

**FIGURE 3**  
A projection transformation.

The next two matrix transformations can be viewed geometrically. They reinforce the dynamic view of a matrix as something that transforms vectors into other vectors. Section 2.7 contains other interesting examples connected with computer graphics.

**EXAMPLE 2** If  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  projects points in  $\mathbb{R}^3$  onto the  $x_1x_2$ -plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

See Figure 3. ■

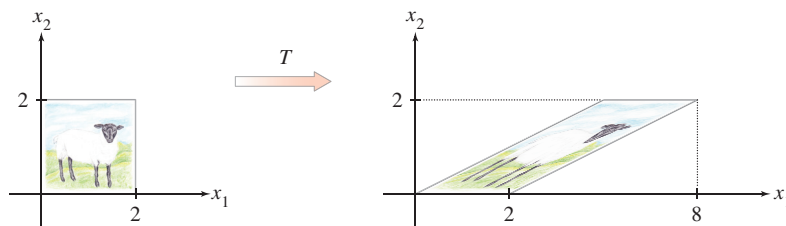


sheep



sheared sheep

**EXAMPLE 3** Let  $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ . The transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is called a **shear transformation**. It can be shown that if  $T$  acts on each point in the  $2 \times 2$  square shown in Figure 4, then the set of images forms the shaded parallelogram. The key idea is to show that  $T$  maps line segments onto line segments (as shown in Exercise 27) and then to check that the corners of the square map onto the vertices of the parallelogram. For instance, the image of the point  $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  is  $T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ , and the image of  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  is  $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ .  $T$  deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and crystallography. ■



**FIGURE 4** A shear transformation.

## Linear Transformations

Theorem 5 in Section 1.4 shows that if  $A$  is  $m \times n$ , then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  has the properties

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} \quad \text{and} \quad A(c\mathbf{u}) = cA\mathbf{u}$$

for all  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$  and all scalars  $c$ . These properties, written in function notation, identify the most important class of transformations in linear algebra.

### DEFINITION

A transformation (or mapping)  $T$  is **linear** if:

- (i)  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in the domain of  $T$ ;
- (ii)  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and all  $\mathbf{u}$  in the domain of  $T$ .

Every matrix transformation is a linear transformation. Important examples of linear transformations that are not matrix transformations will be discussed in Chapters 4 and 5.