SOLUTION TO PRACTICE PROBLEM

The following data are given:

	Inputs Needed per Unit of Output		
Purchased from:	**Goods**	**Services**	**External Demand**
Goods	.2	.4	20
Services	.5	.3	30

The Leontief input–output model is $\mathbf{x} = C\mathbf{x} + \mathbf{d}$, where

$$C = \begin{bmatrix} .2 & .4 \\ .5 & .3 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

2.7 APPLICATIONS TO COMPUTER GRAPHICS

Computer graphics are images displayed or animated on a computer screen. Applications of computer graphics are widespread and growing rapidly. For instance, computer-aided design (CAD) is an integral part of many engineering processes, such as the aircraft design process described in the chapter introduction. The entertainment industry has made the most spectacular use of computer graphics—from the special effects in *Amazing Spider-Man 2* to PlayStation 4 and Xbox One.

Most interactive computer software for business and industry makes use of computer graphics in the screen displays and for other functions, such as graphical display of data, desktop publishing, and slide production for commercial and educational presentations. Consequently, anyone studying a computer language invariably spends time learning how to use at least two-dimensional (2D) graphics.

This section examines some of the basic mathematics used to manipulate and display graphical images such as a wire-frame model of an airplane. Such an image (or picture) consists of a number of points, connecting lines or curves, and information about how to fill in closed regions bounded by the lines and curves. Often, curved lines are approximated by short straight-line segments, and a figure is defined mathematically by a list of points.

Among the simplest 2D graphics symbols are letters used for labels on the screen. Some letters are stored as wire-frame objects; others that have curved portions are stored with additional mathematical formulas for the curves.

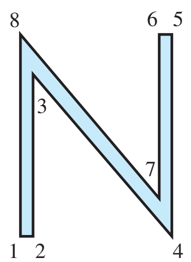


FIGURE 1
Regular N.

EXAMPLE 1 The capital letter N in Figure 1 is determined by eight points, or *vertices*. The coordinates of the points can be stored in a data matrix, D.

Vertex:

$$\begin{array}{r} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & .5 & .5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix} \end{array} = D$$

In addition to D, it is necessary to specify which vertices are connected by lines, but we omit this detail. ■

The main reason graphical objects are described by collections of straight-line segments is that the standard transformations in computer graphics map line segments onto other line segments. (For instance, see Exercise 27 in Section 1.8.) Once the vertices that describe an object have been transformed, their images can be connected with the appropriate straight lines to produce the complete image of the original object.

EXAMPLE 2 Given $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$, describe the effect of the shear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on the letter N in Example 1.

SOLUTION By definition of matrix multiplication, the columns of the product AD contain the images of the vertices of the letter N.

$$AD = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ & \begin{bmatrix} 0 & .5 & 2.105 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.420 & 0 & 8 & 8 & 1.580 & 8 \end{bmatrix} \end{matrix}$$

The transformed vertices are plotted in Figure 2, along with connecting line segments that correspond to those in the original figure. ■

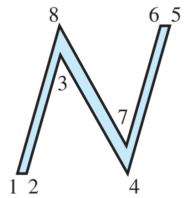


FIGURE 2
Slanted N.

The italic N in Figure 2 looks a bit too wide. To compensate, shrink the width by a scale transformation that affects the x-coordinates of the points.

EXAMPLE 3 Compute the matrix of the transformation that performs a shear transformation, as in Example 2, and then scales all x-coordinates by a factor of .75.

SOLUTION The matrix that multiplies the x-coordinate of a point by .75 is

$$S = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix}$$

So the matrix of the composite transformation is

$$SA = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .75 & .1875 \\ 0 & 1 \end{bmatrix}$$

The result of this composite transformation is shown in Figure 3. ■

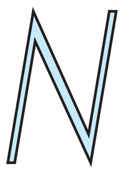
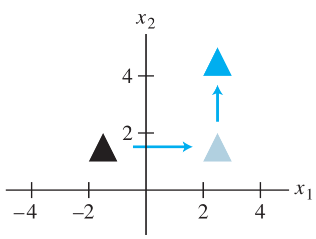


FIGURE 3
Composite transformation of N.

The mathematics of computer graphics is intimately connected with matrix multiplication. Unfortunately, translating an object on a screen does not correspond directly to matrix multiplication because translation is not a linear transformation. The standard way to avoid this difficulty is to introduce what are called *homogeneous coordinates*.

Homogeneous Coordinates

Each point (x, y) in $\mathbb{R}^2$ can be identified with the point $(x, y, 1)$ on the plane in $\mathbb{R}^3$ that lies one unit above the xy-plane. We say that (x, y) has *homogeneous coordinates* $(x, y, 1)$. For instance, the point $(0, 0)$ has homogeneous coordinates $(0, 0, 1)$. Homogeneous coordinates for points are not added or multiplied by scalars, but they can be transformed via multiplication by 3×3 matrices.



Translation by $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

EXAMPLE 4 A translation of the form $(x, y) \mapsto (x + h, y + k)$ is written in homogeneous coordinates as $(x, y, 1) \mapsto (x + h, y + k, 1)$. This transformation can be computed via matrix multiplication:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}$$ ■

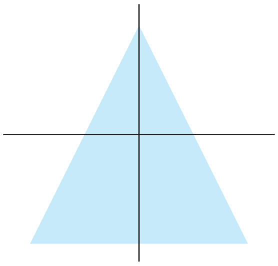
EXAMPLE 5 Any linear transformation on $\mathbb{R}^2$ is represented with respect to homogeneous coordinates by a partitioned matrix of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, where A is a 2×2 matrix. Typical examples are

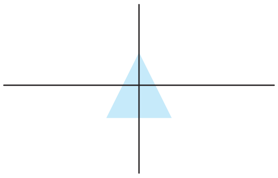
$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

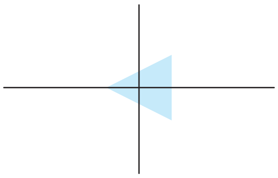
Counterclockwise rotation about the origin, angle φ — Reflection through $y = x$ — Scale x by s and y by t ■

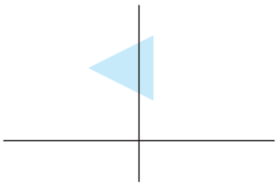
Composite Transformations

The movement of a figure on a computer screen often requires two or more basic transformations. The composition of such transformations corresponds to matrix multiplication when homogeneous coordinates are used.


Original Figure


After Scaling


After Rotating


After Translating

EXAMPLE 6 Find the 3×3 matrix that corresponds to the composite transformation of a scaling by .3, a rotation of 90° about the origin, and finally a translation that adds $(-.5, 2)$ to each point of a figure.

SOLUTION If $\varphi = \pi/2$, then $\sin\varphi = 1$ and $\cos\varphi = 0$. From Examples 4 and 5, we have

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \xrightarrow{\text{Scale}} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{Rotate}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{Translate}} \begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The matrix for the composite transformation is

$$\begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -.5 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -.3 & -.5 \\ .3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$ ■

3D Computer Graphics

Some of the newest and most exciting work in computer graphics is connected with molecular modeling. With 3D (three-dimensional) graphics, a biologist can examine a simulated protein molecule and search for active sites that might accept a drug molecule. The biologist can rotate and translate an experimental drug and attempt to attach it to the protein. This ability to *visualize* potential chemical reactions is vital to modern drug and cancer research. In fact, advances in drug design depend to some extent upon progress