# LOCAL POLAR INVARIANTS FOR PLANE SINGULAR FOLIATIONS 

FELIPE CANO \& NURIA CORRAL \& ROGÉRIO MOL<br>To the memory of Marco Brunella


#### Abstract

In this survey paper, we take the viewpoint of polar invariants to the local and global study of non-dicritical holomorphic foliation in dimension two and their invariant curves. It appears a characterization of second type foliations and generalized curve foliations as well as a description of the GSV-index in terms of polar curves. We also interpret the proofs concerning the Poincaré problem with polar invariants.


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## 1. Introduction

In this paper, we give a look based on the study of intersection properties of polar curves of a foliation to three subjects concerning non-dicritical singular holomorphic foliations:
a) The characterization of second type foliations and generalized curves.

[^0]b) A polar interpretation of the GSV-index.
c) The (non-dicritical) Poincaré problem for foliations in $\mathbb{P}_{\mathbb{C}}^{2}$.

The polar curves of a singular holomorphic foliation $\mathcal{F}$ in dimension two have been studied in a local way by P. Rouillé [ $[7]$ ] and N. Corral [ए]] and in a global way by R. Mol [25]. The definition is quite geometrical. Namely, the polar curve of $\mathcal{F}$ with respect to another foliation $\mathcal{L}$ is just the curve of tangencies between $\mathcal{F}$ and $\mathcal{L}$. It generalizes a classical approach of polar curves by B. Teissier, Lê Dung Tràng, E. Casas-Alvero, and others [20], [20], [10], [2], [7]). Usually $\mathcal{L}$ is a linear foliation of parallel lines (or lines passing through a point of $\mathbb{P}_{\mathbb{C}}^{2}$ ) and in this way the "generic object" is well defined with respect to certain criteria, such as the equisingularity, for instance.

Let us recall that the term generalized curve comes from the results in [6] and denotes a local foliation without saddle-nodes in its desingularization. Such foliations have the outstanding properties of being desingularized by the same sequence of blow-ups that desingularizes the set of separatrices and minimizing Milnor number. The non-dicritical foliations whose reduction of singularities coincides with the reduction of singularities of the set of formal separatrices are those that minimize the algebraic multiplicity. This characterization is due to J.-F. Mattei and E. Salem [23]] and they use the terminology foliations of second type.

The intersection number of a generic polar curve of a non-dicritical foliation $\mathcal{F}$ and a formal invariant curve $S$ at $0 \in \mathbb{C}^{2}$ is what we call polar intersection number $p_{0}(\mathcal{F}, S)$. We have that

$$
\begin{equation*}
p_{0}(\mathcal{F}, S) \leq \nu_{0}(\mathcal{F})+\mu_{0}(\mathcal{F}), \tag{1}
\end{equation*}
$$

where $\nu_{0}(\mathcal{F})$ is the algebraic multiplicity and $\mu_{0}(\mathcal{F})$ the Milnor number. Moreover, equality holds if and only if $\mathcal{F}$ is of second type and $S$ is its set of formal separatrices.

Take a germ of convergent invariant curve $C \subset S$ and a generalized curve $\mathcal{G}$ such that $C$ is its set of separatrices. Then we have

$$
p_{0}(\mathcal{G}, C) \leq p_{0}(\mathcal{F}, S)
$$

and equality holds if and only if $\mathcal{F}$ is a generalized curve and $S=C$ is its set of separatrices.

There is no obstruction to consider formal foliations and formal generalized curves. In this context, we also have that for any formal generalized curve $\hat{\mathcal{G}}$ such that $S$ is its set of separatrices and any formal foliation $\hat{\mathcal{F}}$ such that $S$ is invariant, we have

$$
\Delta_{0}(\hat{\mathcal{F}}, S)=p_{0}(\hat{\mathcal{F}}, S)-p_{0}(\hat{\mathcal{G}}, S)_{0} \geq 0
$$

and equality holds if and only if $\hat{\mathcal{F}}$ is a generalized curve and $S$ is its set of separatrices.

These results express a well known characterization of generalized curves in terms of the Gómez-Mont-Seade-Verjovsky index. Take a convergent curve $C$ and a generalized curve $\mathcal{G}$ such that $C$ is its set of separatrices. Consider a non-dicritical
foliation $\mathcal{F}$ such that $C$ is invariant by $\mathcal{F}$. We have that

$$
G S V_{0}(\mathcal{F}, C)=\Delta_{0}(\mathcal{F}, S) \geq 0
$$

We recover in this way the known result that $\mathcal{F}$ is a generalized curve if and only $G S V_{0}(\mathcal{F}, C)=0([\mathbb{Z},[])$ and in this case $C$ is its set of separatrices. Moreover, the above formula gives a way of generalizing the GSV-index to formal invariant curves, as well as another interpretation of the non-negativity of the GSV-index for non-dicritical foliations (see Proposition 6 of [ [ $]$ ).
Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}_{\mathbb{C}}^{2}$. The degree of $\mathcal{F}$ is the number $\operatorname{deg}(\mathcal{F})$ of tangencies between $\mathcal{F}$ and a generic line $L \subset \mathbb{P}_{\mathbb{C}}^{2}$. The question concerning the existence of a bound for the degree of an algebraic curve $S$ invariant by $\mathcal{F}$ in terms of the degree of $\mathcal{F}$ is known in Foliation Theory as the Poincaré problem, being proposed by Poincaré himself in [26] as a step in finding a rational first integral for a polynomial differential equation in two complex variables. A first answer to this problem was given by D. Cerveau and A. Lins Neto [[2]]: if $S$ has at most nodal singularities, then $\operatorname{deg}(S) \leq \operatorname{deg}(\mathcal{F})+2$. Besides, this bound is reached if and only if $\mathcal{F}$ is a logarithmic foliation, that is, a foliation induced by a closed meromorphic 1-form with simple poles. Later, M. Carnicer obtained in [ [ $]$ ] the same inequality, removing the hypothesis on the singularities of $S$, but admitting that the singularities of $\mathcal{F}$ over $S$ are all non-dicritical, meaning that the number of local separatrices is finite. The Poincaré problem was put in a clearer setting by M. Brunella in the works [T] and [ [7]. Brunella has shown that the bound $\operatorname{deg}(S) \leq$ $\operatorname{deg}(\mathcal{F})+2$ occurs whenever the sum over $S$ of the $G S V$-indices of $\mathcal{F}$ with respect to the local separatrices contained in $S$ is non-negative. This happens in the two cases mentioned above. In the dicritical case, N. Corral and P. Fernández-Sánchez [[.]] proved that the degree of an algebraic invariant curve $S$ of $\mathcal{F}$ is bounded in terms of the degree of $\mathcal{F}$ provided that the germ of $S$ at each singular point of $\mathcal{F}$ is a union of isolated separatrices of $\mathcal{F}$.

The last section is devoted to give a proof of Carnicer's bound in terms of local and global polar curves. We end the paper by a remark on logarithmic foliations, that corresponds to the limit case.

## 2. Recall on local invariants and Reduction of singularities

We recall [[]] that a germ of singular holomorphic foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ is defined by $\omega=0$, where $\omega$ is a 1 -form

$$
\begin{equation*}
\omega=P(x, y) d x+Q(x, y) d y \tag{2}
\end{equation*}
$$

or by the vector field $\mathbf{v}=-Q(x, y) \partial / \partial x+P(x, y) \partial / \partial y$, where $P, Q \in \mathbb{C}\{x, y\}$ are relatively prime. The origin 0 is a singular point if $P(0,0)=Q(0,0)=0$. Note that any nonzero 1 -form defines in a unique way a germ of singular holomorphic foliation, just by taking the common factor of the coefficients; same remark when we consider a vector field.

A formal curve $C$ in $\left(\mathbb{C}^{2}, 0\right)$ is given by a reduced equation $f(x, y)=0$ with $f \in \mathbb{C}[[x, y]]$. When $f$ is irreducible, we say that $C$ is a branch. If $f=f_{1} f_{2} \cdots f_{e}$ is the decomposition of $f$ as a product of irreducible factors, we say that each $B_{i}=\left(f_{i}=0\right)$ is one of the branches of $C$. If we can take a convergent equation $f \in \mathbb{C}\{x, y\}$, then $C$ is a germ of analytic curve, in this case, the branches are also analytic germs of curve.

Let $\sigma:\left(M, \sigma^{-1}(0)\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ be the blow-up of the origin. Given a branch $B$ with $f(x, y)=0$, we know that $f$ has the form

$$
f(x, y)=(\lambda x+\mu y)^{r}+\tilde{f}(x, y), \quad(\lambda, \mu) \neq(0,0)
$$

where all the terms of $\tilde{f}(x, y)$ have degree greater than $r$. We say that $r$ is the multiplicity of $B$ at the origin and we put $r=\nu_{0}(B)$. The line $\lambda x+\mu y=0$ is the tangent cone of $B$ at the origin and it determines a single point $\tau(B)$ in the projective line $\sigma^{-1}(0)$. Up to a linear coordinate change, we can assume that $f=y^{r}+\tilde{f}(x, y)$ and in this case, we have a branch $B^{\prime}$ at $\tau(B)$ given by $f^{\prime}\left(x^{\prime}, y^{\prime}\right)=0$, where

$$
x^{\prime r} f^{\prime}\left(x^{\prime}, y^{\prime}\right)=f\left(x^{\prime}, x^{\prime} y^{\prime}\right)
$$

We say that $B^{\prime}$ is the strict transform of $B$. By taking the union of branches, we define the strict transform of any formal curve $C$ and, by iterating the procedure, we can define the strict transform of $C$ under any sequence

$$
\pi:(M, E) \rightarrow\left(\mathbb{C}^{2}, 0\right)
$$

of punctual blow-ups. Note that each branch $B$ of $C$ gives a point $\tau_{E}(B) \in E$ in the exceptional divisor $E$. We say that the sequence $\pi$ is a reduction of singularities of $C$ if and only if
(1) For any two branches $B_{1}, B_{2}$ of $C$ we have $\tau_{E}\left(B_{1}\right) \neq \tau_{E}\left(B_{2}\right)$.
(2) For any branch $B$ of $C$, the exceptional divisor $E$ has only one irreducible component $D$ through $p=\tau_{E}(B)$ and the strict transform $B^{\prime}$ of $B$ is non singular at $p$ and transversal to $E$. In other words, there are formal coordinates $(x, y)$ at $p$ such that $E=(x=0)$ and $B^{\prime}=(y=0)$.
It is known [30]] that any formal curve has a reduction of singularities. Moreover, there is a minimal one, and any other factorizes through the minimal one by making additional punctual blow-ups. Let us note that doing a reduction of singularities needs at least one blow-up, even in the case that we have a non-singular branch.

The transform $\pi^{*} \mathcal{F}$ of a foliation $\mathcal{F}$ by $\pi$ is locally given by $\pi^{*} \omega$. Note that this definition is also valid for a formal foliation, that is, for a foliation given by $\hat{\omega}=0$, where the coefficients of $\hat{\omega}$ are formal series without common factor.

Take a singular holomorphic foliation $\mathcal{G}$ in $(M, E)$ and a point $p \in E$. We recall [] that $p$ is a simple point for $\mathcal{G}, E$ if there are local coordinates $(x, y)$ centered at $p$ such that $E \subset(x y=0)$ locally at $p$ and one of the following properties holds
(1) $\mathcal{G}$ is locally given by $d x=0$. This is the case when $p$ is non singular. If $(x=0) \subset E$, we say that $p$ is of corner type and otherwise $p$ is of trace type.
(2) Corner type singular points: $\mathcal{G}$ is locally given by $\omega=x y \eta=0$ and the divisor is $E=(x y=0)$, with

$$
\eta=a(x, y) \frac{d x}{x}+b(x, y) \frac{d y}{y}
$$

where $(a(0,0), b(0,0))=(-\mu, \lambda)$, with $\mu \neq 0$ and $\lambda / \mu \notin \mathbb{Q}_{>0}$. We have two possibilities
(a) $\lambda \neq 0$. This is a complex hyperbolic singularity of corner type, following the terminology of $[\nabla]$.
(b) $\lambda=0$. This is a badly oriented saddle-node of corner type.
(3) Trace type singular points: $\mathcal{G}$ is locally given by $\omega=x \eta=0$ and the divisor is $E=(x=0)$, with

$$
\eta=a(x, y) \frac{d x}{x}+b(x, y) d y
$$

where $a(x, y)=-\mu y+\alpha x+\tilde{a}(x, y)$, with $\tilde{a}(x, y)$ having all terms of degree at least two and $b(0,0)=\lambda$. In addition, we ask that one of the following situations holds
(a) $\lambda \mu \neq 0$ and $\lambda / \mu \notin \mathbb{Q}_{>0}$. This is a complex hyperbolic trace singularity.
(b) $\mu \neq 0$ and $\lambda=0$. This is a well oriented saddle node of trace type.
(c) $\mu=0$ and $\lambda \neq 0$. This is a badly oriented saddle node of trace type.

In [L.3], badly oriented saddle nodes are called tangent saddle-nodes. An irreducible component $D$ of $E$ may be invariant or dicritical. If we are dealing with a simple point $p \in M$, a dicritical component $D$ only exists when $p$ is non singular and in this case, up to choosing local coordinates $(x, y)$, we have that $\mathcal{G}$ is given by $d x=0$ and $D=(y=0)$.
As a direct consequence of Seidenberg's Desingularization Theorem [ [ Z , Z] there is a morphism $\pi$ as above for which $\pi^{*} \mathcal{F}$ has only simple singularities. Such $\pi$ is called a reduction of the singularities of $\mathcal{F}$. Note that there is a minimal such $\pi$ and any other reduction of singularities of $\mathcal{F}$ factorizes through the minimal one by an additional sequence of blow-ups.

A separatrix for $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ is an invariant formal irreducible curve. Thus, it is given by an equation $f(x, y)=0$, where $f \in \mathbb{C}[[x, y]]$ is an irreducible formal series such that $f(0,0)=0$ and there is a formal series $h \in \mathbb{C}[[x, y]]$ such that

$$
\omega \wedge d f=(f h) d x \wedge d y
$$

If we can take $f \in \mathbb{C}\{x, y\}$, the separatrix is convergent. We denote $\operatorname{Sep}(\mathcal{F})$ the set of separatrices of $\mathcal{F}$.

Consider a reduction of singularities $\pi:(M, E) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ of $\mathcal{F}$. The following are equivalent
(1) There is a dicritical component $D$ of $E$ for $\pi^{*} \mathcal{F}$.
(2) The foliation $\mathcal{F}$ has infinitely many separatrices.

Such foliations are called dicritical. In this paper we deal with non-dicritical foliations, that is foliations having only finitely many separatrices. In this case, the union $S_{\mathcal{F}}=\bigcup\{B ; B \in \operatorname{Sep}(\mathcal{F})\}$ is a formal curve whose irreducible components are the separatrices.

The separatrices of a non-dicritical $\mathcal{F}$ are in one to one correspondence with the singular points of trace type in the reduction of singularities. To any separatrix $B$ we associate the singular trace point $\tau_{E}(B) \in E$ through which it passes the strict transform of $B$. The separatrix $B$ is called strong or of Briot and Bouquet type (see [ [2]], where this terminology is used) if and only if $\tau_{E}(B)$ is either a complex hyperbolic singularity or a badly oriented saddle node. Such separatrices are convergent by application of the classical Briot and Bouquet Theorem. Note that in the classical Camacho-Sad paper [G] the authors show the existence of a Briot and Bouquet separatrix in order to prove that any non-dicritical foliation has at least one convergent separatrix.

Definition 1. Let $\mathcal{F}$ be a non-dicritical foliation in $\left(\mathbb{C}^{2}, 0\right)$ and consider the minimal desingularization $\pi:(M, E) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ of $\mathcal{F}$.
(a) The foliation $\mathcal{F}$ is a generalized curve (also complex hyperbolic) if all the singularities of $\pi^{*} \mathcal{F}$ are of complex hyperbolic type.
(b) The foliation $\mathcal{F}$ is of second type if all the saddle nodes of $\pi^{*} \mathcal{F}$ are well oriented with respect to $E$.

The terminology comes from previous papers [ [ [ [ [2], 价]. Note that $\mathcal{F}$ is of second type if and only if the strong separatrices correspond to complex hyperbolic trace points and all the corners are also complex hyperbolic. The fact of being a generalized curve or a second type foliation is independent of considering another, may be not minimal, reduction of singularities.
2.1. Local invariants. Let us recall now some of the local invariants frequently used in the local study of singular foliations in dimension two, see also [ [ ] ].

The algebraic multiplicity $\nu_{0}(\mathcal{F})$ is the minimum of the orders $\nu_{0}(P), \nu_{0}(Q)$ at the origin of the coefficients of a local generator of $\mathcal{F}$. The Milnor number $\mu_{0}(\mathcal{F})$ is given by

$$
\mu_{0}(\mathcal{F})=\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(P, Q)}=i_{0}(P, Q)
$$

(where $i_{0}(P, Q)$ stands for the intersection multiplicity).
Take a primitive parametrization $\gamma:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{2}, 0\right), \gamma(t)=(x(t), y(t))$, of a formal irreducible curve $B=(f(x, y)=0)$ at $\left(\mathbb{C}^{2}, 0\right)$. Then $B$ is a separatrix of $\mathcal{F}$ if and only if $\gamma^{*} \omega=0$. In this case, we can consider the Milnor number $\mu_{0}(\mathcal{F}, B)$ of $\mathcal{F}$ along $B$ defined by

$$
\mu_{0}(\mathcal{F}, B)=\operatorname{ord}_{t} \mathbf{w}(t),
$$

where $\mathbf{w}(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_{*} \mathbf{w}(t)=\mathbf{v} \circ \gamma(t)$ (this number is also called multiplicity of $\mathcal{F}$ along $B$, see [ $[\mathbf{[},[\mathbf{[ 6 ]}$ ). We have that

$$
\mu_{p}(\mathcal{F}, B)= \begin{cases}\operatorname{ord}_{t}(Q(\gamma(t)))-\operatorname{ord}_{t}(x(t))+1 & \text { if } x(t) \neq 0  \tag{3}\\ \operatorname{ord}_{t}(P(\gamma(t)))-\operatorname{ord}_{t}(y(t))+1 & \text { if } y(t) \neq 0\end{cases}
$$

If $B$ is not a separatrix, we define the tangency order $\tau_{0}(\mathcal{F}, B)$ to be the $\operatorname{ord}_{t} a(t)$ where $\gamma^{*} w=a(t) d t$.

These invariants have a behavior under blow-up that helps in many of the results we are concerned here. For instance, if $B$ is not a separatrix and we consider the blow-up of the origin $\sigma:\left(M, \sigma^{-1}(0)\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$, we have

$$
\begin{equation*}
\tau_{0}(\mathcal{F}, B)=\nu_{0}(B) \nu_{0}(\mathcal{F})+\tau_{p^{\prime}}\left(\sigma^{*} \mathcal{F}, B^{\prime}\right) \tag{4}
\end{equation*}
$$

where $B^{\prime}$ is the strict transform of $B$ by $\sigma$ and $\left\{p^{\prime}\right\}=B^{\prime} \cap \sigma^{-1}(0)$.
Example 1. For a complex hyperbolic singularity with two transversal separatrices $B_{1}$ and $B_{2}$, we have $\mu_{0}\left(\mathcal{F}, B_{1}\right)=\mu_{0}\left(\mathcal{F}, B_{2}\right)=1$. For a saddle-node, up to reordering, we have $\mu_{0}\left(\mathcal{F}, B_{1}\right)=1, \mu_{0}\left(\mathcal{F}, B_{2}\right)=k+1$ where $B_{1}$ is the strong separatrix and $k \geq 1$ is the Poincaré order of the saddle node.
2.2. Comparison with the hamiltonian foliation. Let $S$ be a formal curve and $f=0$ be a reduced equation for $S$. We denote $\mathcal{G}_{f}$ the "hamiltonian" foliation defined by $d f=0$. Then $\mathcal{G}_{f}$ is non-dicritical and $S$ is its curve of separatrices. Note that

$$
\nu_{0}\left(\mathcal{G}_{f}\right)=\nu_{0}(S)-1
$$

and $\mu_{0}\left(\mathcal{G}_{f}\right)$ corresponds exactly to the usual definition of the Milnor number of $S$ at the origin. We will frequently compare invariants of a non-dicritical $\mathcal{F}$ and $\mathcal{G}_{f}$, where $f=0$ is a reduced equation of $S_{\mathcal{F}}$.

Non-dicritical second type foliations minimize the algebraic multiplicity and have the same reduction of singularities as $S_{\mathcal{F}}$. Moreover, generalized curves also minimize the Milnor number. Let us state these results as Mattei-Salem in [2.3]

Theorem 1 ([回, [23]). Let $\mathcal{F}$ be a non-dicritical foliation and consider $\mathcal{G}_{f}$ where $f=0$ is a reduced equation of $S_{\mathcal{F}}$. Take the minimal reduction of singularities $\pi:(M, E) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ of $\mathcal{F}$. Then
(1) $\pi$ is a reduction of singularities of $S_{\mathcal{F}}$. Moreover $\pi$ is the minimal reduction of singularities of $S_{\mathcal{F}}$ if and only if $\mathcal{F}$ is of second type.
(2) $\nu_{0}(\mathcal{F}) \geq \nu_{0}\left(\mathcal{G}_{f}\right)$. Equality holds if and only if $\mathcal{F}$ is of second type.
(3) $\mu_{0}(\mathcal{F}) \geq \mu_{0}\left(\mathcal{G}_{f}\right)$. Equality holds if and only if $\mathcal{F}$ is a generalized curve.

Next corollary is a more general version of a result in [27].
Corollary 1. Consider a non-dicritical foliation $\mathcal{F}$ and take a branch $B$ which is not a separatrix. Then

$$
i_{0}\left(S_{\mathcal{F}}, B\right) \leq \tau_{0}(\mathcal{F}, B)+1
$$

and equality holds if and only if $\mathcal{F}$ is of second type.

Proof. We make induction on the number $n$ of blow-ups needed to obtain the situation that $\mathcal{F}$ is non singular at the point $p \in B$ we are considering, the branch $B$ is non singular and transversal to $\mathcal{F}$. Note that this number $n$ exists since $B$ is not a separatrix. If $n=0$ we are done since we can assume that $\mathcal{F}$ is given by $d x=0$ and $B=(y=0)$. Let $\sigma:\left(M, \sigma^{-1}(0)\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ be the blow-up of the origin, where the exceptional divisor is the projective line $D=\sigma^{-1}(0)$. Denote by $S_{\mathcal{F}}^{\prime}, B^{\prime}$ the respective strict transforms of $S_{\mathcal{F}}, B$ and let $p^{\prime}$ be the point $D \cap B^{\prime}$. By Noether's formula, we have

$$
\begin{equation*}
i_{0}\left(S_{\mathcal{F}}, B\right)=\nu_{0}\left(S_{\mathcal{F}}\right) \nu_{0}(B)+i_{p^{\prime}}\left(S_{\mathcal{F}}^{\prime}, B^{\prime}\right) . \tag{5}
\end{equation*}
$$

In view of Theorem $\mathbb{\square}$ we have $\nu_{0}\left(S_{\mathcal{F}}\right) \leq \nu_{0}(\mathcal{F})+1$ and equality holds if and only $\mathcal{F}$ is of second type. By the induction hypothesis and recalling that the separatrices of $\sigma^{*} \mathcal{F}$ at $p^{\prime}$ are given by $S_{\mathcal{F}}^{\prime} \cup D$, we have that

$$
i_{p^{\prime}}\left(S_{\mathcal{F}}^{\prime}, B^{\prime}\right)+\nu_{0}(B)=i_{p^{\prime}}\left(S_{\mathcal{F}}^{\prime}, B^{\prime}\right)+i_{p^{\prime}}\left(D, B^{\prime}\right)=i_{p^{\prime}}\left(S_{\mathcal{F}}^{\prime} \cup D, B^{\prime}\right) \leq \tau_{p^{\prime}}\left(\sigma^{*} \mathcal{F}, B^{\prime}\right)+1
$$

where equality holds if and only if $\sigma^{*} \mathcal{F}$ is of second type at $p^{\prime}$. Looking at Equation (四) we conclude that

$$
\begin{aligned}
i_{0}\left(S_{\mathcal{F}}, B\right) & =\nu_{0}\left(S_{\mathcal{F}}\right) \nu_{0}(B)+i_{p^{\prime}}\left(S_{\mathcal{F}}^{\prime}, B^{\prime}\right) \leq \\
& \leq\left(\nu_{0}(\mathcal{F})+1\right) \nu_{0}(B)+\tau_{p^{\prime}}\left(\sigma^{*} \mathcal{F}, B^{\prime}\right)+1-\nu_{0}(B)= \\
& =\tau_{0}(\mathcal{F}, B)+1
\end{aligned}
$$

and equality holds if and only if $\mathcal{F}$ (and hence $\sigma^{*} \mathcal{F}$ ) is of second type.

## 3. Polar intersection numbers

Let $\mathcal{F}$ be a germ of singular foliation in $\left(\mathbb{C}^{2}, 0\right)$ given by $\omega=P d x+Q d y$, where $P, Q$ are without common factors. The polar curve $P_{(a: b)}^{\mathcal{F}}$ of $\mathcal{F}$ with respect to $(a: b) \in \mathbb{P}_{\mathbb{C}}^{1}$ is defined by the equation $a P+b Q=0$. In terms of differential forms, it is given by $\omega \wedge(b d x-a d y)=0$. Note that $P_{(a: b)}^{\mathcal{F}}$ has no invariant branches unless $a x+b y=0$ is an invariant line. The definition of polar curve also makes sense for formal foliations and the invariants to be defined below can be extended to the formal world.

Let us fix a formal curve $C$ invariant by $\mathcal{F}$. There is a non-empty Zariski open set $U_{C} \subset \mathbb{P}_{\mathbb{C}}^{1}$ such that for any $(a: b) \in U_{C}$ the polar $P_{(a: b)}^{\mathcal{F}}$ has no common branches with $C$ and the equisingularity type of $P_{(a: b)}^{\mathcal{F}} \cup C$ is independent of $(a: b) \in U_{C}$. A formal curve $\Gamma$ in $\left(\mathbb{C}^{2}, 0\right)$ is of $C$-generic polar type iff $\Gamma \cup C$ is equisingular to $P_{(a: b)}^{\mathcal{F}} \cup C$ for $(a: b) \in U_{C}$ (see [[ד] $)$. The number

$$
p_{0}(\mathcal{F}, C)=i_{0}(\Gamma, C)
$$

is independent of $\Gamma$ and we call it the $C$-polar intersection number.
Remark 1. Note that, since $(a: b)$ runs in a non empty Zariski open set, we have

$$
\nu_{0}(\Gamma)=\nu_{0}(\mathcal{F})=\min \left\{\nu_{0}(P), \nu_{0}(Q)\right\},
$$

for any $\Gamma$ of $C$-generic polar type. Moreover, taking $\Gamma=P_{(a: b)}^{\mathcal{F}}$ for $(a: b)$ generic enough we have $p_{0}(\mathcal{F}, C)=\min \left\{i_{0}(C, P), i_{0}(C, Q)\right\}$.

Note that if $C \subset C^{\prime}$ are invariant formal curves, any formal curve $\Gamma$ of $C^{\prime}$-generic polar type is also of $C$-generic polar type. In the case that $\mathcal{F}$ is non-dicritical, any invariant formal curve $C$ is contained in the curve $S_{\mathcal{F}}$ of separatrices. In this situation, we say that $\Gamma$ is of generic polar type if it is of $S_{\mathcal{F}}$-generic polar type.

In this section we give two results. The first one concerns the polar intersection number with respect to a single separatrix and the second one relatively to the biggest invariant curve $S_{\mathcal{F}}$. In next section we consider intermediate invariant curves.

Proposition 1. Consider a separatrix $B$ of a non-dicritical foliation $\mathcal{F}$. We have

$$
p_{0}(\mathcal{F}, B)=\mu_{0}(\mathcal{F}, B)+\nu_{0}(B)-1 .
$$

Proof. Let $\gamma(t)=(x(t), y(t))$ be a Puiseux parametrization for $B$ and assume without loss of generality that $x(t) \neq 0$ and hence $\dot{x}(t) \neq 0$. Taking a generic polar $a P+b Q=0$ we know that $p_{0}(\mathcal{F}, B)=\operatorname{ord}_{t}(a P(\gamma(t))+b Q(\gamma(t)))$. Since $B$ is a separatrix, we have $P(\gamma(t)) \dot{x}(t)=-Q(\gamma(t)) \dot{y}(t)$ and applying Equation (B) we obtain

$$
\begin{aligned}
p_{0}(\mathcal{F}, B) & =\operatorname{ord}_{t}\left(-a \frac{Q(\gamma(t)) \dot{y}(t)}{\dot{x}(t)}+b Q(\gamma(t))\right) \\
& =\operatorname{ord}_{t} Q(\gamma(t))-\left(\operatorname{ord}_{t} x(t)-1\right)+\operatorname{ord}_{t}(-a \dot{y}(t)+b \dot{x}(t)) \\
& =\mu_{0}(\mathcal{F}, B)+\operatorname{ord}_{t}(-a y(t)+b x(t))-1 \\
& =\mu_{0}(\mathcal{F}, B)+\nu_{0}(B)-1 .
\end{aligned}
$$

Next result follows applying Corollary [1] to the proof of Proposition 3.7 in [[प]]. We include the proof for the sake of completeness.

Proposition 2. Let $\mathcal{F}$ be a non-dicritical foliation. Then

$$
p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right) \leq \mu_{0}(\mathcal{F})+\nu_{0}(\mathcal{F})
$$

and equality holds if and only if $\mathcal{F}$ is of second type.
Proof. Let $\Gamma=P_{(a: b)}^{\mathcal{F}}$ be a generic polar, with $a=1$ and $b$ generic enough. Denote by $\mathcal{B}(\Gamma)$ the set of irreducible components of $\Gamma$. By Corollary $\mathbb{l}$, we know that

$$
p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=\sum_{B \in \mathcal{B}(\Gamma)} i_{0}\left(B, S_{\mathcal{F}}\right) \leq \sum_{B \in \mathcal{B}(\Gamma)}\left(\tau_{0}(\mathcal{F}, B)+1\right)
$$

and equality holds if and only if $\mathcal{F}$ is of second type. Let us show that the last term is equal to $\mu_{0}(\mathcal{F})+\nu_{0}(\mathcal{F})$. Choose a primitive parametrization $\gamma_{B}(t)=\left(x_{B}(t), y_{B}(t)\right)$ for each $B \in \mathcal{B}(\Gamma)$. If $\omega=P d x+Q d y$ defines the foliation, we recall that $P\left(\gamma_{B}(t)\right)=$
$-b Q\left(\gamma_{B}(t)\right)$, since $B$ is a branch of $\Gamma$. Now

$$
\begin{aligned}
\sum_{B \in \mathcal{B}(\Gamma)}\left(\tau_{0}(\mathcal{F}, B)+1\right) & =\sum_{B \in \mathcal{B}(\Gamma)}\left(\operatorname{ord}_{t}\left\{P\left(\gamma_{B}(t)\right) \dot{x}_{B}(t)+Q\left(\gamma_{B}(t)\right) \dot{y}_{B}(t)\right\}+1\right) \\
& =\sum_{B \in \mathcal{B}(\Gamma)}\left(\operatorname{ord}_{t}\left\{Q\left(\gamma_{B}(t)\right\}+\operatorname{ord}_{t}\left\{-b \dot{x}_{B}(t)+\dot{y}_{B}(t)\right\}+1\right)\right. \\
& =\sum_{B \in \mathcal{B}(\Gamma)}\left(\operatorname{ord}_{t}\left\{Q\left(\gamma_{B}(t)\right\}+\operatorname{ord}_{t}\left\{-b x_{B}(t)+y_{B}(t)\right\}\right)\right. \\
& =i_{0}(Q, \Gamma=P+b Q)+\nu_{0}(\Gamma)=\mu_{0}(\mathcal{F})+\nu_{0}(\mathcal{F}) .
\end{aligned}
$$

Note that $b$ is generic. This ends the proof.
Remark 2. Let $C$ be a formal curve in $\left(\mathbb{C}^{2}, 0\right)$ and $f=0$ be a reduced equation of $C$. Let us consider the hamiltonian foliation $\mathcal{G}_{f}$ given by $d f=0$. We know that $\mathcal{G}_{f}$ is a generalized curve and $C$ its curve of separatrices. Moreover, by definition of Milnor number and multiplicity, we have that $\mu_{0}\left(\mathcal{G}_{f}\right)=\mu_{0}(C)$ and $\nu_{0}\left(\mathcal{G}_{f}\right)=\nu_{0}(C)-1$. Then Proposition $\rrbracket$ gives that

$$
\begin{equation*}
p_{0}\left(\mathcal{G}_{f}, C\right)=\mu_{0}(C)+\nu_{0}(C)-1, \tag{6}
\end{equation*}
$$

and $p_{0}\left(\mathcal{G}_{f}, C\right)$ does not depend on the choice of the reduced equation $f$. More generally, if $\mathcal{G}$ is a generalized curve such that $C=S_{\mathcal{G}}$ we also have that $p_{0}(\mathcal{G}, C)=$ $\mu_{0}(C)+\nu_{0}(C)-1$.

Remark 3. Following Theorem [1] and Proposition [】, we have

$$
\begin{equation*}
p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=\mu_{0}(\mathcal{F})+\nu_{0}\left(S_{\mathcal{F}}\right)-1 \tag{7}
\end{equation*}
$$

for a non-dicritical foliation of second type $\mathcal{F}$. Taking Equation ( $\mathbb{\square}$ ) for $\mathcal{G}_{f}$, where $f=0$ is a reduced equation of $S_{\mathcal{F}}$, we obtain that

$$
\begin{equation*}
p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)-p_{0}\left(\mathcal{G}_{f}, S_{\mathcal{F}}\right)=\mu_{0}(\mathcal{F})-\mu_{0}\left(\mathcal{G}_{f}\right)=\mu_{0}(\mathcal{F})-\mu_{0}\left(S_{\mathcal{F}}\right) \geq 0 \tag{8}
\end{equation*}
$$

Note the positivity of this difference. In particular, a non-dicritical foliation $\mathcal{F}$ of second type is a generalized curve if and only if $p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=p_{0}\left(\mathcal{G}_{f}, S_{\mathcal{F}}\right)$.
Corollary 2. For a non-dicritical foliation $\mathcal{F}$ of second type, we have

$$
\mu_{0}(\mathcal{F})=1-\delta_{\mathcal{F}}+\sum_{B \in \operatorname{Sep}(\mathcal{F})} \mu_{0}(\mathcal{F}, B),
$$

where $\delta_{\mathcal{F}}$ is the number of separatrices of $\mathcal{F}$.
Proof. By summing up polar intersection numbers over all separatrices we get

$$
p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=\sum_{B \in \operatorname{Sep}(\mathcal{F})} p_{0}(\mathcal{F}, B)=\sum_{B \in \operatorname{Sep}(\mathcal{F})} \mu_{0}(\mathcal{F}, B)+\nu_{0}\left(S_{\mathcal{F}}\right)-\delta_{\mathcal{F}},
$$

Since $\mathcal{F}$ is of second type we have $\nu_{0}\left(S_{\mathcal{F}}\right)=\nu_{0}(\mathcal{F})+1=p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)-\mu_{0}(\mathcal{F})+1$ and we are done.

## 4. The formal GSV-index and polar intersection numbers

In this section we give a formal extension of the GSV-index introduced by X. Gómez-Mont, J. Seade and A. Verjovsky in [■]].

Definition 2. Take a germ of singular foliation $\mathcal{F}$ and a formal curve $C$ invariant by $\mathcal{F}$. We define the $C$-polar excess $\Delta_{0}(\mathcal{F}, C)$ by

$$
\Delta_{0}(\mathcal{F}, C)=p_{0}(\mathcal{F}, C)-p_{0}(\mathcal{G}, C)=p_{0}(\mathcal{F}, C)-\mu_{0}(C)-\nu_{0}(C)+1,
$$

where $\mathcal{G}$ is any generalized curve such that $C=S_{\mathcal{G}}$.
Take a non-dicritical foliation $\mathcal{F}$ and a germ of (convergent) curve $C$ invariant by $\mathcal{F}$. Following Brunella [ [Z] we know that $\operatorname{GSV}(\mathcal{F}, C) \geq 0$ and it is equal to 0 in the case of a generalized curve. This positivity is a key argument in bounding degrees for Poincaré Problem in [ $G$ ]. In this section, we show that

$$
G S V(\mathcal{F}, C)=\Delta_{0}(\mathcal{F}, C)
$$

Thus, the polar excess gives a formal extension of the GSV-index.
We start the section by proving that $\Delta_{0}(\mathcal{F}, C) \geq 0$ for any non-dicritical $\mathcal{F}$.
4.1. Positivity of the polar excess. Before proving the positivity of $\Delta_{0}(\mathcal{F}, C)$ we consider the behavior of polar intersection numbers under blow-up.

Consider the blow-up $\pi:(M, E) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ of the origin of $\left(\mathbb{C}^{2}, 0\right)$. Note that if $\mathcal{F}$ is a non-dicritical foliation, then the exceptional divisor $E=\pi^{-1}(0)$ is an invariant projective line for $\pi^{*} \mathcal{F}$.

Proposition 3. Let $B \in \operatorname{Sep}(\mathcal{F})$ be a separatrix of a non-dicritical foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ and let $\widetilde{p} \in E$ be the only point in $\widetilde{B} \cap E$, where $\widetilde{B}$ is the strict transform of B. We have

$$
\begin{equation*}
p_{\widetilde{p}}\left(\pi^{*} \mathcal{F}, \widetilde{B}\right)=p_{0}(\mathcal{F}, B)+\nu_{\widetilde{p}}(\widetilde{B})-\nu_{0}(\mathcal{F}) \nu_{0}(B) . \tag{9}
\end{equation*}
$$

Proof. Up to a linear change of coordinates, we suppose that the tangent cone of $B$ is $y=0$ and thus we have local coordinates $(x, v)$ around $\widetilde{p}$ given by $v=y / x$. Take a Puiseux parametrization $\gamma(t)=\left(t^{n}, \phi(t)\right)$ of $B$, where $n=\nu_{0}(B)$ and $\operatorname{ord}_{t}(\phi(t))>n$. Then, a primitive parametrization of $\widetilde{B}$ is given by $\widetilde{\gamma}(t)=\left(t^{n}, \widetilde{\phi}(t)=\phi(t) / t^{n}\right)$. If $\mathcal{F}$ is given by $\omega=P d x+Q d y=0$, then $\pi^{*} \mathcal{F}$ is locally given at $\widetilde{p}$ by

$$
\omega^{\prime}=x^{-\nu_{0}(\mathcal{F})}\{(P(x, x v)+v Q(x, x v)) d x+x Q(x, x v) d v\} .
$$

We have

$$
p_{\widetilde{p}}\left(\pi^{*} \mathcal{F}, \widetilde{B}\right)=\operatorname{ord}_{t}\left\{P(\gamma(t))+\widetilde{\phi}(t) Q(\gamma(t))+b t^{n} Q(\gamma(t))\right\}-n \nu_{0}(\mathcal{F})
$$

where $b$ is generic. Since $B$ is invariant, we have that $n t^{n-1} P(\gamma(t))=-\phi^{\prime}(t) Q(\gamma(t))$ and thus

$$
p_{\widetilde{p}}\left(\pi^{*} \mathcal{F}, \widetilde{B}\right)=\operatorname{ord}_{t}\{Q(\gamma(t))\}+\operatorname{ord}_{t}\left\{\frac{-t \phi^{\prime}(t)+n \phi(t)}{n t^{n}}+b t^{n}\right\}-n \nu_{0}(\mathcal{F})
$$

For $b$ generic, we have

$$
\operatorname{ord}_{t}\left\{\frac{-t \phi^{\prime}(t)+n \phi(t)}{n t^{n}}+b t^{n}\right\}=\nu_{\widetilde{p}}(\widetilde{B})
$$

Moreover, noting that $n t^{n-1} P(\gamma(t))=-\phi^{\prime}(t) Q(\gamma(t))$ and $\operatorname{ord}_{t}(\phi(t))>n$ we have

$$
\operatorname{ord}_{t}(P(\gamma(t)))>\operatorname{ord}_{t}(Q(\gamma(t)))
$$

and hence $\operatorname{ord}_{t}(Q(\gamma(t)))=p_{0}(\mathcal{F}, B)$. This ends the proof.
Remark 4. If we consider a polar curve $P_{(a: b)}^{\mathcal{F}}$ for a dicritical foliation $\mathcal{F}$ and we define $p_{0}(\mathcal{F}, B)=i_{0}\left(P_{(a: b)}^{\mathcal{F}}, B\right)$ for generic $(a: b)$, we obtain

$$
p_{\widetilde{p}}\left(\pi^{*} \mathcal{F}, \widetilde{B}\right)=p_{0}(\mathcal{F}, B)+\nu_{\widetilde{p}}(\widetilde{B})-\left(\nu_{0}(\mathcal{F})+1\right) \nu_{0}(B),
$$

when the blow-up of the origin is dicritical for the foliation $\mathcal{F}$.
Remark 5. Take a curve $\Gamma$ of generic polar type and consider the situation of


$$
p_{0}(\mathcal{F}, B)=i_{0}(\Gamma, B) .
$$

By Noether's formula, we also have

$$
\left.i_{\widetilde{p}} \widetilde{\Gamma}, \widetilde{B}\right)=i_{0}(\Gamma, B)-\nu_{0}(\Gamma) \nu_{0}(B)=p_{0}(\mathcal{F}, B)-\nu_{0}(\mathcal{F}) \nu_{0}(B) .
$$

We obtain that $p_{\widetilde{p}}\left(\pi^{*} \mathcal{F}, \widetilde{B}\right)-i_{\widetilde{p}}(\widetilde{\Gamma}, \widetilde{B})=\nu_{\widetilde{p}}(\widetilde{B})$. In particular, the strict transform of a curve of generic polar type is not a curve of generic polar type for the transformed foliation.

Theorem 2. Consider a non-dicritical foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ and an invariant curve $C \subset S_{\mathcal{F}}$. For any branch $B \in \mathcal{B}(C)$, we have

$$
p_{0}\left(\mathcal{G}_{f}, B\right) \leq p_{0}(\mathcal{F}, B),
$$

where $f=0$ is a reduced equation of $C$. Moreover, the following statements are equivalent
(1) There is $B \in \mathcal{B}(C)$ such that $p_{0}\left(\mathcal{G}_{f}, B\right)=p_{0}(\mathcal{F}, B)$.
(2) The foliation $\mathcal{F}$ is of second type with $C=S_{\mathcal{F}}$.

Finally, if $\mathcal{F}$ is of second type with $C=S_{\mathcal{F}}$, a separatrix $B \in \mathcal{B}\left(S_{\mathcal{F}}\right)$ is of Briot and Bouquet type if and only if $p_{0}\left(\mathcal{G}_{f}, B\right)=p_{0}(\mathcal{F}, B)$.
Note that in Lemma 2.2. of [G] it is proved that $\mu_{0}(\mathcal{F}, B) \geq \mu_{0}(\mathcal{G}, B)$ and hence the first statement is a consequence of Proposition $\mathbb{\square}$. Anyway we give a complete proof below.
Proof. Consider a sequence of local blow-ups

$$
\pi_{k}:\left(M_{k}, q_{k}\right) \rightarrow\left(M_{k-1}, q_{k-1}\right), \quad k=1,2, \ldots, N
$$

described as follows. First $\left(M_{0}, q_{0}\right)=\left(\mathbb{C}^{2}, 0\right)$ and $\pi_{1}$ is the blow-up of the origin. Next $\pi_{k}:\left(M_{k}, q_{k}\right) \rightarrow\left(M_{k-1}, q_{k-1}\right)$ is the blow-up centered at $q_{k-1}$, where $q_{k-1}$ is the
only point in the strict transform $B_{k-1} \subset M_{k-1}$ of $B$, followed by the localization at the new "infinitesimal near point" $q_{k}$ of $B_{k}$. We denote by $E_{k} \subset M_{k}$ the total exceptional divisor, starting with $E_{0}=\emptyset$. Denote by $C_{k} \subset M_{k}$ the strict transform of $C$ in $M_{k}$ (localized around $q_{k}$ ) and by $\mathcal{F}_{k}$, respectively $\mathcal{G}_{k}$, the transforms of $\mathcal{F}$, respectively $\mathcal{G}_{f}$, at $M_{k}$. At the final step $N \geq 1$ we ask that $C$ is desingularized at $q_{N}$, that is $B_{N}=C_{N}$ locally at $q_{N}$ and $B_{N}$ has normal crossings with $E_{N}$ at $q_{N}$. In particular, there are local coordinates $(x, y)$ at $q_{N}$ such that $E_{N}=(x=0)$ and $B_{N}=(y=0)$. The existence of this sequence is an immediate consequence of the reduction of singularities of plane curves.

Let us denote $\Delta_{k}=p_{q_{k}}\left(\mathcal{F}_{k}, B_{k}\right)-p_{q_{k}}\left(\mathcal{G}_{k}, B_{k}\right)$. Note that $\mathcal{G}_{N}$ is a generalized curve at $q_{N}$ with separatrices $x y=0$, that is, up to a coordinate change, it is given by a linear form $y d x+r x d y$ with $r \in \mathbb{Q}_{>0}$. In particular, we have $p_{q_{N}}\left(\mathcal{G}_{N}, B_{N}\right)=1$. Moreover, $q_{N}$ is a singular point for $\mathcal{F}_{N}$, since $x y=0$ are also invariant curves for $\mathcal{F}_{N}$. Hence $p_{q_{N}}\left(\mathcal{F}_{N}, B_{N}\right) \geq 1$ and thus $\Delta_{N} \geq 0$. Let us show that

$$
\begin{equation*}
\Delta_{0} \geq \Delta_{1} \geq \Delta_{2} \geq \cdots \geq \Delta_{N} \geq 0 \tag{10}
\end{equation*}
$$

Invoking Proposition [], we have

$$
\begin{equation*}
\Delta_{k-1}=\Delta_{k}+m_{k-1}\left(\nu_{q_{k-1}}\left(\mathcal{F}_{k-1}\right)-\nu_{q_{k-1}}\left(\mathcal{G}_{k-1}\right)\right), \tag{11}
\end{equation*}
$$

where $m_{k-1}=\nu_{q_{k-1}}\left(B_{k-1}\right) \geq 1$. Since $\mathcal{G}_{k-1}$ is a generalized curve at $q_{k-1}$ whose separatrices are the irreducible components of $C_{k-1} \cup E_{k-1}$ and $C_{k-1} \cup E_{k-1} \subset S_{\mathcal{F}_{k-1}}$, we have that

$$
\nu_{q_{k-1}}\left(\mathcal{F}_{k-1}\right)-\nu_{q_{k-1}}\left(\mathcal{G}_{k-1}\right) \geq 0 .
$$

This already shows the statement of Equation (四) and in particular $\Delta_{0} \geq 0$.
Now, assume $\Delta_{0}=0$. This implies that $\Delta_{k}=0$ for all $k=0,1, \ldots, N$. In particular, by Equation (四) applied to $k=1$ we deduce that $\nu_{0}(\mathcal{F})=\nu_{0}\left(\mathcal{G}_{f}\right)$, since $C \subset S_{\mathcal{F}}$, by Theorem $\mathbb{m}$, this is only possible if $C=S_{\mathcal{F}}$ and $\mathcal{F}$ is of second type.

Conversely, assume that $\mathcal{F}$ is of second type with $C=S_{\mathcal{F}}$. Take a branch $B \in$ $\mathcal{B}\left(S_{\mathcal{F}}\right)$ and let us consider the local situation at $q_{N}$ where $B_{N}=(y=0)$ and $E_{N}=(x=0)$. Write a generator of $\mathcal{F}_{N}$ as $\omega_{N}=g(x, y) y d x+h(x, y) x d y$. Then $p_{q_{N}}\left(\mathcal{F}_{N}, B_{N}\right) \geq 1$ and $p_{q_{N}}\left(\mathcal{F}_{N}, B_{N}\right)=1$ if and only if $h(0,0) \neq 0$, this is equivalent to say that $B$ is a Briot and Bouquet separatrix. That is we have shown that $B$ is a Briot and Bouquet separatrix if and only if $\Delta_{N}=0$. Moreover, at an intermediate point $q_{k}$ we have that both $\mathcal{F}_{k}$ and $\mathcal{G}_{k}$ are of second type with

$$
S_{\mathcal{F}_{k}}=S_{\mathcal{G}_{k}}=C_{k} \cup E_{k} .
$$

In particular $\nu_{q_{k-1}}\left(\mathcal{F}_{k-1}\right)-\nu_{q_{k-1}}\left(\mathcal{G}_{k-1}\right)=0$ and thus we have that

$$
\Delta_{0}=\Delta_{1}=\Delta_{2}=\cdots=\Delta_{N} \geq 0
$$

Finally, note that by Camacho and Sad arguments in [6] there is always a Briot and Bouquet separatrix. This ends the proof.
Corollary 3. We have $\Delta_{0}(\mathcal{F}, C) \geq 0$ for any non-dicritical foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ and any curve $C \subset S_{\mathcal{F}}$.

Proof．It follows from the additivity of the polar intersection indices，by applying Theorem to all the branches $B \in \mathcal{B}(C)$ ．

Corollary 4．A non－dicritical foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ is a generalized curve if and only if

$$
\Delta_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=0
$$

In this case we have $p_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=\mu_{0}\left(S_{\mathcal{F}}\right)+\nu_{0}\left(S_{\mathcal{F}}\right)-1$ ．
Proof．In view of Theorem 】 and the additivity of the polar intersection numbers the stated equality is equivalent to say that

$$
p_{0}\left(\mathcal{G}_{f}, B\right)=p_{0}(\mathcal{F}, B)
$$

for any separatrix $B \in \operatorname{Sep}(\mathcal{F})$ ．This is also equivalent to say that $\mathcal{F}$ is of second type and all the separatrices are of Briot and Bouquet，what is the same to say that $\mathcal{F}$ is a generalized curve．The second part is a consequence of Proposition $\rrbracket$（see also［［］］）．

4．2．Computation of GSV－index．Let us recall the definition in［Z］of the GSV－ index．Let $C$ be a germ of（convergent）curve invariant by a foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ ． Take a reduced equation $f=0$ of $C$ for $f \in \mathbb{C}\{x, y\}$ and a 1 －form $\omega$ that defines $\mathcal{F}$ ． There is a decomposition

$$
\begin{equation*}
g \omega=k d f+f \eta, \quad g, k \in \mathbb{C}\{x, y\}, \tag{12}
\end{equation*}
$$

where $\eta$ is a holomorphic 1 －form．
Definition 3．The $G S V$－index of $\mathcal{F}$ with respect to $C$ at the origin is defined by

$$
G S V_{0}(\mathcal{F}, C)=\frac{1}{2 \pi i} \int_{\partial C} \frac{g}{k} d\left(\frac{k}{g}\right) .
$$

Here $\partial C=C \cap S_{\epsilon}^{3}$ ，where $S_{\epsilon}^{3}$ is a small sphere centered at $0 \in \mathbb{C}^{2}$ ，oriented as the boundary of $C \cap B_{\epsilon}^{4}$ ，for a ball $B_{\epsilon}^{4}$ such that $S_{\epsilon}^{3}=\partial B_{\epsilon}^{4}$ ．

Next Lemma 四 remakes the behavior of the CSV－index for the union of two sets of separatrices（see［D］，section 3）．

Lemma 1．Let $\mathcal{F}$ be a non－dicritical foliation and consider two curves $C_{1}, C_{2} \subset S_{\mathcal{F}}$ without common branches．Then

$$
\Delta_{0}\left(\mathcal{F}, C_{1} \cup C_{2}\right)=\Delta_{0}\left(\mathcal{F}, C_{1}\right)+\Delta_{0}\left(\mathcal{F}, C_{2}\right)-2 i_{0}\left(C_{1}, C_{2}\right)
$$

Proof．In view of the definition of the polar excess and the additivity of the multi－ plicity of a curve，the statement is equivalent to say that

$$
\mu_{0}\left(C_{1} \cup C_{2}\right)=\mu_{0}\left(C_{1}\right)+\mu_{0}\left(C_{2}\right)+2 i_{0}\left(C_{1}, C_{2}\right)-1 .
$$

This is a classical property of Milnor number of curves（see［［⿴囗

Proposition 4. Let $\mathcal{F}$ be a non-dicritical foliation in $\left(\mathbb{C}^{2}, 0\right)$ and $C \subset S_{\mathcal{F}}$ be a curve union of convergent separatrices of $\mathcal{F}$. Then

$$
G S V_{0}(\mathcal{F}, C)=\Delta_{0}(\mathcal{F}, C)
$$

Proof. It is enough to consider the case that $C$ is a single separatrix. The general case follows from Lemma $\mathbb{D}$ and the similar statement for the GSV-indices.

Take a reduced equation $f=0$ for $C$ and $\gamma=\gamma(t)$ a Puiseux parametrization. For a decomposition $g \omega=k d f+f \eta$ such as in ( [ $\mathbb{D})$, we have

$$
G S V_{0}(\mathcal{F}, C)=\frac{1}{2 \pi i} \int_{\partial C} \frac{g}{k} d\left(\frac{k}{g}\right)=\frac{1}{2 \pi i} \int_{\partial \mathbb{D}_{\epsilon}} \gamma^{*}\left(\frac{g}{k} d\left(\frac{k}{g}\right)\right)=\operatorname{ord}_{t}((k / g) \circ \gamma),
$$

where $\partial \mathbb{D}_{\epsilon}$ is a small circle around $0 \in \mathbb{C}$. But, if $(a: b) \in \mathbb{P}_{\mathbb{C}}^{1}$, we get from ( $\left.\mathbb{R} \mathbb{2}\right)$ that

$$
a P+b Q=\left(\frac{k}{g}\right)\left(a f_{x}+b f_{y}\right)+\frac{f}{g} h
$$

for some $h \in \mathbb{C}\{x, y\}$. This gives

$$
\operatorname{ord}_{t}((a P+b Q) \circ \gamma)=\operatorname{ord}_{t}((k / g) \circ \gamma)+\operatorname{ord}_{t}\left(\left(a f_{x}+b f_{y}\right) \circ \gamma\right)
$$

Hence

$$
G S V_{0}(\mathcal{F}, C)=\operatorname{ord}_{t}((a P+b Q) \circ \gamma)-\operatorname{ord}_{t}\left(\left(a f_{x}+b f_{y}\right) \circ \gamma\right)=\Delta_{0}(\mathcal{F}, C)
$$

Remark 6. In view of this interpretation of the $G S V$-index in terms of polar intersection numbers, Corollary 四 says that a non-dicritical foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ is a generalized curve if and only if $S_{\mathcal{F}}$ is convergent and $G S V_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=0$. This characterization of non-dicritical generalized curves was already known: its necessity has been proved in [ [ ] , whereas its sufficiency in [[]]].

Remark 7. The $G S V$-index of a foliation $\mathcal{F}$ in $\left(\mathbb{C}^{2}, 0\right)$ with respect to a (convergent) curve $C \subset S_{\mathcal{F}}$ is also equal to

$$
G S V_{0}(\mathcal{F}, C)=\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, P, Q)}-\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\left(f, f_{x}, f_{y}\right)}
$$

(see [ [

$$
p_{0}(\mathcal{F}, C)-\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, P, Q)}=\mu_{0}(C)-\tau_{0}(C)+\nu_{0}(C)-1
$$

where $\tau_{0}(C)=\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\left(f, f_{x}, f_{y}\right)}$ is the Tjurina number of $C$. If $S_{\mathcal{F}}$ is convergent and we define the Tjurina number $\tau_{0}(\mathcal{F})$ of the foliation $\mathcal{F}$ as

$$
\tau_{0}(\mathcal{F})=\operatorname{dim}_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, P, Q)}
$$

with $f=0$ an equation of $S_{\mathcal{F}}$, we obtain that

$$
G S V_{0}\left(\mathcal{F}, S_{\mathcal{F}}\right)=\tau_{0}(\mathcal{F})-\tau_{0}\left(S_{\mathcal{F}}\right)
$$

Hence, we have that $\mathcal{F}$ is a generalized curve if and only if $\tau_{0}(\mathcal{F})=\tau_{0}\left(S_{\mathcal{F}}\right)$.

## 5. The Poincaré problem and polar multiplicities

The notion of polar curve of a foliation has a global counterpart. Let $\mathcal{F}$ be a foliation on $\mathbb{P}_{\mathbb{C}}^{2}$ with singular set $\operatorname{Sing}(\mathcal{F})$ and $\operatorname{degree} \operatorname{deg}(\mathcal{F})$. Given $q \in \mathbb{P}_{\mathbb{C}}^{2}$, the polar curve of $\mathcal{F}$ with center $q$ is the closure of the set of tangencies of $\mathcal{F}$ with the lines containing $q$. That is:

$$
P_{q}^{\mathcal{F}}=\overline{\left\{p \in \mathbb{P}_{\mathbb{C}}^{2} \backslash \operatorname{Sing}(\mathcal{F}) ; q \in T_{p}^{\mathbb{P} \mathcal{F}\}}\right.}
$$

where $T_{p}^{\mathbb{P}} \mathcal{F}$ is the line through $p$ with direction $T_{p} \mathcal{F}$. This is a curve of degree $\operatorname{deg}(\mathcal{F})+1$, except for the unique degenerate case where $\mathcal{F}$ is the radial foliation centered at $q$ (see [ [2: $]$ ). This curve contains all points in $\operatorname{Sing}(\mathcal{F})$, as well as the center $q$. As long as $q$ varies through $\mathbb{P}_{\mathbb{C}}^{2}$, the curves $P_{q}^{\mathcal{F}}$ form a two-dimensional linear system, the polar net of $\mathcal{F}$. Its base locus is precisely $\operatorname{Sing}(\mathcal{F})$.

A foliation $\mathcal{F}$ on $\mathbb{P}_{\mathbb{C}}^{2}$ is induced in affine coordinates $\mathbb{C}^{2}$ by a polynomial 1-form $\omega=P(x, y) d x+Q(x, y) d y$. The curves with equation $a P(x, y)+b Q(x, y)=0$, where $(a: b) \in \mathbb{P}_{\mathbb{C}}^{1}$, are polar curves $P_{q}^{\mathcal{F}}$ with $q \in L_{\infty}=\mathbb{P}_{\mathbb{C}}^{1}$, where $L_{\infty}$ denotes the line at infinity with respect to these affine coordinates. Thus the local polar curves can be seen as germs of global polar curves.

In this Section, we revisit the known proofs of the degree bound for Poincaré problem in terms of polar curves. Namely, we will show that if an algebraic curve $S$ is invariant by a foliation $\mathcal{F}$ on $\mathbb{P}_{\mathbb{C}}^{2}$ then $\operatorname{deg}(S) \leq \operatorname{deg}(\mathcal{F})+2$ in the following cases:
(a) The singularities of $\mathcal{F}$ over $S$ are non-dicritical (Carnicer's paper [团]).
(b) The curve $S$ has at most nodal singularities (Cerveau-Lins Neto's paper [[2]]).

Let $S \subset \mathbb{P}_{\mathbb{C}}^{2}$ be a projective curve of degree $d$ invariant by a singular foliation $\mathcal{F}$ of $\mathbb{P}_{\mathbb{C}}^{2}$. Choose a line at infinity $L_{\infty}$ transversal to $S$ that avoids $\operatorname{Sing}(\mathcal{F})$ and $\operatorname{Sing}(S)$. Consider affine coordinates in $\mathbb{C}^{2}=\mathbb{P}_{\mathbb{C}}^{2} \backslash L_{\infty}$ and fix an affine reduced polynomial equation $f(x, y)=0$ for $S \backslash L_{\infty}$. Let $\mathcal{G}$ be the foliation on $\mathbb{P}_{\mathbb{C}}^{2}$ defined by $d f=0$ in the affine part. The foliation $\mathcal{G}$ is the foliation of reference in [ $[\mathbb{G}]$.
Notice that $\operatorname{deg}(\mathcal{G})=d-1$ and $L_{\infty}$ is $\mathcal{G}$-invariant. The only singularities of $\mathcal{G}$ on $L_{\infty}$ are the $d$ points in $S \cap L_{\infty}$. In all such points $\mathcal{G}$ is analytically equivalent to a radial foliation. To see this, it is enough to observe that $\mathcal{G}$ has the rational first integral $F(X, Y, Z) / Z^{d}$, where $X, Y, Z$ are homogeneous coordinates and $F$ is the homogeneous polynomial of degree $d$ such that $F(x, y, 1)=f(x, y)$.

Given $q \in \mathbb{P}_{\mathbb{C}}^{2}$, let us denote $\Gamma_{q}=P_{q}^{\mathcal{F}}$ and $\Sigma_{q}=P_{q}^{\mathcal{G}}$. There is a nonempty Zariski open set $U \subset \mathbb{P}_{\mathbb{C}}^{2}$ such that the following properties hold for any $q \in U$ :
(1) For any $r \in S \cap L_{\infty}$ we have that $r \notin \Gamma_{q}$ and $\Sigma_{q}$ is non singular and transversal to $S$ and $L_{\infty}$ in $r$. In particular $i_{r}\left(\Sigma_{q}, S\right)=1$ and $i_{r}\left(\Gamma_{q}, S\right)=0$.
(2) For any $r \in S \backslash\left(L_{\infty} \cup \operatorname{Sing}(\mathcal{F})\right)$ we have that either $r \notin \Gamma_{q} \cup \Sigma_{q}$ or $r \in \Gamma_{q} \cap \Sigma_{q}$ and then $i_{r}\left(\Gamma_{q}, S\right)=i_{r}\left(\Sigma_{q}, S\right)=1$.
(3) For any $r \in S \backslash\left(L_{\infty} \cup \operatorname{Sing}(S)\right)$ we have that either $i_{r}\left(\Sigma_{q}, S\right)=0$ or

$$
i_{r}\left(\Gamma_{q}, S\right) \geq i_{r}\left(\Sigma_{q}, S\right)=1
$$

In the case $i_{r}\left(\Gamma_{q}, S\right)=i_{r}\left(\Sigma_{q}, S\right)$ then $\mathcal{F}$ is non-singular at $r$.
(4) For any $r \in S \cap \operatorname{Sing}(\mathcal{F})$ we have that $\Gamma_{q}$, respectively $\Sigma_{q}$, is a $S$-polar generic curve for $\mathcal{F}$, respectively for $\mathcal{G}$. In particular, when $\mathcal{F}$ is non-dicritical at $r$ we have

$$
\Delta_{r}(\mathcal{F}, S)=i_{r}\left(\Gamma_{q}, S\right)-i_{r}\left(\Sigma_{q}, S\right) \geq 0
$$

in view of Corollary []. If $r$ is a nodal singularity of $S$, we have $i_{r}\left(\Sigma_{q}, S\right)=2$ (note that $S$ has two branches at $r$ ) and thus $i_{r}\left(\Gamma_{q}, S\right)-i_{r}\left(\Sigma_{q}, S\right) \geq 0$.
For the second statement, let us note that up to choosing a generic $q$, the line joining $q$ and $r$, for $r \in S \backslash\left(L_{\infty} \cup \operatorname{Sing}(\mathcal{F})\right)$ is either transversal to $S$ of tangent with intersection multiplicity equal to 2 . In the first case we have that $r \notin \Gamma_{q} \cup \Sigma_{q}$. Consider the second case. Let us choose affine coordinates $(x, y)$ centered at $r$ such that the lines passing through $q$ are the leaves of $d x=0$. Moreover, the curve $S$ is locally given at $r$ as $y^{2}+x g(x, y)=0$ where $g(x, y) \in \mathbb{C}\{x, y\}$ is such that $g(0,0) \neq 0$. Since $\mathcal{F}$ is non singular at $r$, it is locally given at $r$ by $\eta=0$ where

$$
\eta=d\left(U(x, y)\left(y^{2}+x g(x, y)\right)\right), \quad U(0,0) \neq 0
$$

Hence $\Gamma_{q}$ is given at $r$ by $h(x, y)=0$, for $d x \wedge \eta=h(x, y) d x \wedge d y$. That is

$$
h(x, y)=\frac{\partial\left(U(x, y)\left(y^{2}+x g(x, y)\right)\right)}{\partial y}=2 y U(x, y)+y^{2} \frac{\partial U(x, y)}{\partial y}+x(\cdots) .
$$

This implies that $i_{r}\left(\Gamma_{q}, S\right)=1$. Same argument to see that $i_{r}\left(\Sigma_{q}, S\right)=1$. These kind of computations also show the third statement.

Now, let us consider a generic $q$ as above. By Bézout's theorem applied to the curves $S$ and $\Sigma_{q}$, we have

$$
d^{2}=\sum_{r \in \mathcal{A}} i_{r}\left(\Sigma_{q}, S\right)+\sum_{r \in \mathcal{B}} i_{r}\left(\Sigma_{q}, S\right)+\sum_{r \in \mathcal{C}} i_{r}\left(\Sigma_{q}, S\right),
$$

where $\mathcal{A}=S \cap L_{\infty}, \mathcal{B}=S \backslash\left(L_{\infty} \cup \operatorname{Sing}(\mathcal{F})\right)$ and $\mathcal{C}=S \cap \operatorname{Sing}(\mathcal{F})$. We obtain that

$$
d(d-1)=\sum_{r \in \mathcal{B}} i_{r}\left(\Sigma_{q}, S\right)+\sum_{r \in \mathcal{C}} i_{r}\left(\Sigma_{q}, S\right) .
$$

By Bézout's theorem applied to the curves $S$ and $\Gamma_{q}$, we obtain

$$
d(\operatorname{deg}(\mathcal{F})+1)=\sum_{r \in \mathcal{B}} i_{r}\left(\Gamma_{q}, S\right)+\sum_{r \in \mathcal{C}} i_{r}\left(\Gamma_{q}, S\right) .
$$

Taking the difference, we have

$$
\begin{equation*}
\operatorname{deg}(\mathcal{F})+2-d=\frac{1}{d} \sum_{r \in S \cap \operatorname{Sing}(\mathcal{F})}\left(i_{r}\left(\Gamma_{q}, S\right)-i_{r}\left(\Sigma_{q}, S\right)\right) \geq 0 \tag{13}
\end{equation*}
$$

Note that in the non-dicritical case this equation reads as

$$
\begin{equation*}
\operatorname{deg}(\mathcal{F})+2-d=\frac{1}{d} \sum_{r \in \operatorname{S\cap Sing}(\mathcal{F})} \Delta_{r}(\mathcal{F}, S) \tag{14}
\end{equation*}
$$

Remark 8. By Proposition 四, the number in the right part of Equation ([3]) is $(1 / d) \sum_{p \in \operatorname{Sing}(\mathcal{F}) \cap S} G S V_{p}(\mathcal{F}, S)$, which is obtained in [प] and [7] as $c_{1}\left(N_{\mathcal{F}}\right) \cdot S-S \cdot S$, where $N_{\mathcal{F}}=\mathcal{O}(d+2)$ is the normal bundle of $\mathcal{F}$.
5.1. Logarithmic foliations. The limit case for the bound of degrees is closely related with logarithmic forms, as shown in the following statement

Theorem 3 ([[D2], [ []]). Take a curve $S \subset \mathbb{P}_{\mathbb{C}}^{2}$ given by a homogeneous polynomial equation $P=P_{1} P_{2} \cdots P_{n}=0$, where each polynomial $P_{i}$ is irreducible of degree $d_{i}$. Suppose that $S$ is invariant for a foliation $\mathcal{F}$ of $\mathbb{P}_{\mathbb{C}}^{2}$ that is non-dicritical at each point $q \in S$. The following statements are equivalent
(1) $\operatorname{deg}(S)=\operatorname{deg}(\mathcal{F})+2$.
(2) There are residues $\lambda_{i} \in \mathbb{C}^{*}$ with $\sum_{i=1}^{n} \lambda_{i} d_{i}=0$ such that $\mathcal{F}$ is given by $W=0$, where $W$ is the global closed logarithmic 1-form in $\mathbb{P}_{\mathbb{C}}^{2}$ defined by

$$
W=\sum_{i=1}^{n} \lambda_{i} \frac{d P_{i}}{P_{i}} .
$$

(3) The foliation $\mathcal{F}$ is a generalized curve at any $q \in S$ and $S$ contains all the separatrices of $\mathcal{F}$ at $q$.

The case of nodal singularities has been proved by D. Cerveau and A. Lins Neto in [ [ L ] and it is also valid in a dicritical situation. Later, the result was extended by M. Brunella to the non-dicritical case in ([Z] , Proposition 10). His proof relies on computations involving indices of vector fields for a subsequent application of the following result of Deligne:

Theorem (Deligne [ [6] ). Let $\omega$ be a logarithmic 1-form on a projective variety $M$ whose polar divisor has normal crossings. Then $\omega$ is closed (and hence it defines a codimension one logarithmic foliation on $M$ ).

Recently D. Cerveau gave a very short and elegant proof of Theorem [3] in [[]]]. This approach is based on the fact that a non-dicritical logarithmic meromorphic 1-form $\omega$ (that is $\omega$ and $d \omega$ have at most simple poles) has also this property after a blow-up.

Concerning polar invariants, we can see as a consequence of Corollary 四 and Equation ([4]) that the statements (1) and (3) of Theorem [ are equivalent.

Let us end these notes by providing a proof that (1) implies (2) following the arguments in [ [ $[3]$, [ $\mathbb{Z}]$ and [3] of D. Cerveau, M. Brunella and L. G. Mendes, but avoiding the direct use of Deligne's statement.

Let us consider a homogeneous 1-form

$$
T=\sum_{i=0}^{2} A_{i}\left(X_{0}, X_{1}, X_{2}\right) d X_{i} ; \quad \sum_{i=0}^{2} X_{i} A_{i}=0
$$

defining $\mathcal{F}$ (see [ [ $]$ ), where $\operatorname{deg} A_{i}=\operatorname{deg} \mathcal{F}+1$ and consider the meromorphic 1-form $\Omega=T / P$. Since $\operatorname{deg}(P)=\operatorname{deg} \mathcal{F}+2$, we see that $\Omega$ defines a global meromorphic 1 -form $\omega$ on $\mathbb{P}_{\mathbb{C}}^{2}$. Looking at each point $p \in S$, by Proposition 2.1 in [[]:3], the form $\omega$ is logarithmic at $p$. We apply the stability by non-dicritical blowing-up (Proposition 2.2 in [ [ [ ] ) of being logarithmic, to see that after a reduction of singularities

$$
\pi: \widetilde{\mathbb{P}}_{\mathbb{C}}^{2} \rightarrow \mathbb{P}_{\mathbb{C}}^{2}
$$

of $\mathcal{F}$ along $S$, we obtain a logarithmic 1 -form $\pi^{*} \omega$ that has locally one of the following expressions at a point $p$ in the total transform $\tilde{S}$ of $S$ :

$$
U(x, y)\left(\lambda_{p} \frac{d x}{x}+b(x, y) d y\right) ; \quad U(x, y)\left(\lambda_{p} \frac{d x}{x}+b(x, y) \frac{d y}{y}\right)
$$

with $U(0,0)=1$, where $x=0$ is a selected irreducible component of the total transform $\tilde{S}$ of $S$. The first case corresponds to a non-singular point in $\tilde{S}$ and in the second one we have $S=(x y=0)$ (note that we can do the same argument for the component $y=0$ ). The functions $\lambda_{p}$ are holomorphic, hence they are constant. So we can attach a residue $\lambda_{i}$ to each irreducible component $P_{i}=0$ of $S$. Moreover, taking a general line $\ell$ in $\mathbb{P}_{\mathbb{C}}^{2}$ avoiding the singular locus of $P$, the sum of residues in $\left.\omega\right|_{\ell}$ gives

$$
\sum_{i=1}^{n} \lambda_{i} \operatorname{deg}\left(P_{1}\right)=0
$$

Now, we consider the global meromorphic 1-form

$$
W=\sum_{i=1}^{n} \lambda_{i} \frac{d P_{i}}{P_{i}}
$$

It follows that $\Omega-W$ is holomorphic (the residues coincide) and hence $\Omega=c W$ for a non-null constant $c \in \mathbb{C}$.

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Felipe Cano<br>Departamento de Álgebra, Análisis Matemático, Geometría y Topología Universidad de Valladolid<br>Paseo de Belén 7, 47011 - Valladolid, SPAIN<br>fcano@agt.uva.es<br>Nuria Corral<br>Departamento de Matemáticas, Estadística y Computación Universidad de Cantabria<br>Avda. de los Castros s/n, 39005 - Santander, SPAIN<br>nuria.corral@unican.es<br>Rogério Mol<br>Departamento de Matemática<br>Universidade Federal de Minas Gerais<br>Av. Antônio Carlos, 6627 C.P. 702<br>30123-970 - Belo Horizonte - MG, BRAZIL<br>rsmol@mat.ufmg.br


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