship between these two approaches is explained in [31, Section 7.1].

We are interested in the description of the relationship between the dual graphs $G(C)$ and $G(\widetilde{C})$ of the minimal reduction of singularities of $C$ and $\widetilde{C}$ respectively. This relationship will be a very useful tool to prove the results in Sections 1.5 and 1.6.

Note that two consecutive vertices $\widetilde{E}$, $\widetilde{E}'$ of $G(\widetilde{C})$ with $\widetilde{E} < \widetilde{E}'$ verify that $\upsilon(\widetilde{E}') = \upsilon(\widetilde{E})+1$. This implies that $G(\widetilde{C})$ is completely determined by the bifurcation divisors, the order relations among them and the number of edges which leave from each bifurcation divisor.

Let $K_i$ be the geodesic in $G(C)$ of a branch $C_i$ of $C$ and let $\widetilde{K}_i$ be the sub-graph of $G(\widetilde{C})$ corresponding to the geodesics of the irreducible components $\{\sigma^i_l\}_{l=1}^{n^i}$ of $\rho^{-1}C_i$. Let us explain how to construct $\widetilde{K}_i$ from $K_i$. Denote by $B(\widetilde{K}_i)$ and $B(K_i)$ the bifurcation vertices of $\widetilde{K}_i$ and $K_i$ respectively. We say that a vertex $\widetilde{E}$ of $B(\widetilde{K}_i)$ is *associated* to a vertex $E$ of $B(K_i)$ if $\upsilon(\widetilde{E}) = n\upsilon(E)$. Note that there can be other bifurcation vertices in $G(\widetilde{C}) \setminus B(\widetilde{K}_i)$ with valuation equal to $n\upsilon(E)$ but they are not associated to $E$.

If $E$ is the first bifurcation divisor of $B(K_i)$ and $E'$ is its consecutive vertex in $B(K_i)$, then $E$ has only one associated vertex $\widetilde{E}$ in $B(\widetilde{K}_i)$ and there are two possibilities for the number of edges which leave from it:

- If $E$ is a Puiseux divisor for $C_i$ (that is, $E$ is a Puiseux divisor of $G(C)$ with $\upsilon(E) = \frac{\beta^i_1}{n^i}$), then there are $n^i_1$ edges which leave from $\widetilde{E}$ in $\widetilde{K}_i$; then $E'$ has $n^i_1$ associated vertices in $B(\widetilde{K}_i)$;
- otherwise $E$ is a contact divisor for $C_i$ and there is only one edge which leave from $\widetilde{E}$ in $\widetilde{K}_i$; thus $E'$ has only one associated vertex in $B(\widetilde{K}_i)$.

Recall that, if $E$ is a Puiseux divisor for $C$, then $E$ is a Puiseux divisor for at least one irreducible component $C_i$ but it can be a contact divisor for all the other irreducible components (see [31]). Consider now any vertex $E$ of $B(K_i)$ and assume that we have constructed the part of $\widetilde{K}_i$ corresponding to the vertices of $K_i$ with valuation $\leq \upsilon(E)$. Then there are $\underline{n}_E = n^i_1 \cdots n^i_{k_E}$ vertices $\{\widetilde{E}^\ell\}_{\ell=1}^{n_E}$ associated to $E$ and

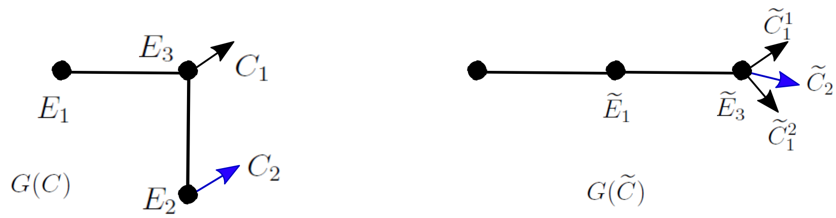
- If $E$ is a Puiseux divisor for $C_i$, then there are $n^i_{k_E+1}$ edges which leave from each vertex $\widetilde{E}_\ell$ in $\widetilde{K}_i$.
- If $E$ is a contact divisor for $C_i$, then there is only one edge which leaves from each vertex $\widetilde{E}_\ell$ in $\widetilde{K}_i$.

The dual graph $G(\widetilde{C})$ is constructed by gluing the graphs $\widetilde{K}_i$. Thus we deduced that, if $\widetilde{E}$ is a divisor of $G(\widetilde{C})$ associated to a divisor $E$ of $G(C)$, then

$$b_{\widetilde{E}} = \begin{cases} b_E, & \text{if } E \text{ is a contact divisor for } C; \\ (b_E - 1)n_E, & \text{if } E \text{ is a Puiseux divisor for } C \text{ which belongs} \\ & \text{to a dead arc;} \\ (b_E - 1)n_E + 1, & \text{if } E \text{ is a Puiseux divisor for } C \text{ which does not} \\ & \text{belong to a dead arc.} \end{cases}$$

Note that all vertices in $G(\widetilde{C})$ associated to a divisor $E$ of $G(C)$ have the same valence. Observe also that there are non-bifurcation divisors of $G(C)$ without associated divisors in $G(\widetilde{C})$.

*Example 1.4.8* Let us consider the curve $C = C_1 \cup C_2$ with $C_1 = (y^2 - x^3 = 0)$ and $C_2 = (y = 0)$ whose dual graph was given in Example 1.4.2. If we take the ramification $\rho(u, v) = (u^2, v)$, the curve $\rho^{-1}C = \widetilde{C}$ is given by $v(v - u^3)(v + u^3) = 0$. Next figure represents the dual graphs $G(C)$ and $G(\widetilde{C})$




where $\widetilde{E}_1$ and $\widetilde{E}_3$ are the associated divisors to $E_1$ and $E_3$ respectively, and $\widetilde{C}_1 = \widetilde{C}_1^1 \cup \widetilde{C}_1^2$.

Other examples describing $G(C)$ and $G(\widetilde{C})$ can be found in [28, Example 2].

Given two divisors $\widetilde{E}^\ell$ and $\widetilde{E}^k$ associated to the same bifurcation divisor $E$ of $G(C)$, there exists a bijection $\rho_{\ell,k} : \widetilde{E}^\ell_{red} \to \widetilde{E}^k_{red}$ which maps the points in $\pi^*_{\widetilde{E}^\ell}\widetilde{C}_i \cap \widetilde{E}^\ell_{red}$ to the points $\pi^*_{\widetilde{E}^k}\widetilde{C}_i \cap \widetilde{E}^k_{red}$ (see [31, Appendix A.2]). Moreover, there is a morphism $\rho_{\widetilde{E}^\ell,E} : \widetilde{E}^\ell_{red} \to E_{red}$, which is a ramification of order $n_E$ (see [28, Lemma 8]), such that $\rho_{\widetilde{E}^k,E} \circ \rho_{\ell,k} = \rho_{\widetilde{E}^\ell,E}$. Note that the morphism $\rho_{\widetilde{E}^\ell,E}$ maps the points in $\pi^*_{\widetilde{E}^\ell}\widetilde{C}_i \cap \widetilde{E}^\ell_{red}$ to the only point in $\pi^*_E C_i \cap E_{red}$. We say that the infinitely near points of $\widetilde{C}_i$ in $\widetilde{E}^\ell$, that is $\pi^*_{\widetilde{E}^\ell}\widetilde{C}_i \cap \widetilde{E}^\ell_{red}$, are associated to the infinitely near point $\pi^*_E C_i \cap E_{red}$ of $C_i$ in $E$.

In [31, Appendix A.3] we can found a description of the behaviour of logarithmic foliations under ramifications.

## 1.5 Topological properties of polar curves of foliations

In this section we will state the results concerning the topological properties of polar curves of singular foliations that depend on the local invariants of the foliations. The notations introduced in Section 1.4 will be useful to state the results concerning the decomposition results of polar and jacobian curves of foliations.

Recall that if $\mathcal{F}$ is a foliation in $(\mathbb{C}^2, 0)$ defined by a 1-form $\omega = 0$, with $\omega = A(x,y)dx + B(x,y)dy$, a *generic polar curve* $\mathcal{P}^{\mathcal{F}}$ is given by

$$aA(x,y) + bB(x,y) = 0$$

with $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$.

One of the main tools used in the study of properties of polar curves is the Newton polygon. Let us recall its definition. Let $\mathcal{F}$ be a foliation in $(\mathbb{C}^2, 0)$ defined by a 1-form $\omega = 0$, with $\omega = A(x,y)dx + B(x,y)dy$, $A, B \in \mathbb{C}\{x,y\}$. The *Newton polygon* $\mathcal{N}(\mathcal{F}; x, y) = \mathcal{N}(\omega; x, y)$ is defined as the Newton polygon of the ideal generated by $xA$ and $yB$. More precisely, $\mathcal{N}(\mathcal{F}; x, y)$ is the convex envelop of $\Delta(\omega) + (\mathbb{R}_{\geq 0})^2$ where $\Delta(\omega) = \{(i,j) \;:\; \omega_{ij} \neq 0\}$ is the *support* of $\omega$ and $\omega = \sum_{i,j} \omega_{ij}$ with

$$\omega_{ij} = A_{ij} x^{i-1} y^j dx + B_{ij} x^i y^{j-1} dy. \tag{1.11}$$

For a plane curve $C$ defined by the reduced equation $f = 0$, we can write $f = \sum_{ij} f_{ij} x^i y^j$ and define the support $\Delta(f) = \{(i,j) \;:\; f_{ij} \neq 0\}$. Hence the Newton polygon $\mathcal{N}(C; x, y)$ of the curve $C$ is the convex envelop of $\Delta(f) + (\mathbb{R}_{\geq 0})^2$. Note that $\mathcal{N}(C; x, y) = \mathcal{N}(df; x, y)$. It is known that if $\mathcal{F}$ is a generalized curve foliation with $C$ as curve of separatrices we have that

$$\mathcal{N}(\mathcal{F}; x, y) = \mathcal{N}(C; x, y)$$

(see [70] or [71, Proposition 3.8]).

Since the knowledge of the Newton polygon is useful to describe the infinitely near points of a curve (see [26]), we wonder if it is possible to determine the Newton polygon of $\mathcal{P}^{\mathcal{F}}$ in terms of the one of $\mathcal{F}$. As we will see, the Camacho-Sad indices of $\mathcal{F}$ have a great influence in that relationship. A first result in this direction is the following

**Lemma 1.5.1** *[26, Proposition 3.5] Let $\mathcal{F}$ be a singular foliation in $(\mathbb{C}^2, 0)$ and consider a side $L$ of $\mathcal{N}(\mathcal{F}; x, y)$ with slope $-1/\mu$ with $\mu \in \mathbb{Q}$ and $\mu \geq 1$. If $i + \mu j = k$ is the equation of the line which contains $L$, then*

$$\mathcal{N}(\mathcal{P}^{\mathcal{F}}; x, y) \subset \{(i,j) \;:\; i + \mu j \geq k - \mu\}.$$

Note that the lemma above does not provide enough information to describe the slopes of $\mathcal{N}(\mathcal{P}^{\mathcal{F}}; x, y)$ as the following example shows.

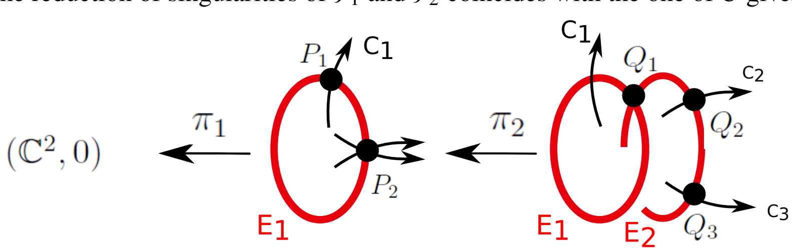
*Example 1.5.2* Consider the foliations $\mathcal{F}_i$ defined by the 1-forms $\omega_i = 0$, $i = 1, 2$, with $\omega_1 = (y^2 - 4xy^2 - 4x^2y - x^4)dx + (y^2 + 2x^2y + 2x^3 - x^4)dy$ and $\omega_2 = (y^2 -$

$4x^3y - 5x^4)dx + (3y^2 + 2xy - x^4)dy$. Both foliations are generalized curve foliations with curve of separatrices $C$ given by $(y+x)(y-x^2)(y+x^2) = 0$ (this can be checked using Theorem 1.2.1 or from the description of their reduction of singularities given below). Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same Newton polygon. However, $\mathcal{N}(\mathcal{P}^{\mathcal{F}_1})$ and $\mathcal{N}(\mathcal{P}^{\mathcal{F}_2})$ are different as it is shown in the figure below:


The black points denote points in $\Delta(\omega_i)$ while the points marked with $\circ$ are points in the support of $\mathcal{P}^{\mathcal{F}_i}$.

In fact, the reader can check that the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-dicritical logarithmic foliations with $f_1 = (y+x)$, $f_2 = (y-x^2)$, $f_3 = (y+x^2)$, $\lambda(\mathcal{F}_1) = (1, 1, -1)$ and $\lambda(\mathcal{F}_2) = (1, 1, 1)$. From the Newton polygons of $\mathcal{P}^{\mathcal{F}_1}$ and $\mathcal{P}^{\mathcal{F}_2}$ we can deduce that the curve $\mathcal{P}^{\mathcal{F}_1}$ is an irreducible curve with one Puiseux pair $\{(3, 2)\}$ while $\mathcal{P}^{\mathcal{F}_2}$ have two non-singular irreducible components. If $\pi_1 : M_1 \to (\mathbb{C}^2, 0)$ is the blow-up of the origin, there are two infinitely near points $P_1, P_2$ of $C$ in $E_1$. Assume that $P_1$ is the infinitely near point of the curve $y + x = 0$. Hence, the curve $\mathcal{P}^{\mathcal{F}_1}$ is tangent to the exceptional divisor $E_1 = \pi_1^{-1}(0)$ at the point $P_2$ and $\mathcal{P}^{\mathcal{F}_1}$ does not satisfy the statement of Theorem 1.5.6 below.

The reduction of singularities of $\mathcal{F}_1$ and $\mathcal{F}_2$ coincides with the one of $C$ given by




The computation of the Camacho-Sad indices for a logarithmic foliation $\mathcal{L}_{\lambda,f}$ gives (see the proof of [26, Proposition 4.4]) :

$$\mathcal{I}_{P_1}(\mathcal{L}_{\lambda,f}, E_1) = -\tfrac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}; \qquad \mathcal{I}_{Q_1}(\mathcal{L}_{\lambda,f}, E_2) = -\tfrac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+2(\lambda_2+\lambda_3)};$$
$$\mathcal{I}_{Q_2}(\mathcal{L}_{\lambda,f}, E_2) = -\tfrac{\lambda_2}{\lambda_1+2(\lambda_2+\lambda_3)}; \qquad \mathcal{I}_{Q_3}(\mathcal{L}_{\lambda,f}, E_2) = -\tfrac{\lambda_3}{\lambda_1+2(\lambda_2+\lambda_3)};$$

where, by abuse of notation, we put $\mathcal{L}_{\lambda,f}$ to refer to the strict transform of the foliation by the corresponding morphism. Hence, for the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ we get

$$\mathcal{I}_{P_1}(\mathcal{F}_1, E_1) = -1; \quad \mathcal{I}_{Q_1}(\mathcal{F}_1, E_1) = -1; \quad \mathcal{I}_{Q_2}(\mathcal{F}_1, E_1) = -1; \quad \mathcal{I}_{Q_3}(\mathcal{F}_1, E_1) = 1;$$

$$\mathcal{I}_{P_1}(\mathcal{F}_2, E_1) = -\tfrac{1}{3}; \quad \mathcal{I}_{Q_1}(\mathcal{F}_2, E_1) = -\tfrac{3}{5}; \quad \mathcal{I}_{Q_2}(\mathcal{F}_2, E_1) = -\tfrac{1}{5}; \quad \mathcal{I}_{Q_3}(\mathcal{F}_2, E_1) = -\tfrac{1}{5}.$$

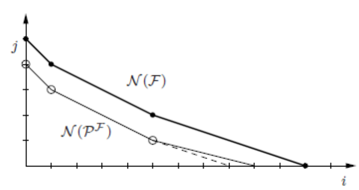
In order have a more detailed description of $\mathcal{N}(\mathcal{P}^{\mathcal{F}}; x, y)$, we have to control the "contribution" to the Newton polygon of $\mathcal{F}$ of the coefficients $A$ and $B$ of the 1-form $\omega = Adx + Bdy$ defining $\mathcal{F}$. More precisely, a point $(i, j) \in \Delta(\omega)$ is said to be a *contribution* of $B$ if $B_{ij} \neq 0$ in the expression (1.11). We have the following results that can be found in [26]

**Proposition 1.5.3** *[26, Proposition 3.4] Let $\mathcal{F}$ be a generalized curve foliation in $(\mathbb{C}^2, 0)$ and take a side $L$ of the Newton polygon $\mathcal{N}(\mathcal{F})$ with slope $-1/p$, $p \in \mathbb{N}$. If $L$ has no contribution of $B$ in his highest vertex, then there is a corner in the reduction of singularities of $\mathcal{F}$ with Camacho-Sad index equal to $-1$.*

*Example 1.5.4* In Example 1.5.2, the side $L_2$ of $\mathcal{N}(\mathcal{F}_1)$ does not have contribution of $B$ and hence $\mathcal{F}_1$ has Camacho-Sad index equal to $-1$ at the corner $Q_2$.

If $\mathcal{L}$ is a logarithmic foliation with non-singular separatrices, we can explicitly compute the Camacho-Sad indices at all the infinitely near points in the reduction of singularities in terms of the exponent vector $\lambda(\mathcal{L})$ and conclude that, if $\mathcal{L}$ is non-resonant, then there is no Camacho-Sad index equal to $-1$ in any corner of the reduction of singularities of $\mathcal{L}$ (see [26, Proposition 4.4]). This result implies the contribution of $B$ in the highest vertex of all the sides of $\mathcal{N}(\mathcal{L})$. We deduce that, if $\mathcal{F}$ is a non-dicritical generalized curve such that its logarithmic model is non-resonant and such that the irreducible components of its curve of separatrices are non-singular, we have

($\star$) if $\mathcal{N}(\mathcal{F}; x, y)$ has $s$ sides with slopes $-1/p_j$, $p_j \in \mathbb{N}$, $j = 1, \ldots, s$ and $p_1 < p_2 < \cdots < p_s$, then the first $s-1$ sides of $\mathcal{N}(\mathcal{P}^{\mathcal{F}}; x, y)$ are obtained from the ones of $\mathcal{N}(\mathcal{F}; x, y)$ by a vertical translation of one unit and the others sides have slope $\geq -1/p_s$.




Behaviour of the Newton polygon of $\mathcal{F}$ and $\mathcal{P}^{\mathcal{F}}$ when they verify property ($\star$).

*Remark 1.5.5* Note that if $\mathcal{G}_f$ is the hamiltonian foliation given by $df = 0$ and $C = (f = 0)$, we have the following property:

If the Newton polygon $\mathcal{N}(\mathcal{G}_f; x, y)$ has $s$ sides $L_1, L_2, \ldots, L_s$ with $L_\ell$ contained in the line given by $i + \alpha_\ell j = k_\ell$ with $\alpha_\ell \in \mathbb{Q}$, $\alpha_\ell \geq 1$, then the first $s-1$ sides of $\mathcal{N}(\mathcal{P}^{\mathcal{G}_f}; x, y) = \mathcal{N}(\mathcal{P}^{C}; x, y)$ are obtained from the ones of $\mathcal{N}(\mathcal{G}_f; x, y)$ by a vertical translation of one unit and the other sides are contained in the region $\{(i, j) \ : \ i + \alpha_s j \geq k_s - \alpha_s\}$.

This property can be deduce from the fact that the hamiltonian foliation $\mathcal{G}_f$ is a non-resonant logarithmic foliation, or directly from the expression of the 1-form defining $\mathcal{G}_f$. In fact, since $\mathcal{G}_f$ is defined by $df = 0$, then the coefficients of the

1-form that defines $\mathcal{G}_f$ are $A = \frac{\partial f}{\partial x}$ and $B = \frac{\partial f}{\partial y}$. Hence, if $(i, j) \in \Delta(f)$ with $j \geq 1$, then $(i, j-1) \in \Delta(\frac{\partial f}{\partial y})$, and thus each side $L$ of $\mathcal{N}(\mathcal{G}_f; x, y)$ has contribution of $B$ in its highest vertex and we get that

$$\text{if } (i,j) \in \Delta(df) = \Delta(f) \text{ with } j \geq 1, \quad \text{then} \quad (i, j-1) \in \Delta\left(\frac{\partial f}{\partial y}\right) \subset \Delta\left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right),$$

for generic $a, b$ (see the proof of [26, Proposition 3.5] for more details).

In particular, hamiltonian foliations always satisfy property $(\star)$ when its curve of separatrices has non-singular irreducible components (and hence all the slopes of the Newton polygon are of the type $-1/p$ with $p \in \mathbb{N}$).

Property $(\star)$ is key to prove the decomposition result for generic polar curves of foliations. The strategy is to prove first the decomposition theorem for foliations whose curve of separatrices has non-singular irreducible components and then study the general case considering a ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of the type $\rho(u, v) = (u^n, v)$ such that the curve of separatrices $\rho^{-1}C$ of $\rho^*\mathcal{F}$ has non-singular irreducible components.

### 1.5.1 The case of non-singular separatrices

Let us consider first the case of a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ such that its curve of separatrices $C$ has only non-singular irreducible components. We have that

**Theorem 1.5.6** *[26, Theorem 6.1] Let $\mathcal{F}$ be a non-dicritical generalized curve foliation in $(\mathbb{C}^2, 0)$ with non-resonant logarithmic model. Assume that all the irreducible components of the curve $C$ of separatrices of $\mathcal{F}$ are non-singular curves. Let $\pi : (N, P) \to (\mathbb{C}^2, 0)$ be a finite sequence of punctual blow-ups such that $P$ is an infinitely near point of $C$, then*

$$m_P(\pi^*\mathcal{P}^{\mathcal{F}}) = m_P(\pi^*C) - 1,$$

*where $\pi^*\mathcal{P}^{\mathcal{F}}$, $\pi^*C$ denote the strict transforms of $\mathcal{P}^{\mathcal{F}}$, $C$ by $\pi$. Moreover, the curve $\pi^*\mathcal{P}^{\mathcal{F}}$ is transversal to the exceptional divisor $\pi^{-1}(0)$ at $P$.*

Consider $\pi_C : X_C \to (\mathbb{C}^2, 0)$ the minimal reduction of singularities of $C$. Given a component $E$ of the divisor $\pi_C^{-1}(0)$, we consider the morphism $\pi_E : X_E \to (\mathbb{C}^2, 0)$ reduction of $\pi_C$ to $E$ and we write $\pi_C = \pi_E \circ \pi'_E$. In the conditions of Theorem 1.5.6, given a bifurcation divisor $E$ of $G(C)$, we define the curve $\mathcal{P}^E$ as the union of the branches $\xi$ of $\mathcal{P}^{\mathcal{F}}$ such that

- $\pi_E^*\xi \cap \pi_E^*C = \emptyset$;
- if $E' < E$, then $\pi_E^*\xi \cap \pi'_E(E') = \emptyset$.

Thus, if $B(C)$ is the set of bifurcation vertices of $G(C)$, there is a unique decomposition $\mathcal{P}^{\mathcal{F}} = \cup_{E \in B(C)} P^E$ such that

d1. $m_0(P^E) = b_E - 1$;
d2. $\pi_E^* P^E \cap \pi_E^* C = \emptyset$;
d3. if $E' < E$, then $\pi_E^* P^E \cap \pi'_E(E') = \emptyset$;
d4. if $E' > E$, then $\pi_{E'}^* P^E \cap E'_{red} = \emptyset$.

In particular, if $E$ is not a bifurcation divisor, then $\pi_E^* \mathcal{P} \cap E_{red} = \pi_E^* C \cap E_{red}$. Consequently, properties d1-d4 above imply the following ones stated in terms of the coincidences and the data in $G(C)$.

**Corollary 1.5.7** *[26, Corollaire 6.2] Let $\mathcal{F}$ be a non-dicritical generalized curve foliation in $(\mathbb{C}^2, 0)$ with non-resonant logarithmic model. Let $C = \cup_{i=1}^r C_i$ be the curve of separatrices of $\mathcal{F}$ and assume that all the irreducible components $C_i$ of $C$ are non-singular curves. If $\mathcal{P}^{\mathcal{F}}$ is a generic polar curve of $\mathcal{F}$, there is a unique decomposition $\mathcal{P}^{\mathcal{F}} = \cup_{E \in B(C)} P^E$ of such that*

*1. $m_0(P^E) = b_E - 1$;*
*2. for each irreducible component $\xi$ of $P^E$, we have that*

*(i) $C(C_i, \xi) = \upsilon(E)$ if $E$ belongs to the geodesic of $C_i$;*
*(ii) $C(C_j, \xi) = C(C_j, C_i)$ if $E$ belongs to the geodesic of $C_i$ but not to the one of $C_j$.*

### 1.5.2 General case

Let us consider now the general case. Let $C$ be the curve of separatrices of the foliation $\mathcal{F}$. Assume that $x = 0$ is not in the tangent cone of $C$. Then the morphism $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $\rho(u, v) = (u^n, v)$ is a *ramification transverse* to $C$ (see Section 1.4.3). If $n = m_0(C)$, then all the irreducible components of $\rho^{-1}C$ are non-singular curves. Note that, if $\mathcal{F}$ is a non-dicritical generalized curve foliation with $C$ as curve of separatrices, then $\rho^*\mathcal{F}$ is also a non-dicritical generalized curve foliation with $\rho^{-1}C$ as curve of separatrices. Moreover, logarithmic models of foliations also behave well under ramifications (see [26, Section 7]).

In general, the curve $\rho^{-1}\mathcal{P}^{\mathcal{F}}$ is not equisingular to the curve $\mathcal{P}^{\rho^*\mathcal{F}}$: it is enough to consider the foliation $\mathcal{F}$ given by $d(y^3 - x^{11}) = 0$ and $\rho(u, v) = (u^3, v)$. The curve $\rho^{-1}\mathcal{P}^{\mathcal{F}} = \{-11au^{30} + 3bv^2 = 0\}$ has two non-singular branches $\gamma_1, \gamma_2$ with coincidence equal to $C(\gamma_1, \gamma_2) = 15$ while the curve $\mathcal{P}^{\rho^*\mathcal{F}} = \{-11au^{32} + bv^2 = 0\}$ has also two non-singular branches $\delta_1, \delta_2$ but with coincidence $C(\delta_1, \delta_2) = 16$.

However, we can prove the following result. Let us denote $\widetilde{C} = \rho^{-1}C$ and let $\pi_{\widetilde{C}} : X_{\widetilde{C}} \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\widetilde{C}$, then

**Lemma 1.5.8** *[28, Lemma 6] Let $\mathcal{F} \in \mathbb{G}_C$ with non-resonant logarithmic model. For each irreducible component $\widetilde{E}$ of $\pi_{\widetilde{C}}^{-1}(0)$ with $\upsilon(\widetilde{E}) > n$, we have that*

$$\pi_{\widetilde{E}}^*(\rho^{-1}\mathcal{P}^{\mathcal{F}}) \cap \widetilde{E}_{red} = \pi_{\widetilde{E}}^* \mathcal{P}^{\rho^*\mathcal{F}} \cap \widetilde{E}_{red}.$$

*Moreover, $m_P(\pi_{\widetilde{E}}^*(\rho^{-1}\mathcal{P}^{\mathcal{F}})) = m_P(\pi_{\widetilde{E}}^* \mathcal{P}^{\rho^*\mathcal{F}})$ for each $P \in \pi_{\widetilde{E}}^*(\rho^{-1}\mathcal{P}^{\mathcal{F}}) \cap \widetilde{E}_{red}$.*

Since $\mathcal{P}^{\rho^*\mathcal{F}}$ satisfies Theorem 1.5.6, the previous proposition allows to prove next result

**Theorem 1.5.9** *[26, Proposition 7.6] Let $\mathcal{F}$ be a non-dicritical generalized curve foliation with non-resonant logarithmic model and $C$ as curve of separatrices. Consider any ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ transverse to $C$ such that $\rho^{-1}C$ has non-singular irreducible components. If $\pi : (N, P) \to (\mathbb{C}^2, 0)$ is a sequence of punctual blow-ups such that $P$ is an infinitely near point of $\rho^{-1}C$, then*

$$m_P(\pi^*(\rho^{-1}\mathcal{P}^{\mathcal{F}})) = m_P(\pi^*(\rho^{-1}C)) - 1$$

*and the curve $\pi^*(\rho^{-1}\mathcal{P}^{\mathcal{F}})$ is transversal to the exceptional divisor $\pi^{-1}(0)$ at $P$.*

As we have already explained in Section 1.5.1, the result above allows to give a decomposition

$$\rho^{-1}\mathcal{P}^{\mathcal{F}} = \bigcup_{\widetilde{E} \in B(\widetilde{C})} \widetilde{P}^{\widetilde{E}}$$

as the one described in Corollary 1.5.7, where $\widetilde{C} = \rho^{-1}C$. Thanks to the relationship between the dual graphs $G(C)$ and $G(\widetilde{C})$ (see Section 1.4.3 and [28, Appendix B]), we get a decomposition

$$\mathcal{P}^{\mathcal{F}} = \bigcup_{E \in B(C)} P^E$$

where $\rho^{-1}P^E = \cup_{i=1}^{n_E} \widetilde{P}^{\widetilde{E}^j}$ with $\{\widetilde{E}^j\}_{j=1}^{n_E}$ the divisors of $G(\widetilde{C})$ associated to $E$ in $G(\widetilde{C})$. Hence, we recover the usual statement of the decomposition result for a generic polar curve:

**Theorem 1.5.10** *(Decomposition theorem [25],[26, Theorem 5.1]) Let $\mathcal{F}$ be a non-dicritical generalized curve foliation with non-resonant logarithmic model and $C$ as curve of separatrices. Let $B(C)$ be the set of bifurcation divisors of $G(C)$. Then a generic polar curve $\mathcal{P}^{\mathcal{F}}$ has a decomposition*

$$\mathcal{P}^{\mathcal{F}} = \bigcup_{E \in B(C)} P^E$$

*such that*

*D1.* $m_0(P^E) = \begin{cases} \underline{n}_E n_E(b_E - 1), & \textit{if } E \textit{ does not belong to a dead arc;} \\ \underline{n}_E n_E(b_E - 1) - \underline{n}_E, & \textit{otherwise.} \end{cases}$

*D2. For each irreducible component $\xi$ of $P^E$, we have that*

- *i)* $C(C_i, \xi) = \upsilon(E)$ *if $E$ belongs to the geodesic of $C_i$;*
- *ii)* $C(C_k, \xi) = C(C_k, C_i)$ *if $E$ belongs to the geodesic of $C_i$ but not to the one of $C_k$.*

The result above generalizes the decomposition theorem given by E. García-Barroso in [39, 40] for a generic polar curve of a plane curve with several irreducible components (see also [55]). Note that the above description does not determine completely the topological type of $\mathcal{P}^{\mathcal{F}}$ as it is showed in next example.

*Example 1.5.11* ([28, Example 1]). Consider the foliations $\mathcal{F}_i$ given by $\omega_i = 0$ where

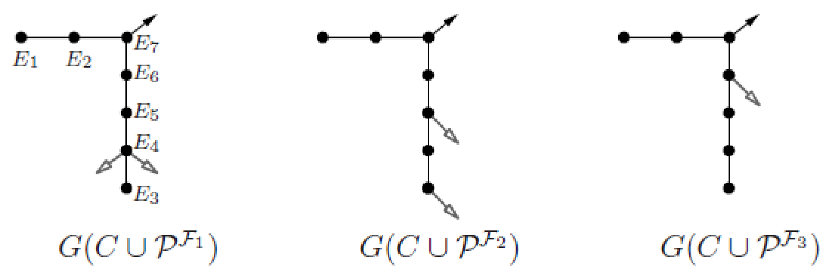
$$\begin{aligned} \omega_1 &= d(y^5 - x^{11}), \\ \omega_2 &= 11(-x^{10} + y^2x^6)dx + 5(y^4 - x^7y)dy, \\ \omega_3 &= 11(-x^{10} + yx^8)dx + 5(y^4 - x^9)dy. \end{aligned}$$

The curve $C = (y^5 - x^{11} = 0)$ is the only separatrix of each $\mathcal{F}_i$. In fact, if we consider $\gamma(t) = (t^5, t^{11})$, we have that $\gamma^*\omega_i \equiv 0$. Moreover, the computation of Milnor number gives $\mu_0(\mathcal{F}_i) = 40 = \mu_0(C)$ for $i = 1, 2, 3$, hence by Theorem 1.2.1, the foliations $\mathcal{F}_i$ are non-dicritical generalized curve foliations in $\mathbb{G}_C$.

The generic polar curves $\mathcal{P}^{\mathcal{F}_1}$, $\mathcal{P}^{\mathcal{F}_2}$ and $\mathcal{P}^{\mathcal{F}_3}$ are not equisingular:

- $\mathcal{P}^{\mathcal{F}_1}$ have two branches $\gamma_1, \gamma_2$, each of them with characteristic exponents $\{2, 5\}$ and coincidence $C(\gamma_1, \gamma_2) = \frac{5}{2}$.
- $\mathcal{P}^{\mathcal{F}_2}$ has two irreducible components $\delta_1$, $\delta_2$. One branch $\delta_1$ has characteristic exponents $\{3, 7\}$ and the branch $\delta_2$ is a non-singular curve. The coincidence between both branches is given by $C(\delta_1, \delta_2) = \frac{7}{3}$.
- $\mathcal{P}^{\mathcal{F}_3}$ is an irreducible curve with characteristic exponents $\{4, 9\}$.

In these examples, the minimal reduction of singularities of $C$ gives also a reduction of singularities of a generic polar curve as shows next figure.




The grey arrows correspond to the branches of a generic polar curve. The black arrow represents the curve $C$.

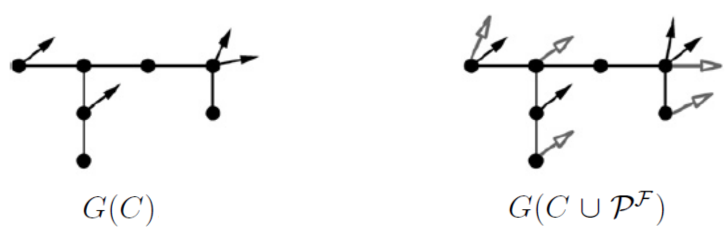
Note that, in general, the minimal reduction of singularities of $\mathcal{F}$ is not a reduction of singularities of $\mathcal{P}^{\mathcal{F}}$, although we can deduce some properties of $\pi_C^*\mathcal{P}^{\mathcal{F}}$. For instance, if $C$ is an irreducible curve, from the properties of the decomposition of $\mathcal{P}^{\mathcal{F}}$, we get that, given a bifurcation divisor $E$ of $G(C)$, the curve $\pi_E^*P^E$ does not intersect $E_{red}$ since $m_0(P^E) < m(E)$. The branches of $P^E$ go through the divisors in the dead arc corresponding to $E$ as shown in the previous example. In [40, Section 5.1], for the generic polar curve $\mathcal{P}^C$ of a plane curve $C$, E. García Barroso gives a description of some the properties of the branches of the curves $P^E$ which appear in the decomposition theorem in terms of the Eggers diagram of the curve $C$.

However, we are able to determine completely the topological type of a generic polar curve of a foliation $\mathcal{F}$ provided we impose conditions over the equisingularity type of the curve $C$ of separatrices of $\mathcal{F}$ and we assume that the exponents vector $\lambda(\mathcal{F})$ belongs to a Zariski open set $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ which depends on the analytic type of the curve $C$ (a detailed description of the set $U_C$ can be found in [28]: in Section 4 when the curve $C$ has non-singular irreducible components, and in Section 6 for the general case). Foliations $\mathcal{F}$ with $\lambda(\mathcal{F}) \in U_C$ are called *Zariski-general foliations*. We say that a curve $C$ has a *kind equisingularity type* if for each dead arc of $G(C)$ with bifurcation divisor $E$ and terminal divisor $F$ we have that $m(E) = 2m(F)$, that is, $n_E = 2$ for any bifurcation divisor $E$ of $G(C)$ which belongs to a dear arc. We can state now the result which describes the equisingularity type of $\mathcal{P}^{\mathcal{F}}$ in terms of the dual graph $G(C)$:

**Theorem 1.5.12** *[28, Proposition 5] Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F}$. Then the minimal reduction of singularities $\pi_C$ of $C$ is also a reduction of singularities of $\mathcal{P}^{\mathcal{F}} \cup C$. Moreover, the irreducible components of $\mathcal{P}^{\mathcal{F}}$ intersect an irreducible component $E$ of the exceptional divisor $\pi_C^{-1}(0)$ as follows:*

- *if $E$ is a bifurcation divisor of $G(C)$, the number of branches of $\mathcal{P}^{\mathcal{F}}$ that intersect $E$ is equal to $b_E - 2$ if $E$ belongs to a dead arc and to $b_E - 1$ otherwise;*
- *if $E$ is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of $\mathcal{P}^{\mathcal{F}}$ that intersects $E$;*
- *otherwise, no branches of $\mathcal{P}^{\mathcal{F}}$ intersect $E$.*

In the conditions of theorem above, we can obtain the dual graph $G(C \cup \mathcal{P}^{\mathcal{F}})$ from $G(C)$. We only need to add the arrows corresponding to the branches of $\mathcal{P}^{\mathcal{F}}$ as shown in next example (see [27]).




The black arrows represents the branches of $C$. The grey arrows correspond to the branches of a generic polar curve.

In particular, a complete description of the characteristic exponents of the branches of $\mathcal{P}^{\mathcal{F}}$ can be given in terms of the equisingularity data of $C$ as follows

**Lemma 1.5.13** *[28, Lemma 4] Let $C$ be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F}$. Let $\mathcal{P}^{\mathcal{F}} = \cup_{E \in B(C)} P^E$ be the decomposition of $\mathcal{P}^{\mathcal{F}}$ given in Theorem 1.5.10. Then, for each $E \in B(C)$, we have that* *(i) if $E$ is a contact divisor, then the curve $P^E$ has $b_E - 1$ irreducible components. Each irreducible component $\xi$ of $P^E$ has characteristic exponents $\{\nu_0^\xi, \nu_1^\xi, \dots, \nu_{k_E}^\xi\}$ given by*

$$\nu_0^\xi = m_0(\xi) = \underline{n}_E, \quad \nu_\ell^\xi = \underline{n}_E \beta_\ell^i / n^i \qquad \text{for } \ell = 1, 2, \dots, k_E,$$

*where $C_i$ is any irreducible component of $C$ such that $E$ belongs to its geodesic;*

*(ii) if $E$ is a Puiseux divisor which belongs to a dead arc, the curve $P^E$ has one irreducible component $\xi_0$ with characteristic exponents $\{\nu_0^{\xi_0}, \nu_1^{\xi_0}, \dots, \nu_{k_E}^{\xi_0}\}$ given by*

$$\nu_0^{\xi_0} = m_0(\xi_0) = \underline{n}_E, \quad \nu_\ell^{\xi_0} = \underline{n}_E \beta_\ell^i / n^i \qquad \text{for } \ell = 1, 2, \dots, k_E,$$

*and $b_E - 2$ irreducible components such that each branch $\xi$ of $P^E$, $\xi \neq \xi_0$, has characteristic exponents $\{\nu_0^\xi, \nu_1^\xi, \dots, \nu_{k_E}^\xi, \nu_{k_E+1}^\xi\}$ given by*

$$\nu_0^\xi = m_0(\xi) = \underline{n}_E n_E, \quad \nu_\ell^\xi = \underline{n}_E n_E \beta_\ell^i / n^i \qquad \text{for } \ell = 1, 2, \dots, k_E + 1,$$

*where $C_i$ is any irreducible component of $C$ such that $E$ belongs to its geodesic and $\upsilon(E) = \beta_{k_E+1}^i / \beta_0^i$;*

*(iii) if $E$ is a bifurcation divisor which does not belong to a dead arc, then $P^E$ has $b_E - 1$ irreducible components. Each irreducible component $\xi$ of $P^E$ with characteristic exponents $\{\nu_0^\xi, \nu_1^\xi, \dots, \nu_{k_E}^\xi, \nu_{k_E+1}^\xi\}$ given by*

$$\nu_0^\xi = m_0(\xi) = \underline{n}_E n_E, \quad \nu_\ell^\xi = \underline{n}_E n_E \beta_\ell^i / n^i \qquad \text{for } \ell = 1, 2, \dots, k_E + 1,$$

*where $C_i$ is any irreducible component of $C$ such that $E$ belongs to its geodesic and $\upsilon(E) = \beta_{k_E+1}^i / \beta_0^i$.*

Note that hamiltonian foliations $df = 0$ have vector of exponents $\lambda = \underline{1}$ and it can happen that $\underline{1} \notin U_C$. For instance, if we consider $f(x, y) = y(y - x^2)(2y - (1 + \sqrt{3}i)x^2)$, then a generic polar curve $\mathcal{P}^{df}$ is an irreducible curve with a Puiseux pair $(5, 2)$ (see [27] for other examples).

Nevertheless, E. Casas-Alvero proved that it is possible to determine the equisingularity type of generic polar curves of a plane curve $C$ provided that the curve $C$ is generic in its equisingularity class (see [16] and [18, Section 6.6], for the irreducible case and [17] for the general case). Recently, some papers also deal with the study of the equisingularity type of generic polar curves of irreducible plane curves (see [53, 50, 51, 52, 54]).

Moreover, with the hypothesis of Theorem 1.5.12, it is possible to describe the minimal resolution of singularities $\sigma : X \to (\mathbb{C}^2, 0)$ of the polar pencil $\{aA + bB = 0 \ : \ a, b \in \mathbb{C}\}$ of $\mathcal{F}$, that is, $\sigma$ is a morphism which gives a partial reduction of singularities of the polar foliation defined by $d(A/B) = 0$ in the sense that the minimal reduction of singularities $\pi : \mathfrak{X} \to (C^2, 0)$ of the polar foliation factorizes as $\pi = \sigma \circ \tau$ where $\tau : \mathfrak{X} \to X$ is a finite sequence of non-dicritical punctual blow-ups (see [30, Theorem 1]).

## 1.6 Topological properties of jacobian curves of foliations

The aim of this section is to describe properties of the equisingularity type of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ of two singular foliations $\mathcal{F}$ and $\mathcal{G}$ in terms of invariants of the foliations. A first remark is that any curve $h(x,y)=0$ can be the jacobian curve of two non-singular foliations: it is enough to consider the foliations given by $dx=0$ and $dx+h(x,y)dy=0$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two singular foliations defined by the 1-forms $\omega = A(x,y)dx + B(x,y)dy$ and $\eta = P(x,y)dy + Q(x,y)dy$ respectively. The *jacobian curve* $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is the curve of contact of both foliations $\omega \wedge \eta = 0$, that is, the curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is given by $J(x,y)=0$ where

$$J(x,y) = \begin{vmatrix} A(x,y) & B(x,y) \\ P(x,y) & Q(x,y) \end{vmatrix}.$$

Note that if $\mathcal{F}$ and $\mathcal{G}$ have a common separatrix, then this invariant curve is a branch of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$. Hence, from now on, we will assume that $\mathcal{F}$ and $\mathcal{G}$ have no common separatrix.

The topology of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ depends on "how similar" are the foliations $\mathcal{F}$ and $\mathcal{G}$ in terms of its singularities and Camacho-Sad indices at the common singularities. Let us illustrate this fact explaining the behaviour of the multiplicity $m_0(\mathcal{J}_{\mathcal{F},\mathcal{G}})$.

The first topological invariant of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ that we can consider is the multiplicity $m_0(\mathcal{J}_{\mathcal{F},\mathcal{G}})$. Note that $m_0(\mathcal{J}_{\mathcal{F},\mathcal{G}}) \geq \nu_0(\mathcal{F}) + \nu_0(\mathcal{G})$. In particular, $m_0(\mathcal{J}_{\mathcal{F},\mathcal{G}}) > \nu_0(\mathcal{F}) + \nu_0(\mathcal{G})$ implies that the jets $j^{\nu_0(\mathcal{F})}(\omega)$ and $j^{\nu_0(\mathcal{G})}(\eta)$ are proportional since, if we write $J(x,y) = \sum_{\ell \geq \nu_0(\mathcal{F})+\nu_0(\mathcal{G})} J_\ell(x,y)$ with $J_\ell$ homogeneous polynomials of degree $\ell$ or zero, we have that

$$J_{\nu_0(\mathcal{F})+\nu_0(\mathcal{G})}(x,y) = A_{\nu_0(\mathcal{F})}(x,y) Q_{\nu_0(\mathcal{G})}(x,y) - B_{\nu_0(\mathcal{F})}(x,y) P_{\nu_0(\mathcal{G})}(x,y)$$

where $j^{\nu_0(\mathcal{F})}(\omega) = A_{\nu_0(\mathcal{F})}(x,y)dx + B_{\nu_0(\mathcal{F})}(x,y)dy$ and $j^{\nu_0(\mathcal{G})}(\eta) = P_{\nu_0(\mathcal{G})}(x,y)dx + Q_{\nu_0(\mathcal{G})}(x,y)dy$. Consequently the condition $m_0(\mathcal{J}_{\mathcal{F},\mathcal{G}}) > \nu_0(\mathcal{F}) + \nu_0(\mathcal{G})$ also implies that the tangent cones of the foliations $\mathcal{F}$ and $\mathcal{G}$ are equal (see Equation (1.3)). Hence, if we consider the blow-up $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ of the origin, both foliations have the same singularities in $E_1 = \pi^{-1}(0)$ and same Camacho-Sad indices (by Remark 1.2.2 and Example 1.6.8) at these singularities provided that $\pi_1$ is non-dicritical (see Section 1.2). For this reason, if we want that the multiplicity of jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is minimal we need that $\mathcal{F}$ and $\mathcal{G}$ have different Camacho-Sad index at any singular point in the exceptional divisor $E_1$ obtained after one blow-up (see [31, Lemma 2.3]). This condition implies that the divisor $E_1$ is a non-collinear divisor following the notations in [31] (see also Definition 1.6.1 below). These are the type of conditions that we need to impose over $\mathcal{F}$ and $\mathcal{G}$ in order to be able to describe some properties of the topology type of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$. Let us introduce some notations to state the result which gives such description (Theorem 1.6.12 below).

Consider two singular foliations $\mathcal{F}$ and $\mathcal{G}$ in $(\mathbb{C}^2, 0)$ that are non-dicritical generalized curve foliations. Let $C$ and $D$ be the curves of separatrices of $\mathcal{F}$ and $\mathcal{G}$ respectively and assume that $C$ and $D$ have no common branches. As in the study of polar curves of foliations, ramifications and logarithmic models will play a key role in the study of jacobian curves. The strategy used in [31] is to study first the case of $\mathcal{F}$ and $\mathcal{G}$ having separatrices with non-singular irreducible components and the general case is reduced to this one using a ramification. Moreover, logarithmic models will allow to control the influence of Camacho-Sad indices in the behaviour of jacobian curves.

### 1.6.1 The case of non-singular separatrices

Let $C$ and $D$ be two plane curves in $(\mathbb{C}^2, 0)$ without common branches and assume that the curve $Z = C \cup D$ has only non-singular irreducible components. Consider two foliations $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$. Let $\pi_Z : X_Z \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $Z$. Note that $Z$ gives a common reduction of singularities of the foliations $\mathcal{F}$ and $\mathcal{G}$.

We will use the notations introduced in Section 1.4 concerning the dual graph $G(Z)$ of $Z$. Let $E$ be an irreducible component of $\pi_Z^{-1}(0)$ an denote by $\pi_E : X_E \to (\mathbb{C}^2, 0)$ the morphism reduction of $\pi_Z$ to $E$. Denote $b_E = b_E^Z$ the number associated to each divisor $E$ of $G(Z)$ and let $\{R_1^E, R_2^E, \ldots, R_{b_E}^E\}$ be the infinitely near points of $Z$ in $E$, that is, $\pi_E^* Z \cap E_{red} = \{R_1^E, R_2^E, \ldots, R_{b_E}^E\}$ (see Remark 1.4.6). We denote

$$\Delta_E(R_i^E) = \Delta_E^{\mathcal{F},\mathcal{G}}(R_i^E) = \mathcal{I}_{R_i^E}(\pi_E^*\mathcal{G}, E_{red}) - \mathcal{I}_{R_i^E}(\pi_E^*\mathcal{F}, E_{red})$$

where $\mathcal{I}_{R_i^E}(\pi_E^*\mathcal{G}, E_{red})$ is the Camacho-Sad index of $\pi_E^*\mathcal{G}$ relative to $E_{red}$ at the point $R_i^E$ (whose definition was given in Section 1.2). Let us introduce some definitions which will be essential in the rest of the section (see [31, Section 4] for more details).

**Definition 1.6.1** [31, Definition 4.2] An infinitely near point $R_\ell^E$ of $Z$ is a *collinear point* for the foliations $\mathcal{F}$ and $\mathcal{G}$ in $E$ if $\Delta_E(R_\ell^E) = 0$; otherwise $R_\ell^E$ is a *non-collinear point*.

A divisor $E$ is *collinear* (for the foliations $\mathcal{F}$ and $\mathcal{G}$) if $\Delta_E(R_\ell^E) = 0$ for all $\ell = 1, 2, \ldots, b_E$; otherwise $E$ is a *non-collinear divisor*. A divisor $E$ is *purely non-collinear* if $\Delta_E(R_\ell^E) \neq 0$ for all $\ell \in \{1, 2, \ldots, b_E\}$.

The set of collinear points (resp. non-collinear points) of $E$ will be denoted by $\mathrm{Col}(E)$ (resp. $\mathrm{NCol}(E)$). Note that $\mathrm{Col}(E) \cup \mathrm{NCol}(E) = \{R_1^E, R_2^E, \ldots, R_{b_E}^E\}$. In [31, Section 4.1], it is explained the behaviour of collinear and non-collinear infinitely near points by blow-up.

*Remark 1.6.2* The definitions above coincide with the notions given by Kuo and Parusiński in [56, 57] in the case of curves (see [31, Section 7.1]).

Let $E$ be an irreducible component of $\pi_Z^{-1}(0)$ with $\upsilon(E) = p$. We can take coordinates $(x, y)$ in $(\mathbb{C}^2, 0)$ and $(x_p, y_p)$ in the first chart of $E_{red}$ such that $\pi_E(x_p, y_p) = (x_p, x_p^p y_p)$ and $E_{red} = (x_p = 0)$. The coordinates $(x, y)$ are called *coordinates in* $(\mathbb{C}^2, 0)$ *adapted to* $E$ (see [31, Remark 4.1]). In these coordinates, the infinitely near points of $C$ in $E_{red}$ are given by $R_\ell^E = (0, c_\ell^E)$, $\ell = 1, 2, \ldots, b_E$. We define the *rational function* $\mathcal{M}_E(z) = \mathcal{M}_E^{\mathcal{F},\mathcal{G}}(z)$ *associated to the divisor* $E$ (*for the foliations* $\mathcal{F}$ *and* $\mathcal{G}$ by

$$\mathcal{M}_E(z) = \sum_{\ell=1}^{b_E} \frac{\Delta_E(R_\ell^E)}{z - c_\ell^E}.$$

We have that

- if $E$ is a non-collinear divisor, then $\mathcal{M}_E(z) \not\equiv 0$;
- if $\mathcal{L}_{\lambda,f}$, $\mathcal{L}_{\mu,g}$ are the logarithmic models of $\mathcal{F}$, $\mathcal{G}$ respectively, then

$$\Delta_E^{\mathcal{F},\mathcal{G}}(R_\ell^E) = \Delta_E^{\mathcal{L}_{\lambda,f},\mathcal{L}_{\mu,g}}(R_\ell^E); \qquad \mathcal{M}_E^{\mathcal{F},\mathcal{G}}(z) = \mathcal{M}_E^{\mathcal{L}_{\lambda,f},\mathcal{L}_{\mu,g}}(z).$$

Let $\{q_1, \ldots, q_{s_E}\}$ be the set of zeros of $\mathcal{M}_E(z)$. We denote by $M(E) = \{Q_1^E, \ldots, Q_{s_E}^E\}$ the set of points in $E_{red}$ given by $Q_\ell^E = (0, q_\ell)$ in coordinates $(x_p, y_p)$. If $t_{Q_\ell^E}$ is the multiplicity of $q_\ell$ as a zero of $\mathcal{M}_E(z)$, we put $t(E) = \sum_{\ell=1}^{s_E} t_{Q_\ell^E}$ the degree of the numerator of the rational function $\mathcal{M}_E(z)$. We put $t_P = 0$ for any $P \in E \setminus M(E)$. Note that $M(E)$ can be an empty set (see example in [57, p. 584]). Moreover, if $E$ is a non-collinear divisor, we have ([31, Lemma 4.6])

(i) $\mathrm{NCol}(E) \cap M(E) = \emptyset$
(ii) $\sharp\mathrm{NCol}(E) \geq 1 + t(E)$.
(iii) if $\sum_{R_\ell^E \in \mathrm{NCol}(E)} \Delta_E(R_\ell^E) \neq 0$, then the equality in (ii) holds.

Consider a non-collinear divisor $E$ and a point $P \in E_{red}$, we put

$$\tau_E(P) = \begin{cases} t_P, & \text{if } P \in M(E) \\ -1, & \text{if } P \in \mathrm{NCol}(E) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{P \in E_{red}} \tau_E(P) = t(E) - \sharp\mathrm{NCol}(E)$ is a negative integer which coincides with the degree of the rational function $\mathcal{M}_E(z)$.

If $E$ is a non-collinear divisor with $\upsilon(E) = p$ and $(x, y)$ are adapted coordinates to $E$, the weighted initial form $\mathrm{In}_p(J; x, y)$, where $J(x, y) = 0$ is an equation of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$, can be determined from the weighted initial forms $\mathrm{In}_p(\omega)$ and $\mathrm{In}_p(\eta)$ of the 1-forms defining $\mathcal{F}$ and $\mathcal{G}$ respectively (see [31, Lema 4.13]). This is a key property to determine the points $\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}} \cap E_{red}$.

Hence, with the notations stated in this section, we can determine the infinitely near points of the jacobian curve in a non-collinear divisor. More precisely, if we denote by $E_{red}^*$ the points in the first chart of $E_{red}$ (that is, all the points of $E_{red}$ except the corner one), we have

**Theorem 1.6.3** *[31, Theorem 5.2] Let $\mathcal{F}$ and $\mathcal{G}$ be two non-dicritical generalized curve foliations with $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$. Assume that $Z = C \cup D$ has non-singular irreducible components. Let $E$ be a non-collinear divisor of $\pi_Z^{-1}(0)$. Given any $P \in E_{red}^*$, we have that*

$$m_P(\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) = m_P(\pi_E^* C) + m_P(\pi_E^* D) + \tau_E(P).$$

*In particular, if $P \in E^*_{red}$ with $m_P(\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) > 0$, then $P$ is either an infinitely near point of $Z$ or a point in $M(E)$.*

To prove this result, we first prove it for logarithmic foliations and the general case for $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$ is consequence of the facts ([31, Lemma 4.16])

$$\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}} \cap E_{red} = \pi_E^* \mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{L}_{\mu,g}} \cap E_{red}$$

and

$$m_P(\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) = m_P(\pi_E^* \mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{L}_{\mu,g}}) \text{ at each } P \in \pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}} \cap E_{red},$$

when $E$ is a non-collinear divisor and $\mathcal{L}_{\lambda,f}$, $\mathcal{L}_{\mu,g}$ are the logarithmic models of $\mathcal{F}$, $\mathcal{G}$ respectively.

Consider now two consecutive bifurcation divisors $E$ and $E'$ in $G(Z)$, this means that there exists a chain of consecutive divisors

$$E_0 = E < E_1 < \cdots E_{k-1} < E_k = E'$$

with $b_{E_\ell} = 1$ for $\ell = 1, \ldots, k-1$ and the morphism $\pi_{E'} = \pi_E \circ \sigma$ with $\sigma : X_{E'} \to (X_E, P)$ a composition of $k$ punctual blow-ups

$$(X_E, P) \overset{\sigma_1}{\longleftarrow} (X_{E_1}, P_1) \overset{\sigma_2}{\longleftarrow} \cdots \overset{\sigma_{k-1}}{\longleftarrow} (X_{E_{k-1}}, P_{k-1}) \overset{\sigma_k}{\longleftarrow} X_{E'}. \tag{1.12}$$

When $E$ and $E'$ are two consecutive bifurcation divisors as above, we say that $E'$ *arises from $E$ at $P$* and we denote $E <_P E'$. Next result shows that, if $\delta$ is a branch of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ going through a non-collinear point $P$ in a bifurcation divisor, then $\delta$ goes also through the points $P_1, P_2, \ldots, P_{k-1}$ of the sequence (1.12), that is, the divisor $E'$ is in the geodesic of the branches of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ going through $P$ in $E_{red}$. More precisely, we have

**Corollary 1.6.4** *[31, Corollary 5.4] Let $E$ and $E'$ be two consecutive bifurcation divisors in $G(Z)$ with $E <_P E'$. If $P \in \mathrm{NCol}(E)$, we have that*

$$m_P(\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) = \sum_{Q \in E'_{red}} m_Q(\pi_{E'}^* \mathcal{J}_{\mathcal{F},\mathcal{G}}).$$

*In particular, there is no irreducible component $\delta$ of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ such that $\pi_{E'}^* \delta$ is attached to some intermediate component $E_i$, $1 \leq i \leq k-1$, in the chain $E < E_1 < \cdots < E_{k-1} < E'$. Moreover, we have*

$$1 + \sum_{Q \in M(E')} t_Q = \sharp\, \mathrm{NCol}(E'), \tag{1.13}$$

*and hence $E'$ is a non-collinear divisor.*

From the previous result, we get that there is no irreducible component $\delta$ of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ such that

$$\upsilon(E) < C(\delta, \gamma_{E'}) < \upsilon(E')$$

where $\gamma_{E'}$ is a $E'$-curvette.

Let us now explain the behaviour of the branches of the jacobian curve going through a collinear point. We need to introduce the notion of cover of a divisor at a point.

**Definition 1.6.5** [31, Definition 5.6] Let $E$ be a bifurcation divisor of $G(Z)$ and take a collinear point $P$ of $E$. A set of non-collinear bifurcation divisors $\{E_1, \ldots, E_u\}$ is a *(non-collinear) cover* of $E$ at $P$ if:

(i) $E$ is in the geodesic of each $E_\ell$;
(ii) if $\{E_1^\ell, \ldots, E_{r(\ell)}^\ell\}$ is the set of all bifurcation divisors in the geodesic of $E_\ell$ with

$$E <_P E_1^\ell < \ldots < E_{r(\ell)}^\ell < E_\ell$$

then either $r(\ell) = 0$ or else each $E_j^\ell$ is collinear;
(iii) if $Z_j$ is an irreducible component of $Z$ with $\pi_E^* Z_j \cap E_{red} = \{P\}$, then there exists a divisor $E_\ell$ in the cover such that $\pi_{E_\ell}^* Z_j \cap E_\ell \neq \emptyset$, that is, there is a divisor $E_\ell$ in the cover which is in the geodesic of $Z_j$.

Note that, given a collinear point $P$ in a divisor $E$, there is a unique cover of $E$ at $P$. To find it we take an irreducible component $Z_\ell$ of $Z$ with $\pi_E^* Z_\ell \cap E_{red} = \{P\}$. Let $E'$ be the first bifurcation divisor after $E$ in the geodesic of $Z_\ell$ (necessarily $E <_P E'$). If $E'$ is a non-collinear divisor, then $E'$ is one of the divisors of the cover of $E$ at $P$. Otherwise, we repeat the process (see [31] for more details). Then we have

**Theorem 1.6.6** *[31, Theorem 5.7] Consider a non-collinear bifurcation divisor $E$ of $G(Z)$ and a collinear point $P$ of $E$. Let $\{E_1, \ldots, E_u\}$ be a cover of $E$ at $P$. Then*

$$m_P(\pi_E^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) - \sum_{\ell=1}^{u} \sum_{Q \in E_{\ell,red}} m_Q(\pi_{E_\ell}^* \mathcal{J}_{\mathcal{F},\mathcal{G}}) = t_P + \sum_{\ell=1}^{u} (\sharp\, \mathrm{NCol}(E_\ell) - t(E_\ell)).$$

*Consequently, there is a curve $J_P^E$ composed by irreducible components of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ such that, if $\delta$ is a branch of $J_P^E$,*

- $\pi_E^* \delta \cap E_{red} = \{P\}$
- $C(\delta, \gamma_{E_\ell}) < \upsilon(E_\ell)$ *for $\ell = 1, \ldots, u$, where $\gamma_{E_\ell}$ is any $E_\ell$-curvette.*

*Moreover, we have that*

$$m_0(J_P^E) = t_P + \sum_{\ell=1}^{u} (\sharp\, \mathrm{NCol}(E_\ell) - t(E_\ell)).$$

The results in this section allow to give a decomposition of the jacobian curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ in the sense of the decomposition theorem for polar curves. Given any bifurcation divisor $E \in B(Z)$ which is a non-collinear divisor for $\mathcal{F}$ and $\mathcal{G}$, we define $J_{nc}^E$ as the union of the branches $\xi$ of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ such that

- $\pi_E^*\xi \cap \pi_E^* Z = \emptyset$;
- if $E' < E$, then $\pi_E^*\xi \cap \pi'_E(E') = \emptyset$;
- if $E' > E$, then $\pi_{E'}^*\xi \cap E'_{red} = \emptyset$.

For a non-collinear bifurcation divisor $E$, we denote $J_c^E = \cup_{P\in \mathrm{Col}(E)} J_P^E$ (with $J_c^E = \emptyset$ if $\mathrm{Col}(E) = \emptyset$).

Given a non-collinear bifurcation divisor $E$ of $G(Z)$, we denote

$$t^*(E) = \sum_{Q\in M(E)\smallsetminus \mathrm{Col}(E)} t_Q,$$

that is, the number of zeros of $\mathcal{M}_E(z)$ (counting with multiplicities) which do not correspond to collinear points. The previous results allow to give a decomposition of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ as follows:

**Theorem 1.6.7** *[31, Theorem 5.8] Consider $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$ such that $Z = C\cup D$ is a curve with only non-singular irreducible components. Let $B_N(Z)$ be the set of non-collinear bifurcation divisors of $G(Z)$. Then there is a unique decomposition $\mathcal{J}_{\mathcal{F},\mathcal{G}} = J^* \cup (\cup_{E\in B_N(Z)} J^E)$ where $J^E = J_{nc}^E \cup J_c^E$ with the following properties*

*(1) $m_0(J_{nc}^E) = t^*(E)$. In particular, $m_0(J_{nc}^E) \le \sharp\,\mathrm{NCol}(E) - 1 \le b_E - 1$;*
*(2) $\pi_E^* J_{nc}^E \cap \pi_E^* Z = \emptyset$;*
*(3) if $E' < E$, then $\pi_E^* J_{nc}^E \cap \pi'_E(E') = \emptyset$;*
*(4) if $E' > E$, then $\pi_{E'}^* J_{nc}^E \cap E'_{red} = \emptyset$;*
*(5) if $\delta$ is a branch of $J_c^E$, then $\pi_E^*\delta \cap E_{red}$ is a point in $\mathrm{Col}(E)$;*
*(6) $m_0(J_c^E) = \sum_{P\in C(E)}(t_P + \sum_{\ell=1}^{u(P)}(\sharp\,\mathrm{NCol}(E_\ell^P) - t(E_\ell^P)))$ where $\{E_1^P,\dots,E_{u(P)}^P\}$ is a cover of $E$ at $P$.*

*Moreover, if $E$ is a purely non-collinear divisor with $\sum_{R_\ell^E\in \mathrm{NCol}(E)} \Delta_E(R_\ell^E) \neq 0$, then*

$$m_0(J^E) = m_0(J_{nc}^E) = b_E - 1. \tag{1.14}$$

Note that, in the decomposition above, there is a certain control of the topology of the irreducible components of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ in terms of the data of the foliations $\mathcal{F}$ and $\mathcal{G}$ when the component of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is attached either to a non-collinear divisor or to a chain of collinear divisors which are between two non-collinear bifurcation divisors. The irreducible components corresponding to $J^*$ are the one attached to “isolated” collinear divisors for which no control is possible.

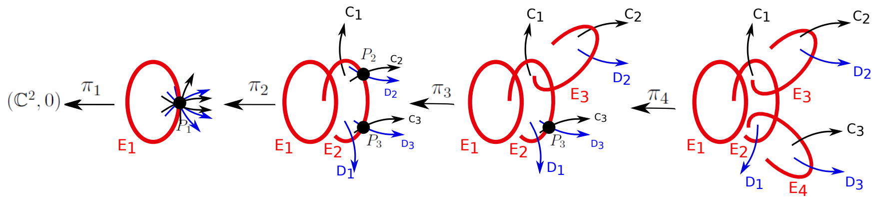
Let us give some examples where we describe different possibilities of decomposition of the jacobian curve although the curves of separatrices of the foliations are fixed.

*Example 1.6.8* Consider the curves $C = C_1 \cup C_2 \cup C_3$ and $D = D_1 \cup D_2 \cup D_3$ with $C = (f = 0)$ and $D = (g = 0)$ where

$$f = (y - 2x^2)(y - x^2 + x^3)(y + x^2 + x^3),$$

$$g = (y + 2x^2)(y - x^2 - x^3)(y + x^2 - x^3).$$

The reduction of singularities of the curve $C \cup D$ is given by




The jacobian curve $\mathcal{J}_{\mathcal{G}_f,\mathcal{G}_g}$ has three branches given by

$$\mathcal{J}_{\mathcal{G}_f,\mathcal{G}_g}: \quad \begin{cases} x = 0 \\ y = x^2 + ax^{5/2} + \cdots \\ y = -x^2 + bx^{5/2} + \cdots \end{cases}$$

If we consider the logarithmic foliation $\mathcal{L}_{\lambda,f}$ given by

$$f_1 f_2 f_3 \sum_{i=1}^{3} \lambda_i \frac{df_i}{f_i} = 0$$

with $\lambda_1 = i, \lambda_2 = 1 - 2i, \lambda_3 = 2 + i$, we get that the jacobian curve $\mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{G}_g}$ has five branches given by

$$\mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{G}_g}: \quad \begin{cases} x = 0 \\ y = \alpha x^2 + \cdots \\ y = \beta x^2 + \cdots \\ y = x^2 + \gamma x^3 + \cdots \\ y = -x^2 + \delta x^3 + \cdots \end{cases}$$

Let us compute the Camacho-Sad indices of the foliations above. Consider the blow-up of the origin $\pi_1 : X_1 \to (\mathbb{C}^2, 0)$ with $E_1 = \pi^{-1}(0)$ and put $\{P_1\} = \pi_1^* C \cap E_1 = \pi_1^* D \cap E_1$. Then $\mathcal{I}_{P_1}(\pi_1^* \mathcal{G}_f, E_1) = \mathcal{I}_{P_1}(\pi_1^* \mathcal{G}_g, E_1) = \mathcal{I}_{P_1}(\pi_1^* \mathcal{L}_{\lambda,f}, E_1) = -1$.

In particular, this implies all the foliations $\mathcal{G}_f$, $\mathcal{G}_g$ $\mathcal{L}_{\lambda,f}$ have the same tangent cone and that the divisor $E_1$ is a collinear divisor for $\mathcal{G}_f$ and $\mathcal{G}_g$ (and also for $\mathcal{L}_{\lambda,f}$ and $\mathcal{G}_g$). Moreover, note that $\nu_0(\mathcal{G}_f) = \nu_0(\mathcal{G}_g) = \nu_0(\mathcal{L}_{\lambda,f}) = 2$ but $m_0(\mathcal{J}_{\mathcal{G}_f,\mathcal{G}_g}) = m_0(\mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{G}_g}) = 5$.

Take now $\pi_2 : X_2 \to (X_1, P_1)$ the blow-up with center at $P_1$ and put $\sigma_2 = \pi_1 \circ \pi_2$. Denote $R_1^{E_2}$ the infinitely near point of $C_1$ in $E_2$, $R_2^{E_2}$ the infinitely near point of $D_1$ in $E_2$ and $R_3^{E_2} = P_2$ the point where the branches $C_2$ and $D_2$ intersect $E_2$ and $R_4^{E_2} = P_3$ the point where the branches $C_3$ and $D_3$ meet $E_2$. For the logarithmic foliation $\mathcal{L}_{\lambda,f}$ we have that (see [26, Proposition 4.4])

$$\mathcal{I}_{R_1^{E_2}}(\sigma_2^*\mathcal{L}_{\lambda,f},E_2)=-\tfrac{\lambda_1}{2(\lambda_1+\lambda_2+\lambda_3)}=-\tfrac{i}{6};$$
$$\mathcal{I}_{R_2^{E_2}}(\sigma_2^*\mathcal{L}_{\lambda,f},E_2)=0;$$
$$\mathcal{I}_{R_3^{E_2}}(\sigma_2^*\mathcal{L}_{\lambda,f},E_2)=-\tfrac{\lambda_2}{2(\lambda_1+\lambda_2+\lambda_3)}=-\tfrac{1-2i}{6};$$
$$\mathcal{I}_{R_4^{E_2}}(\sigma_2^*\mathcal{L}_{\lambda,f},E_2)=-\tfrac{\lambda_3}{2(\lambda_1+\lambda_2+\lambda_3)}=-\tfrac{2+i}{6}.$$

Taking into account that hamiltonian foliations are logarithmic foliations with exponent vector equal to $\lambda(\mathcal{G}_f)=\lambda(\mathcal{G}_g)=(1,1,1)$, similar computations give that

$$\mathcal{I}_{R_1^{E_2}}(\sigma_2^*\mathcal{G}_f,E_2)=\mathcal{I}_{R_3^{E_2}}(\sigma_2^*\mathcal{G}_f,E_2)=\mathcal{I}_{R_4^{E_2}}(\sigma_2^*\mathcal{G}_f,E_2)=-\tfrac{1}{6};\qquad \mathcal{I}_{R_2^{E_2}}(\sigma_2^*\mathcal{G}_f,E_2)=0;$$

$$\mathcal{I}_{R_1^{E_2}}(\sigma_2^*\mathcal{G}_g,E_2)=0;\qquad \mathcal{I}_{R_2^{E_2}}(\sigma_2^*\mathcal{G}_g,E_2)=\mathcal{I}_{R_3^{E_2}}(\sigma_2^*\mathcal{G}_g,E_2)=\mathcal{I}_{R_4^{E_2}}(\sigma_2^*\mathcal{G}_g,E_2)=-\tfrac{1}{6}.$$

Consider now the blow-up $\pi_3:X_3\to(X_2,P_2)$ with center at $P_2$ and let $R_1^{E_3}$ (resp. $R_2^{E_3}$) be the point where $C_2$ (resp. $D_2$ intersects $E_3$). Put $\sigma_3=\sigma_2\circ\pi_3$, then

$$\mathcal{I}_{R_1^{E_3}}(\sigma_3^*\mathcal{L}_{\lambda,f},E_3)=-\tfrac{\lambda_2}{2(\lambda_1+\lambda_3)+3\lambda_2}=-\tfrac{1-2i}{7-2i};\qquad \mathcal{I}_{R_2^{E_3}}(\sigma_3^*\mathcal{L}_{\lambda,f},E_3)=0;$$
$$\mathcal{I}_{R_1^{E_3}}(\sigma_3^*\mathcal{G}_f,E_3)=-\tfrac{1}{7};\qquad \mathcal{I}_{R_2^{E_3}}(\sigma_3^*\mathcal{G}_f,E_3)=0;$$
$$\mathcal{I}_{R_1^{E_3}}(\sigma_3^*\mathcal{G}_g,E_3)=0;\qquad \mathcal{I}_{R_2^{E_3}}(\sigma_3^*\mathcal{G}_g,E_3)=-\tfrac{1}{7}.$$

Finally, if $\pi_4:X_4\to(X_3,P_3)$ is the blow-up with center at $P_3$, the point $R_1^{E_4}$ (resp. $R_2^{E_4}$) is intersection of $C_3$ (resp. $D_3$) with $E_4$ and we put $\sigma_4=\sigma_3\circ\pi_4$, then

$$\mathcal{I}_{R_1^{E_4}}(\sigma_4^*\mathcal{L}_{\lambda,f},E_4)=-\tfrac{\lambda_3}{2(\lambda_1+\lambda_2)+3\lambda_3}=-\tfrac{2+i}{8+i};\qquad \mathcal{I}_{R_2^{E_4}}(\sigma_4^*\mathcal{L}_{\lambda,f},E_4)=0;$$
$$\mathcal{I}_{R_1^{E_4}}(\sigma_4^*\mathcal{G}_f,E_4)=-\tfrac{1}{7};\qquad \mathcal{I}_{R_2^{E_4}}(\sigma_4^*\mathcal{G}_f,E_4)=0;$$
$$\mathcal{I}_{R_1^{E_4}}(\sigma_4^*\mathcal{G}_g,E_4)=0;\qquad \mathcal{I}_{R_2^{E_4}}(\sigma_4^*\mathcal{G}_g,E_4)=-\tfrac{1}{7}.$$

For the foliations $\mathcal{G}_f$ and $\mathcal{G}_g$ we have that $E_1$ is a collinear divisor, $E_2,E_3,E_4$ are non-collinear bifurcation divisors. In the divisor $E_2$, the points $R_1^{E_2}$ and $R_2^{E_2}$ are non-collinear points while $R_3^{E_2}$ and $R_4^{E_2}$ are collinear points. The points $R_1^{E_3},R_2^{E_3},R_1^{E_4},R_2^{E_4}$ are non-collinear. Let us compute the rational function associated to the non-collinear divisors for $\mathcal{G}_f$ and $\mathcal{G}_g$. We have that

$$\mathcal{M}_{E_2}(z)=\frac{1}{6}\frac{1}{z-2}-\frac{1}{6}\frac{1}{z+2}=\frac{2}{3(z^2-4)};$$
$$\mathcal{M}_{E_3}(z)=\mathcal{M}_{E_4}(z)=\frac{1}{7}\frac{1}{z+1}-\frac{1}{7}\frac{1}{z-1}=-\frac{2}{7(z^2-1)}.$$

Hence $M(E_2)=M(E_3)=M(E_4)=\emptyset$ and then in the decomposition of $\mathcal{J}_{\mathcal{G}_f,\mathcal{G}_g}$ we have that $J_{nc}^{E_2}=J_{nc}^{E_3}=J_{nc}^{E_4}=\emptyset$. We have that $\mathcal{J}_{\mathcal{G}_f,\mathcal{G}_g}=J^*\cup J_c^{E_2}$ where

$J^* = (x = 0)$ and $J_c^{E_2}$ is the union of the other two branches. Note that $m_0(J_c^{E_2}) = \sharp\mathrm{NCol}(E_3) + \sharp\mathrm{NCol}(E_4) = 4$ as shown in (6) of Theorem 1.6.7.

For the foliations $\mathcal{L}_{\lambda,f}$ and $\mathcal{G}_g$ we have that $E_1$ is a collinear divisor, $E_2, E_3, E_4$ are purely non-collinear bifurcation divisors since all the infinitely near points of $C \cup D$ in these divisors are non-collinear points. Let us compute the rational function associated to the non-collinear divisors for $\mathcal{L}_{\lambda,f}$ and $\mathcal{G}_g$. We have that

$$\begin{aligned}
\mathcal{M}_{E_2}(z) &= \frac{i}{6}\frac{1}{z-2} - \frac{1}{6}\frac{1}{z+2} + \left(-\frac{1}{6} + \frac{1-2i}{6}\right)\frac{1}{z-1} + \left(-\frac{1}{6} + \frac{2+i}{6}\right)\frac{1}{z+1} \\
&= \frac{(1-i)z^2 + 3(i-1)z + 10i + 2}{6(z^2-4)(z^2-1)}; \\
\mathcal{M}_{E_3}(z) &= \frac{1-2i}{7-2i}\frac{1}{z+1} - \frac{1}{7}\frac{1}{z-1} = \frac{-12iz + 16i - 14}{7(7-2i)(z^2-1)}; \\
\mathcal{M}_{E_4}(z) &= \frac{2+i}{8+i}\frac{1}{z+1} - \frac{1}{7}\frac{1}{z-1} = \frac{6(i+1)z - 2(4i+11)}{7(7+i)(z^2-1)}.
\end{aligned}$$

Now we have that the decomposition of the jacobian curve is given by $\mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{G}_g} = J^* \cup J_{nc}^{E_2} \cup J_{nc}^{E_3} \cup J_{nc}^{E_4}$ with $J^* = (x = 0)$, the curve $J_{nc}^{E_2}$ is the union of the branches $y = \alpha x^2 + \cdots$ and $y = \beta x^2 + \ldots$ and $J_{nc}^{E_3}$ and $J_{nc}^{E_4}$ are non-singular curves which correspond to the other two branches of $\mathcal{J}_{\mathcal{L}_{\lambda,f},\mathcal{G}_g}$.

### 1.6.2 Jacobian curve: general case

The strategy to prove the decomposition result of $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ in the general case is to consider a ramification such that $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$ are foliations which satisfy the hypothesis of the previous section. First, let us study the relationship between the curves $\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}} = \rho^{-1}\mathcal{J}_{\mathcal{F},\mathcal{G}}$ and $\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}}$.

Consider $\mathcal{F}$ and $\mathcal{G}$ two foliations in $(\mathbb{C}^2, 0)$ given by $\omega = 0$ and $\eta = 0$ respectively, with

$$\omega = A(x,y)dx + B(x,y)dy; \qquad \eta = P(x,y)dx + Q(x,y)dy.$$

If $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a ramification given in coordinates by $\rho(u,v) = (u^n, v)$, then the foliations $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$ are given by $\rho^*\omega = 0$ and $\rho^*\eta = 0$ respectively, where

$$\rho^*\omega = A(u^n,v)nu^{n-1}du + B(u^n,v)dv; \qquad \rho^*\eta = P(u^n,v)nu^{n-1}du + Q(u^n,v)dv.$$

Then the curve $\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is defined by $J(x,y) = 0$, with $J(x,y) = A(x,y)Q(x,y) - B(x,y)P(x,y)$, and consequently, the curve $\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}} = \rho^{-1}\mathcal{J}_{\mathcal{F},\mathcal{G}}$ is given by $J(u^n, v) = 0$ while the curve $\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}}$ is given by $nu^{n-1}J(u^n,v) = 0$.

Assume that $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$ with $C = \cup_{i=1}^r C_i$ and $D = \cup_{j=1}^s D_j$ two plane curves in $(\mathbb{C}^2, 0)$ that can have singular branches. Denote $Z = C \cup D$ and take $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ a $Z$-transversal ramification such that $\widetilde{Z} = \rho^{-1}Z$ has non-singular irreducible components. Let us study the relationship between the infinitely near points of $\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}}$ and $\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}}$ which appear when we consider the reduction of singularities of the curve $\widetilde{Z}$.

**Lemma 1.6.9** *[31, Lemma 6.1] Let $\pi_{\widetilde{Z}} : X_{\widetilde{Z}} \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\widetilde{Z}$. Take $\widetilde{E}$ an irreducible component of $\pi_{\widetilde{Z}}^{-1}(0)$ and consider $\pi_{\widetilde{E}} : X_{\widetilde{E}} \to (\mathbb{C}^2, 0)$ the morphism reduction of $\pi_{\widetilde{Z}}$ to $\widetilde{E}$. Then we have that*

$$\pi^*_{\widetilde{E}}\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}} \cap \widetilde{E}^*_{red} = \pi^*_{\widetilde{E}}\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}} \cap \widetilde{E}^*_{red},$$

*where $\widetilde{E}^*_{red}$ denote the points in the first chart of $\widetilde{E}_{red}$. Moreover, for each $P \in \pi^*_{\widetilde{E}}\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}} \cap \widetilde{E}^*_{red}$, we have*

$$m_P(\pi^*_{\widetilde{E}}\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}}) = m_P(\pi^*_{\widetilde{E}}\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}}).$$

In particular, if we consider a non-collinear divisor $\widetilde{E}$ of $\pi_{\widetilde{Z}}^{-1}(0)$ for the foliations $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$, the curve $\mathcal{J}_{\rho^*\mathcal{F},\rho^*\mathcal{G}}$ satisfies Theorem 1.6.3 with respect to the curve $\widetilde{Z} = \rho^{-1}Z$. Thus, the previous lemma implies the following result

**Corollary 1.6.10** *[31, Corollary 6.2] Take $\widetilde{E}$ an irreducible component of $\pi_{\widetilde{Z}}^{-1}(0)$ which is a non-collinear divisor for the foliations $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$. Given any $P \in \widetilde{E}^*_{red}$, we have that*

$$m_P(\pi^*_{\widetilde{E}}\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}}) = m_P(\pi^*_{\widetilde{E}}\widetilde{C}) + m_P(\pi^*_{\widetilde{E}}\widetilde{D}) + \tau_{\widetilde{E}}(P).$$

*In particular, if $P \in \widetilde{E}^*_{red}$ with $m_P(\pi^*_{\widetilde{E}}\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}}) > 0$, then $P$ is an infinitely near point of $\widetilde{Z}$ or a point in $M(\widetilde{E})$.*

The relationship among the bifurcation divisors of $G(Z)$ and the ones of $G(\widetilde{Z})$, the infinitely near points of $Z$ and $\widetilde{Z}$ and the Camacho-Sad indices of $\mathcal{F}$, $\mathcal{G}$ and the ones of $\rho^*\mathcal{F}$, $\rho^*\mathcal{G}$ allows to give the following definition (see Section 1.4.3 and [31, Appendix A.2 and A3])

**Definition 1.6.11** A bifurcation divisor $E$ of $G(Z)$ is *collinear (resp. non-collinear)* for the foliations $\mathcal{F}$ and $\mathcal{G}$ when any of its associated divisors $\widetilde{E}^\ell$ is collinear (resp. non-collinear) for the foliations $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$.

An infinitely near point $R^E$ of $Z$ in $E_{red}$ is a *collinear point* (resp. *non-collinear point*) for the foliations $\mathcal{F}$ and $\mathcal{G}$ if for any associated divisor $\widetilde{E}^\ell$ and any infinitely near point $R_t^{\widetilde{E}^\ell}$ of $\rho^{-1}Z$ in $\widetilde{E}^\ell_{red}$ associated to $R^E$, the point $R_t^{\widetilde{E}^\ell}$ is collinear (resp. non-collinear) for the foliations $\rho^*\mathcal{F}$ and $\rho^*\mathcal{G}$.

By Corollary 1.6.10, we get a decomposition of $\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}}$ as the one given in Theorem 1.6.7. Thus we can write

$$\widetilde{\mathcal{J}}_{\mathcal{F},\mathcal{G}} = \widetilde{J}^* \cup (\cup_{\widetilde{E}\in B_N(\widetilde{Z})} J^{\widetilde{E}})$$

with $J^{\widetilde{E}} = J^{\widetilde{E}}_{nc} \cup J^{\widetilde{E}}_{c}$.

In order to obtain the decomposition result for $\mathcal{J}_{\mathcal{F},\mathcal{G}}$, given a non-collinear bifurcation divisor $E$ of $G(Z)$, we define $J^E = J^E_{nc} \cup J^E_c$ to be the curve with

$$\rho^{-1} J^E_{nc} = \bigcup_{\ell=1}^{\underline{n}_E} J^{\widetilde{E}^\ell}_{nc}; \qquad \rho^{-1} J^E_{c} = \bigcup_{\ell=1}^{\underline{n}_E} J^{\widetilde{E}^\ell}_{c},$$

where $\{\widetilde{E}^\ell\}_{\ell=1}^{\underline{n}_E}$ are the divisors of $G(\widetilde{Z})$ associated to $E$ and $J^*$ to be such that $\rho^{-1} J^* = \widetilde{J}^*$. Hence, the decomposition theorem for the jacobian curve can be stated as follows

**Theorem 1.6.12** *[31, Theorem 6.4] Consider two generalized curve foliations $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{G} \in \mathbb{G}_D$ and denote $Z = C \cup D$. Let us write $Z = \cup_{i=1}^{r+s} Z_i$ with $Z_i$ irreducible and denote by $B_N(Z)$ the set of non-collinear bifurcation divisors of $G(Z)$. Then there is a decomposition*

$$\mathcal{J}_{\mathcal{F},\mathcal{G}} = J^* \cup (\cup_{E \in B_N(Z)} J^E)$$

*with $J^E = J^E_{nc} \cup J^E_c$ such that*

*(i)* $m_0(J^E_{nc}) \leq \begin{cases} \underline{n}_E n_E (b_E - 1), & \text{if } E \text{ does not belong to a dead arc;} \\ \underline{n}_E n_E (b_E - 1) - \underline{n}_E, & \text{otherwise.} \end{cases}$

*(ii) For each irreducible component $\delta$ of $J^E_{nc}$ we have that*

- *$C(\delta, Z_i) = \upsilon(E)$ if $E$ belongs to the geodesic of $Z_i$;*
- *$C(\delta, Z_j) = C(Z_i, Z_j)$ if $E$ belongs to the geodesic of $Z_i$ but not to the one of $Z_j$.*

*(iii) For each irreducible component $\delta$ of $J^E_c$, there exists an irreducible component $Z_i$ of $Z$ such that $E$ belongs to its geodesic and*

$$C(\delta, Z_i) > \upsilon(E).$$

*Moreover, if $E'$ is the first non-collinear bifurcation divisor in the geodesic of $Z_i$ after $E$, then*

$$C(\delta, Z_i) < \upsilon(E').$$

In particular, this theorem implies the results of T.-C. Kuo and A. Parusiński for the jacobian curve of two plane curves (see [56, 57]) or the result of E. García Barroso and J. Gwoździewicz (see [41, Theorem 1]) about the jacobian curve of a plane curve and its characteristic approximate roots. In [72, 38], the authors proved a particular case of Theorem 1.6.12 when $\mathcal{F}$ has only an irreducible separatrix $C = (f = 0)$ and $\mathcal{G}$ is a hamiltonian foliation associated to a characteristic approximate root of $C$. In [31, Section 7], we explain in a detailed way how to recover all these results from Theorem 1.6.12.

## 1.7 Analytic invariants of irreducible plane curves

The study of the analytic classification of plane curves in a systematic way started with the work of O. Zariski in [82] although some previous works were published before (see [20] for references about the subject). The complete classification for the irreducible case was first given by A. Hefez and M. Hernandes in [48] (see also [49] for a more detailed description of the results concerning the analytic classification of irreducible plane branches). There are two types of invariants which appear in the classification given in [48, 49]: discrete invariants provided by the set of differential values of the curve and continuous invariants given by some coefficients in a Puiseux parametrization of the curve.

As we mention in the introduction, given a plane curve $C$, the topological type of a generic polar curve $\mathcal{P} = \mathcal{P}^C$ of $C$ depends on the analytic type of $C$. We have showed that, if $C$ is irreducible, the set of quotient polars $Q(C)$ can be computed in terms of the equisingularity data of $C$ and that the set $Q(C) \cup \{m_0(\mathcal{P})\}$ determines the topological type of $C$, where $m_0(\mathcal{P})$ is the multiplicity of a generic polar curve of $C$. Moreover, let us see that there are some analytic invariants of the curve $C$ that can easily be obtained from a generic polar curve.

If $\{\beta_0, \beta_1, \dots, \beta_g\}$ are the characteristic exponents of the curve $C$, by Merle's decomposition theorem, we know that the number of irreducible components $\delta(\mathcal{P})$ of $\mathcal{P}$ verifies that $g \leq \delta(\mathcal{P}) \leq m_0(C) - 1 = \beta_0 - 1$ but $\delta(\mathcal{P})$ depends on the analytic type of $C$. Moreover, if we write $\mathcal{P} = \cup_{i=1}^{\delta} \gamma_i$ the decomposition of $\mathcal{P}$ into irreducible components, the set of polar quotients $Q(C) = \left\{ \frac{(\gamma_i, C)_0}{m_0(\gamma_i)} \ : \ i = 1, \dots, \delta \right\}$ is equal to $\left\{ \bar{\beta}_1, \frac{\bar{\beta}_2}{n_1}, \dots, \frac{\bar{\beta}_g}{n_1 \cdots n_{g-1}} \right\}$ where $\{\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g\}$ is the minimal system of generators of the semigroup $\Gamma_C$ of $C$ (see Section 1.4). Moreover, if we denote by $t_i$ the number of branches $\gamma_j$ such that $\frac{(\gamma_j, C)_0}{m_0(\gamma_j)} = \frac{\bar{\beta}_i}{n_1 \cdots n_{i-1}}$ then the vector $(t_1, t_2, \dots, t_g)$ is also an analytic invariant of the curve $C$.

By the results of P. Rouillé (see [71, Corollary 4.5]) and Lemma 1.3.10, we have that if $C$ is an irreducible curve, $\mathcal{F} \in \mathbb{G}_C$ and $\mathcal{P}^{\mathcal{F}}$ is a generic polar curve of $\mathcal{F}$, the set of polar quotients $Q(\mathcal{F})$ for the foliation $\mathcal{F}$ can be computed as

$$
\begin{aligned}
Q(\mathcal{F}) &= \left\{ \frac{(C, \xi_j)_0}{m_0(\xi_j)} \quad : \ \xi_j \text{ is an irreducible component of } \mathcal{P}^{\mathcal{F}} \right\} \\
&= \left\{ \frac{\tau_0(\mathcal{F}, \xi_j) + 1}{m_0(\xi_j)} \quad : \ \xi_j \text{ is an irreducible component of } \mathcal{P}^{\mathcal{F}} \right\}
\end{aligned}
$$

where $\tau_0(\mathcal{F}, \xi)$ denotes the tangency order of $\mathcal{F}$ with $\xi$ (see page 16) and the set $Q(\mathcal{F})$ is also equal to $\left\{ \bar{\beta}_1, \frac{\bar{\beta}_2}{n_1}, \dots, \frac{\bar{\beta}_g}{n_1 \cdots n_{g-1}} \right\}$. Hence, we can associate to $\mathcal{F}$ a vector $t(\mathcal{F}) = (t_1^{\mathcal{F}}, t_2^{\mathcal{F}}, \dots, t_g^{\mathcal{F}})$ where $t_i^{\mathcal{F}}$ is the number of branches $\xi_j$ of $\mathcal{P}^{\mathcal{F}}$ such that $\frac{\tau_0(\mathcal{F}, \xi_j)+1}{m_0(\xi_j)} = \frac{\bar{\beta}_i}{n_1 \cdots n_{i-1}}$. Note that $t(\mathcal{F})$ is not determined by $C$, it depends on the foliation: if we consider the foliations in Example 1.5.11, we have that $t_1^{\mathcal{F}_1} = t_1^{\mathcal{F}_2} = 2$ but $t_1^{\mathcal{F}_3} = 1$. Consequently $t(\mathcal{F})$ is an analytic invariant of the foliation $\mathcal{F}$.

As we mentioned before, one of the main ingredients in the analytic classification of plane curves is the set of values of differentials (see [82, 48, 49]). Let $C$ be a germ of irreducible plane curve in $(\mathbb{C}^2, 0)$ and let $\gamma(t) = (x(t), y(t))$ be a parametrization of the curve $C$. The *set of differential values* $\Lambda_C$ of the curve $C$ is given by

$$\Lambda_C = \{\mathrm{ord}_t(\gamma^*\omega) + 1 \ : \ \omega \in \Omega^1\},$$

where $\Omega^1$ is the $\mathbb{C}\{x, y\}$-module of 1-forms in $(\mathbb{C}^2, 0)$.

Note that if $\mathcal{F}$ is the foliation defined by $\omega = 0$ and $C$ is not a separatrix of $\mathcal{F}$, then $\mathrm{ord}_t(\gamma^*\omega)$ coincides with the tangency order $\tau_0(\mathcal{F}, C)$ defined in Section 1.3. Moreover, if $\mathcal{F}$ is non-dicritical and $\mathcal{G}_f$ is the hamiltonian foliation defined by $df = 0$, where $f$ is a reduced equation of the curve $S_{\mathcal{F}}$ of separatrices of $\mathcal{F}$, then we have that

$$\tau_0(\mathcal{F}, C) \geq \tau_0(\mathcal{G}_f, C) = (S_{\mathcal{F}}, C)_0 - 1 \tag{1.15}$$

and the equality holds if and only if $\mathcal{F}$ is a non-dicritical second type foliation (see Lemma 1.3.10 and [14, Corollary 1]).

In particular, if we consider 1-forms $\omega$ which define non-dicritical second type foliations, the values $\mathrm{ord}_t(\gamma^*\omega) + 1$ allow to recover the values of the semigroup $\Gamma_C$ of the curve. Thus, if we want to describe the set $\Lambda_C \setminus \Gamma_C$, we have to study the values $\mathrm{ord}_t(\varphi^*\omega) + 1$ for 1-forms such that the foliation defined by $\omega = 0$ is either dicritical or it is not a second type foliation.

Note that given $h \in \mathbb{C}\{x, y\}$ and a 1-form $\omega$, we have that $\mathrm{ord}_t(\gamma^*(h\omega)) = \mathrm{ord}_t(h(\gamma(t))) + \mathrm{ord}_t(\gamma^*\omega)$, then we obtain $\Lambda_C + \Gamma_C \subset \Lambda_C$: this property means that the set of differential values $\Lambda_C$ is a $\Gamma_C$*-semimodule*.

If $C$ is a curve with only one Puiseux pair $\{(m, n)\}$ (we will refer to this type of curves as *cusps*) there exists a unique sequence of increasing non-negative integer numbers $(n, m, \lambda_1, \ldots, \lambda_s)$, called a *basis* of $\Lambda_C$, with $\Lambda_C = \cup_{i=1}^{s}(\lambda_i + \Gamma_C)$ and $\lambda_j \notin \cup_{i=1}^{j-1}(\lambda_i + \Gamma_C)$ ([15, Proposition 6.2]). In [35], C. Delorme gives a description of the structure of the semimodule of differential values when $C$ is a cusp. In [15, Appendices B and C], some of the results of Delorme concerning the structure of $\Lambda_C$ are proved with a different approach.

As well properties of the foliations defined by the 1-forms $\omega_i$ with $\mathrm{ord}_t(\gamma^*\omega_i)+1 = \lambda_i$ are described in [15] for cusps. In particular, the bifurcation divisor $E$ of $G(C)$ is dicritical for these foliations defined by $\omega_i = 0$. For any free point $P \in E$, let $C_P^i$ be the invariant curve of $\omega_i = 0$ corresponding to the dicritical component and passing through $P$. Then $C_P^i$ is also an irreducible curve with only one Puiseux pair $\{(m, n)\}$ and semimodule of differentials values given by $\cup_{i=1}^{j-1}(\lambda_i + \Gamma_C)$ ([15, Theorem 8.8]). If $P_C$ is the infinitely near point of $C$ in $E$, the curves $C_{P_C}^i$ are called *analytic semiroots* of the cusp $C$.

There are two recent papers by E. Casas-Alvero addressing the study of the continuous analytic invariants in the classification of irreducible plane curves with only one Puiseux pair. In [20], E. Casas-Alvero presents some results determining which coefficients of the Puiseux series of the Puiseux parametrization affect the analytic type of an irreducible plane curve with only one Puiseux pair; he uses the Newton polygon of the curve $C$ in order to determine which coefficients of the equation of $C$ are "relevant" for the analytic classification of the curve. In [21], the author tries to go deeper in the relationship between polar curves and analytic classification of irreducible curves with only one Puiseux pair showing that the topological type of generic polars and base points of polar curves provide only partial information on the analytic type of the curve, and that generalized polars contain more information.

Another recent paper [33] presents a method to construct dicritical foliations at the bifurcation divisors of the reduction of singularities of an irreducible curve $C$ using the semiroots of the curve $C$. Moreover, the authors describe the relationship between the 1-forms defining these foliations and the semimodule of differential values $\Lambda_C$. It is also shown (Proposition 4.9) how to compute the *Zariski invariant* $\lambda$ of $\Lambda_C$, that is, $\lambda = \min(\Lambda_C \setminus \Gamma_C) - n$, considering dicritical foliations in the first triple point of the resolution dual graph of $C$ (this proposition extends a result proved by O. Gómez-Martínez [44] for cusps). Recall that the Zariski invariant is an analytic invariant introduced by O. Zariski in [80].

We would like to mention as well the paper [2] where the authors study the relationship between analytic invariants, as the set of differential values $\Lambda_C$ and the Tjurina number $\tau(C)$ of a plane irreducible curve $C$, and the corresponding analytic invariants of its semiroots. In particular, they show how to determine part of $\Lambda_C$ from the sets of differential values $\Lambda_{C_k}$ of its semiroots $C_k$.

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