Linear biseparating maps between spaces of vector-valued differentiable functions and automatic continuity

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Abstract

We give a complete description of linear biseparating maps between spaces of vector-valued differentiable functions. This description automatically implies continuity of such maps.

Key words: biseparating map, disjointness preserving, automatic continuity, vector-valued diffrentiable functions
1991 MSC: Primary 47B33; Secondary 46H40, 47B38, 46E40, 46E25

1 Introduction

It is well known that an algebraic link between spaces of continuous functions may lead to a topological link between the spaces on which the functions are defined. For instance, it turns out that if there exists a ring isomorphism $T : C(X) \to C(Y)$, then the realcompactifications of X and Y are homeomorphic ([16, pp. 115-118]). Also if h is the resultant homeomorphism from the realcompactification of Y onto that of X, then $Tf = f \circ h$ for every $f \in C(X)$, so we have a complete description of it. As a result, when X and Y are realcompact, we deduce that if both spaces of continuous functions $C(X)$ and

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[1] Research partially supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT, PB98-1102).

$C(Y)$ are endowed with the compact-open topology, then every ring isomorphism between them is continuous. In this result, the key point is that every ring isomorphism sends maximal ideals into maximal ideals. This implies that a good description of maximal ideals lead to the definition of a map from Y onto X.

Of course, the pattern above has been successfully applied to many other algebras of functions. However the situation becomes more complicated if we consider spaces of functions which take values in *arbitrary* Banach spaces. In this context and unlike algebra or ring homomorphisms, we can still use mappings satisfying the property $\|Tf\| \, \|Tg\| \equiv 0$ if and only if $\|f\| \, \|g\| \equiv 0$. These maps are called *biseparating*, and coincide with disjointness preserving mappings whose inverses preserve disjointness too ([1]). In general these maps turn out to be efficacious substitutes for homomorphisms. Indeed, in [2], we prove that the existence of a biseparating mapping between a large class of spaces of vector-valued continuous functions $A(X, E)$ and $A(Y, F)$ (E, F are Banach spaces) yields homeomorphisms between some compactifications (and even the realcompactifications) of X and Y. The automatic continuity of a *linear* biseparating mapping is also accomplished in some cases (see [3,4]). Related results have also been given recently, for some other families of scalar-valued functions, for instance, in [5,15,19] and [20]. In this paper, we go a step beyond and work in a context which does not seem to have made its way into the literature yet, namely, linear operators between spaces of differentiable functions taking values in arbitrary Banach spaces.

As for spaces (indeed algebras) of *scalar-valued* differentiable functions, Myers ([25]) showed that the structure of a compact differentiable manifold of class C^n is determined by the algebra of all real-valued functions on M of class C^n. In this line, Pursell ([28]) checked that the ring structure of infinitely differentiable functions defined on an open convex set of $\mathbb{R}^n$ determines such set up to a diffeomorphism. Homomorphisms between algebras of differentiable functions defined on real Banach spaces have been studied by Aron, Gómez and Llavona ([6]); in their paper a description of homomorphisms is given and the automatic continuity is obtained as a corollary in quite a general setting, which includes in particular the case when the Banach spaces are finite-dimensional (see also [17]).

On the other hand, automatic continuity results for algebras of differentiable functions have been also given for instance in [7,21,24,26] and [27]. In particular, automatic continuity of separating maps defined on spaces of differentiable functions has been studied by Kantrowitz and Neumann in [22]. For classical results and techniques in the study of automatic continuity, see [10] and [29], and the recent book [11] by H. G. Dales.

2 Preliminaries and notation

Let E be a *real* Banach space. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a p-tuple of non-negative integers, we set $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_p$. If Ω is a nonempty open subset of $\mathbb{R}^p$, then $C^n(\Omega, E)$ consists of the E-valued functions f in Ω that are of class C^n, that is, those functions whose partial derivatives

$$\partial^\lambda f := \frac{\partial^{\lambda_1+\lambda_2+\ldots+\lambda_p} f}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \ldots \partial x_p^{\lambda_p}}$$

exist and are continuous for each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda$, where $\Lambda := \{\lambda \in (\mathbb{N} \cup \{0\})^p : |\lambda| \leq n\}$. It is well known that, if for $L^k(\mathbb{R}^p, E)$ we denote the space of continuous k-$\mathbb{R}$-linear maps of $\mathbb{R}^p$ into E, then $C^n(\Omega, E)$ coincides with the space of maps $f : \Omega \to E$ such that the differential $D^k f : \Omega \to L^k(\mathbb{R}^p, E)$ exists and is continuous for each $k = 0, \ldots, n$. It is well known that, when it exists at a point $a \in \Omega$, the differential $D^k f(a) \in L^k(\mathbb{R}^p, E)$ is a symmetric form of degree k.

Now, given a map $f \in C^n(\Omega, E)$, we define its Taylor polynomial function of degree n at $a \in \Omega$ as

$$\begin{aligned} T_a(x) := & f(a) + Df(a)(x-a) + \frac{1}{2} D^2 f(a)(x-a, x-a) + \ldots \\ & + \frac{1}{n!} D^n f(a)(\underbrace{x-a, x-a, \ldots, x-a}_{n}). \end{aligned}$$

In the case when $E = \mathbb{R}$, $C_c^n(\Omega, \mathbb{R})$ will denote the subring of $C^n(\Omega, \mathbb{R})$ of functions with compact support.

On the other hand, in the case when Ω is also bounded $C^n(\bar{\Omega}, E)$ denotes the subspace of $C^n(\Omega, E)$ of those functions whose partial derivatives up to order n admit continuous extension to the boundary of Ω.

For a set $C \subset \mathbb{R}^p$, $\mathrm{cl}_{\mathbb{R}^p} C$ and $\mathrm{int}_{\mathbb{R}^p} C$ denote its closure and its interior in $\mathbb{R}^p$, respectively. Given $x_0 \in \mathbb{R}^p$ and $\delta > 0$, $B(x_0, \delta)$ and $\bar{B}(x_0, \delta)$ stand for the open and closed balls of center x_0 and radius δ, respectively. As for the norm in $\mathbb{R}^p$, if $x = (x_1, x_2, \ldots, x_p)$ belongs to $\mathbb{R}^p$, we set $|x| := \max_j |x_j|$. For $i \in \{1, 2, \ldots, p\}$, $x_i : \Omega \to \mathbb{R}$ will be the projection on the i-th coordinate, that is, $x_i(t_1, t_2, \ldots, t_p) = t_i$ for every $(t_1, t_2, \ldots, t_p) \in \Omega$.

Notice also that if E is a Banach space over $\mathbb{C}$, it can also be viewed as a *real* space, and in this sense we consider defined the space $C^n(\Omega, E)$. It is immediate that all results valid in the real setting hold also in the complex case. But being also complex ensures that $C^n(\Omega, E)$ is both real and complex as a linear space, and consequently we can consider both real and complex linear maps from $C^n(\Omega, E)$ into some other vector spaces. This is the reason why in this paper we will assume that E and F are $\mathbb{K}$-Banach spaces, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

As for the spaces of linear functions, we will denote by $L'(E, F)$ and by $I'(E, F)$ the sets of (not necessarily continuous) linear maps and bijective linear maps from E into F, respectively. $L(E, F)$ and $I(E, F)$ will denote the spaces of *continuous* linear maps and bijective *continuous* linear maps from E into F.

The context. From now on we will assume that we are in one of the following two situations. All definitions, results and comments given in this paper apply to these two situations unless otherwise stated.

- **Situation 1.** Ω and Ω' are (not necessarily bounded) open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively ($p, q \in \mathbb{N}$). $A^n(\Omega, E) = C^n(\Omega, E)$ and $A^m(\Omega', F) = C^m(\Omega', F)$ $(n, m \geq 1)$. $A^n(\Omega, \mathbb{K}) = C^n(\Omega, \mathbb{K})$ and $A^m(\Omega', \mathbb{K}) = C^m(\Omega', \mathbb{K})$.
- **Situation 2.** Ω and Ω' are *bounded* open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively ($p, q \in \mathbb{N}$), with the property that $\operatorname{int}_{\mathbb{R}^p} \operatorname{cl}_{\mathbb{R}^p} \Omega = \Omega$ and $\operatorname{int}_{\mathbb{R}^q} \operatorname{cl}_{\mathbb{R}^q} \Omega' = \Omega'$. $A^n(\Omega, E) = C^n(\bar{\Omega}, E)$ and $A^m(\Omega', F) = C^m(\bar{\Omega}', F)$ $(n, m \geq 1)$. $A^n(\Omega, \mathbb{K}) = C^n(\bar{\Omega}, \mathbb{K})$ and $A^m(\Omega', \mathbb{K}) = C^m(\bar{\Omega}', \mathbb{K})$.

This means that when we refer to spaces Ω, Ω', $A^n(\Omega, E)$, $A^m(\Omega', F)$, $A^n(\Omega, \mathbb{K})$, $A^m(\Omega', \mathbb{K})$, we assume that all of them are included at the same time in one of the above two situations.

The topologies. One of the goals of this paper is to provide some results of automatic continuity. This will be done when the spaces of functions are endowed with some natural topologies.

Definition 2.1 We say that a locally convex topology in $A^n(\Omega, E)$ is *compatible with the pointwise convergence* if the following two conditions are satisfied:

- 1. when endowed with it, $A^n(\Omega, E)$ is a Fréchet (or Banach) space, and
- 2. if (f_n) is a sequence in $A^n(\Omega, E)$ converging to zero, then $(f_n(x))$ converges to zero for every $x \in \Omega$.

Biseparating maps. For a function $f \in A^n(\Omega, E)$, we denote by $c(f)$ the cozero set of f, that is, the set $\{x \in \Omega : f(x) \neq 0\}$.

Definition 2.2 A map $T : A^n(\Omega, E) \to A^m(\Omega', F)$ is said to be *separating* if it is additive and $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in A^n(\Omega, E)$ satisfy $c(f) \cap c(g) = \emptyset$. Besides T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, we see that an additive map $T : A^n(\Omega, E) \to A^m(\Omega', F)$ is separating if $\|(Tf)(y)\| \, \|(Tg)(y)\| = 0$ for all $y \in \Omega'$ whenever $f, g \in A^n(\Omega, E)$ satisfy $\|f(x)\| \, \|g(x)\| = 0$ for all $x \in \Omega$.

Let $\Omega_1 := \Omega$, $\Omega'_1 := \Omega'$ when we are in Situation 1, and $\Omega_1 := \bar{\Omega}$, $\Omega'_1 := \bar{\Omega}'$ if we are in Situation 2. A point $x \in \Omega_1$ is said to be a support point of $y \in \Omega'_1$ if, for every neighborhood U of x in Ω_1, there exists $f \in A^n(\Omega, E)$ satisfying $c(f) \subset U$ such that $(Tf)(y) \neq 0$.

In [2], biseparating maps are studied in a more general setting. In particular, applied to our situation, we have that $A^n(\Omega, E)$ and $A^m(\Omega', F)$ are modules over the strongly regular rings $C^n(\Omega_1, \mathbb{R})$ and $C^n(\Omega'_1, \mathbb{R})$, respectively (for the definition of strongly regular ring, see [2]; see also [30, Corollary 1.2] or [18, Proposition 2.12.5]). According to [2, Corollary 3.2], we conclude that there exists a homeomorphism h from Ω'_1 onto Ω_1 (which implies that in both Situation 1 and Situation 2 h is a homeomorphism from Ω' onto Ω). This map h sends each point in Ω'_1 into its support point in Ω_1, and is called *support map* for T. It turns out that the support map for T^{-1} is h^{-1} (see the proof of [2, Theorem 3.1]). Rephrasing Lemma 4.4 in [2], we have the following property.

Lemma 2.1 *If $y \in \Omega'_1$ and $f \in A^n(\Omega, E)$ vanishes on a neighborhood of $h(y)$, then $(Tf)(y) = 0$.*

Functions of class $\mathrm{s} - C^n$. Suppose that $K : \Omega \to L(E, F)$ is a continuous map, where $L(E, F)$ is endowed with the topology of the norm. For each $\mathbf{e} \in E$, we define $K_{\mathbf{e}} : \Omega \to F$ as $K_{\mathbf{e}}(y) := (Ky)(\mathbf{e})$ for every $y \in \Omega$. We say that K is of class $\mathrm{s} - C^1$ if, for every $\mathbf{e} \in E$, the map $K_{\mathbf{e}}$ admits all partial derivatives of order 1 in Ω, and for each $i = 1, \ldots, p$, the map

$$\frac{\partial_{\mathrm{s}}}{\partial x_i} K : \Omega \to L(E, F),$$

sending each $y \in \Omega$ and each $\mathbf{e} \in E$ into $\frac{\partial}{\partial x_i} K_{\mathbf{e}}(y)$, is continuous when considering in $L(E, F)$ the strong operator topology, that is, the coarsest topology such that the mapping $A \in L(E, F) \hookrightarrow A\mathbf{e} \in F$ is continuous for every $\mathbf{e} \in E$.

Definition 2.3 Let $n \geq 2$. A map $J : \Omega \to L(E, F)$ is said to be of class $\mathrm{s} - C^n$ if the following two statements are satisfied:

- 1. J is of class C^{n-1} (considering $L(E, F)$ as a Banach space).
- 2. All partial derivatives $K : \Omega \to L(E, F)$ of order $n - 1$ of J are of class $\mathrm{s} - C^1$.

Examples. Next we provide two examples of biseparating linear maps between spaces of vector-valued functions of class C^1. The first example will give the most basic form of a biseparating linear map, which is essentially the only possible when E and F are finite-dimensional spaces, that is, the case when the operator through which it is defined is itself of class C^1. In the second example, the biseparating map is defined through an operator which is of class $\mathrm{s}-C^1$ but it is not of class C^1.

Example 2.4 Let $\Omega = \Omega' = (-1, 1)$, $E = F = \mathbb{R}^2$. For each $t \in (-1, 1)$, let us consider $Jt : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$(Jt)(x_1, x_2) := ((2 + t)x_1, (2 - t)x_2)$$

for $(x_1, x_2) \in \mathbb{R}^2$. Clearly each Jt is linear, bijective and continuous, being its inverse $Kt : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$(Kt)(x_1, x_2) = \left(\frac{x_1}{2 + t}, \frac{x_2}{2 - t}\right)$$

for every $(x_1, x_2) \in \mathbb{R}^2$.

It is easy to see that the derivatives of $J : (-1, 1) \to L(\mathbb{R}^2, \mathbb{R}^2)$ and $K : (-1, 1) \to L(\mathbb{R}^2, \mathbb{R}^2)$ exist and, for $(x_1, x_2) \in \mathbb{R}^2$, $(J't)(x_1, x_2) = (x_1, -x_2)$ and

$$(K't)(x_1, x_2) = \left(\frac{-tx_1}{(2 + t)^2}, \frac{tx_2}{(2 - t)^2}\right).$$

In this way we see that J and K are of class C^1. Consequently the map $T; C^1((-1, 1), \mathbb{R}^2)$, defined as $(Tf)(t) = (Jt)(f(t))$ is biseparating (see Proposition 6.1).

Example 2.5 Let $\Omega = \Omega' = (-1, 1)$, $E = F = c_0$ be the space of sequences in $\mathbb{K}$ converging to zero, endowed with the sup norm. For each $t \in (-1, 1)$, let us define the map $Jt : c_0 \to c_0$ as

$$(Jt)(x_n) := \left(\left(2 + \frac{2n - 1}{2n} t^{\frac{2n}{2n-1}}\right) x_n\right)$$

for each $(x_n) \in c_0$.

It is easy to see that each $Jt \in I(c_0, c_0)$, that is, it is linear, continuous and bijective, and that its inverse is the map $Kt \in I(c_0, c_0)$ defined as

$$(Kt)(x_n) := \frac{x_n}{\left(2 + \frac{2n-1}{2n} t^{\frac{2n}{2n-1}}\right)}$$

for each $(x_n) \in c_0$.

Let us see that the map $J : (-1, 1) \to L(c_0, c_0)$ is continuous when $L(c_0, c_0)$ is endowed with the norm topology. We are going to show that if $0 < t < t' < 1$, then $\|Jt - Jt'\| \leq 2\,|t - t'|$. It is clear that we just need to prove that if $n \in \mathbb{N}$, then $\left|t'^{2n/2n-1} - t^{2n/2n-1}\right| \leq 2\,|t' - t|$. Notice that

$$
\begin{aligned}
0 &\leq t'^{\frac{2n}{2n-1}} - t^{\frac{2n}{2n-1}} \\
&\leq t'^{\frac{2n}{2n-1}} + t' t^{\frac{1}{2n-1}} - t t'^{\frac{1}{2n-1}} - t^{\frac{2n}{2n-1}} \\
&= (t' - t)(t'^{\frac{1}{2n-1}} + t^{\frac{1}{2n-1}}) \\
&\leq 2(t' - t).
\end{aligned}
$$

Since a similar reasoning also holds for any $t, t' \in (-1, 1)$, we conclude that J is continuous. In the same way we can prove that $K : (-1, 1) \to L(c_0, c_0)$ is continuous.

On the other hand, we have that, for each $(x_n) \in c_0$, the maps $J_{(x_n)}, K_{(x_n)} : (-1, 1) \to c_0$, defined as $J_{(x_n)}(t) := (Jt)(x_n)$ and $K_{(x_n)}(t) := (Kt)(x_n)$ are derivable, and it can be seen that for each $t \in (-1, 1)$,

$$
J'_{(x_n)}(t) = (t^{\frac{1}{2n-1}} x_n)
$$

and

$$
K'_{(x_n)}(t) = \left(-\frac{t^{\frac{1}{2n-1}} x_n}{\left(2 + \frac{2n-1}{2n} t^{\frac{2n}{2n-1}}\right)^2} \right) = \left(-\frac{J'_{(x_n)}(t)}{\left(2 + \frac{2n-1}{2n} t^{\frac{2n}{2n-1}}\right)^2} \right).
$$

Now, let us define $J'_{\mathrm{s}}, K'_{\mathrm{s}} : (-1, 1) \to L(c_0, c_0)$ as $J'_{\mathrm{s}}(t)(x_n) := J'_{(x_n)}(t)$ and $K'_{\mathrm{s}}(t)(x_n) := K'_{(x_n)}(t)$ for each $t \in (-1, 1)$ and each $(x_n) \in c_0$. It is straightforward to see that, for every $(x_n) \in c_0$, the maps $J'_{(x_n)}$ and $K'_{(x_n)}$ are continuous, which is to say that J'_{s} and K'_{s} are continuous when $L(c_0, c_0)$ is endowed with the strong operator topology. We conclude that J and K are of class $\mathrm{s} - C^1$.

Finally notice that, when $L(c_0, c_0)$ is endowed with the topology of the norm, then $\|J'_{\mathrm{s}}(t)\| = 1$ for every $t \in (-1, 1) \setminus \{0\}$, which implies that J'_{s} is not continuous at $t = 0$.

If we define now the map $T : C^1((-1, 1), c_0) \to C^1((-1, 1), c_0)$ as

$$
(Tf)(t) := (Jt)(f(t)),
$$

it turns out that T is well-defined (see Proposition 6.1), and it is routine matter to check that it is biseparating.

3 Some previous results

Lemma 3.1 *Suppose that $a_0, a_1, \ldots, a_k \in C^n(\Omega, E)$, and that $f : \Omega \times \mathbb{R} \to E$ is a polynomial in t defined as*

$$f(x,t) = \sum_{i=0}^{k} a_i(x) t^i$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Then $f \in C^n(\Omega \times \mathbb{R}, E)$.

Proof. It is immediate from the fact that all partial derivatives up to order n exist and are continuous. □

Lemma 3.2 *Suppose that $f, g \in C^n(\Omega, E)$ and $k : \Omega \to \mathbb{R}$ satisfy*

$$f(x) = k(x)g(x)$$

for every $x \in \Omega$. If $g(x) \neq 0$ for every $x \in \Omega$, then $k \in C^n(\Omega, \mathbb{R})$.

Proof. Fix $x_0 \in \Omega$. We are going to prove that k is of class C^n in a neighborhood of x_0. First we take $\mathbf{f}' : E \to \mathbb{R}$ linear and continuous, and such that $\mathbf{f}'(g(x_0)) \neq 0$. Since $\mathbf{f}'$ and g are continuous, then there exists an open neighborhood U of x_0 such that

$$\mathbf{f}'(g(x)) \neq 0$$

for every $x \in U$.

Now

$$\mathbf{f}'(f(x)) = k(x)\mathbf{f}'(g(x))$$

for every $x \in U$. This implies that, for every $x \in U$,

$$k(x) = \frac{\mathbf{f}'(f(x))}{\mathbf{f}'(g(x))},$$

which is the quotient of two real-valued functions of class C^n.

This proves that k is of class C^n. □

The proof of the following result is straightforward.

Lemma 3.3 *Suppose that $f \in C^n(\Omega, E)$, $a \in \Omega$, and $1 \leq k \leq n$. Then the k-th derivative of the Taylor polynomial function T_a of degree n of f is equal to the Taylor polynomial function of degree $n-k$ of $D^k f$ at a.*

The following theorem, known as *Whitney's extension theorem*, can be found, for instance, in [12, Theorem 3.1.14].

Theorem 3.4 *Suppose $n \in \mathbb{N}$, A is a closed subset of $\mathbb{R}^p$, and to each $a \in A$ corresponds a polynomial function*

$$P_a : \mathbb{R}^p \to E$$

with degree $P_a \leq n$. Whenever $C \subset A$ and $\delta > 0$ let $\rho(C, \delta)$ be the supremum of the set of all numbers

$$\left\| D^i P_a(b) - D^i P_b(b) \right\| \cdot |a-b|^{i-n} \cdot (n-i)!$$

corresponding to $i = 0, \ldots, n$ and $a, b \in C$ with $0 < |a-b| \leq \delta$.

If $\rho(C, \delta) \to 0$ as $\delta \to 0+$ for each compact subset C of A, then there exists a map $g : \mathbb{R}^p \to E$ of class C^n such that

$$D^i g(a) = D^i P_a(a)$$

for $i = 0, \ldots, n$ and $a \in A$.

Proposition 3.5 *Let $p \geq 2$. For $s \in \mathbb{R}$, $s > 0$, consider the following compact subsets of $\mathbb{R}^p$:*

$$A := \{(x_1, x_2, \ldots, x_p) \in \mathbb{R}^p : \sum_{i=2}^{p} x_i^2 \leq x_1^2/9, |x_1| \leq s\},$$

$$A^+ := \{(x_1, x_2, \ldots, x_p) \in A : x_1 \geq 0\},$$

and

$$A^- := \{(x_1, x_2, \ldots, x_p) \in A : x_1 \leq 0\}.$$

Suppose that Ω is an open subset of $\mathbb{R}^p$ containing A and that f belongs to $C^n(\Omega, E)$. If

$$\partial^\lambda f(0, 0, \ldots, 0) = 0$$

for every $\lambda \in \Lambda$, then there exists a function $f_+ \in C^n(\Omega, E)$ with compact support such that, for $\lambda \in \Lambda$,

$$\partial^\lambda f_+(x) = \partial^\lambda f(x)$$

for every $x \in A^+$, and

$$\partial^\lambda f_+(x) = 0$$

for every $x \in A^-$.

Proof. Suppose that for $x \in A$, T_x stands for the polynomial function of degree n given in the Taylor formula for f at x. Now for $a \in A^-$, we consider as P_a the polynomial identically zero, and for $a \in A^+$, we consider as P_a the polynomial T_a. As it is seen for instance in [9, Theorem 2.71] or [23, p. 350], if $r > 0$ and $|b-a| < r$, we have that

$$\|T_a(b) - f(b)\| \leq \frac{|b-a|^n}{n!} \sup_{|x-a|\leq r} \|D^n f(x) - D^n f(a)\| .$$

Now it is easy to see that, by Lemma 3.3, for $i \in \{1, 2, \ldots, n\}$,

$$\left\|D^i T_a(b) - D^i f(b)\right\| \leq \frac{|b-a|^{n-i}}{(n-i)!} \sup_{|x-a|\leq r} \|D^n f(x) - D^n f(a)\| .$$

This proves that if $|b-a| < r$,

$$\left\|D^i T_a(b) - D^i f(b)\right\| \cdot |b-a|^{i-n} \cdot (n-i)! \leq \sup_{|x-a|\leq r} \|D^n f(x) - D^n f(a)\|$$

for $i \in \{0, 1, \ldots, n\}$. Then we define $\tau(r)$ as the supremum of the set of all numbers

$$\sup_{|x-a|\leq r} \|D^n f(x) - D^n f(a)\|$$

for $x, a \in A$, which is a real number, because A is compact. Clearly, since $D^n f$ is continuous, if r tends to zero, $\tau(r)$ tends to zero. Now suppose that a and b belong to A and $0 < |b-a| \leq r$. Then we have the following possibilities:

- $a, b \in A^+$. Then we have that, by Lemma 3.3, for $i \in \{0, 1, \ldots, n\}$,

$$D^i P_b(b) = D^i f(b)$$

and consequently

$$\left\|D^iP_a(b) - D^iP_b(b)\right\| = \left\|D^iT_a(b) - D^if(b)\right\|.$$

- $a, b \notin A^+$. Then

$$\left\|D^iP_a(b) - D^iP_b(b)\right\| = 0.$$

- $a \notin A^+$, $b \in A^+$. Note that since we are assuming by hypothesis that $D^if(0,0,\ldots,0) = 0$, then $D^iP_0(b) = 0$ for all $i \in \{0,1,\ldots,n\}$. Consequently

$$\begin{aligned}\left\|D^iP_a(b) - D^iP_b(b)\right\| &= \left\|D^iP_b(b)\right\| \\ &= \left\|D^iP_0(b) - D^iP_b(b)\right\| \\ &= \left\|D^iT_0(b) - D^if(b)\right\|.\end{aligned}$$

- $a \in A^+$, $b \notin A^+$. Then we have that

$$\begin{aligned}\left\|D^iP_a(b) - D^iP_b(b)\right\| &= \left\|D^iT_a(b)\right\| \\ &\leq \left\|D^iT_a(b) - D^if(b)\right\| + \left\|D^if(b)\right\| \\ &= \left\|D^iT_a(b) - D^if(b)\right\| + \left\|D^if(b) - D^iT_0(b)\right\|.\end{aligned}$$

Note that in the third and forth cases above, $|a|, |b| \leq |b-a| \leq r$. This implies that in these two cases

$$\begin{aligned}\left\|D^iP_a(b) - D^iP_b(b)\right\| \cdot |b-a|^{i-n} \cdot (n-i)! \leq &\sup_{|x-a|\leq r} \|D^nf(x) - D^nf(a)\| \\ &+ \sup_{|x|\leq r} \|D^nf(x) - D^nf(0)\|.\end{aligned}$$

On the other hand it is easy to see that in the other two cases

$$\left\|D^iP_a(b) - D^iP_b(b)\right\| \cdot |b-a|^{i-n} \cdot (n-i)! \leq \sup_{|x-a|\leq r} \|D^nf(x) - D^nf(a)\|.$$

This facts imply that, if ρ is defined as in Theorem 3.4, $\rho(A,r) \leq 2\tau(r)$. Also, it is clear that if C is a compact subset of A, then $\rho(C,r) \leq \rho(A,r)$. Consequently by Theorem 3.4, we have that there exists $f_0 \in C^n(\Omega, E)$ such that, given any $\lambda \in \Lambda$,

$$\partial^\lambda f_0(x) = \partial^\lambda f(x)$$

for every $x \in A^+$, and

$$\partial^\lambda f_0(x) = 0$$

for every $x \in A^-$. Also it is clear that if we take $g_0 \in C_c^n(\Omega, \mathbb{R})$ such that $g_0 \equiv 1$ on an open neighborhood of A, then $f_+ := g_0 f_0 \in C^n(\Omega, E)$ satisfies the requirements of the theorem. $\square$

4 Biseparating maps: a first approach

In this section we make a first attempt to describe all biseparating maps, but we do not take into account some important details which will be discussed in Section 6. In this way we characterize all biseparating linear maps from $A^n(\Omega, E)$ onto $A^m(\Omega', F)$ as weighted composition bijective maps. Notice that we assume no continuity properties on T. In fact, we will suppose that our spaces $A^n(\Omega, E)$ and $A^m(\Omega', F)$ are not endowed with any topologies.

Lemma 4.1 *Suppose that Ω contains the origin, and that $T : A^n(\Omega, E) \to A^m(\Omega', F)$ is a biseparating map. Assume also that $f \in A^n(\Omega, E)$ satisfies that for all $\lambda \in \Lambda$,*

$$\partial^\lambda f(0, 0, \ldots, 0) = 0.$$

If $(0, 0, \ldots, 0) \in \Omega$ is the support point of $y \in \Omega'$, then $(Tf)(y) = 0$.

Proof. First suppose that $p > 1$ and that the closed ball of center 0 and radius s is contained in Ω. If we take A^+, A^- and f_+ as in Proposition 3.5, then f_+ and $f - f_+$ belong to $A^n(\Omega, E)$ and satisfy $f_+(x) = 0$ and $(f - f_+)(x) = 0$ for every $x \in A^-$ and for every $x \in A^+$ respectively. We have that $(Tf)(y) = (Tf_+)(y) + (T(f - f_+))(y)$. Also, since for any neighborhood U of the origin there exists an open subset V of U such that $f_+(x) = 0$ for every $x \in V$, then we have that, taking into account that the support map for T^{-1} is h^{-1}, by Lemma 2.1, $(Tf_+)(h^{-1}(x)) = 0$. Since $h : \Omega' \to \Omega$ is a homeomorphism, we deduce that $(Tf_+)(y) = 0$ and, in the same way, $(T(f - f_+))(y) = 0$. We conclude that $(Tf)(y) = 0$.

Consider now the case when $p = 1$. Note that if $f \in A^n(\Omega, E)$ satisfies $f(0) = 0$ and $0 = f'(0) = \cdots = f^{(n)}(0)$, then it is clear that $f\xi_{(-\infty,0)}$ and $f\xi_{(0,+\infty)}$ belong to $A^n(\Omega, E)$ (where ξ_A stands for the characteristic function of A) and, as above, $(T(f\xi_{(0,+\infty)}))(y) = 0 = (T(f\xi_{(-\infty,0)}))(y)$. We conclude that $(Tf)(y) = 0$. $\square$

Proposition 4.2 *Suppose that $T : A^n(\Omega, E) \to A^m(\Omega', F)$ is a $\mathbb{K}$-linear biseparating map. Then $p = q$, $n = m$, and there exist a diffeomorphism h of class C^n from Ω' onto Ω and a map $J : \Omega' \to I'(E, F)$ such that for every*

$y \in \Omega'$ *and every* $f \in A^n(\Omega, E)$,

$$(Tf)(y) = (Jy)(f(h(y))).$$

Proof. First, the existence of the homeomorphism h (the support map) between Ω' and Ω implies that $p = q$ (see for instance [13, p. 120]).

Note that if $f \in A^n(\Omega, E)$ and $y \in \Omega'$ satisfy

$$\partial^\lambda f(h(y)) = 0$$

for all $\lambda \in \Lambda$, then by Lemma 4.1 we have that

$$(Tf)(y) = 0.$$

Now take $y \in \Omega'$ and fix $\mathbf{e} \in E$, $\mathbf{e} \neq 0$. If $\#\Lambda$ stands for the cardinal of Λ, then we can define a linear map $S_y : \mathbb{R}^{\#\Lambda} \to F$ as follows. Given $(a_{\lambda_1}, \ldots, a_{\lambda_{\#\Lambda}}) \in \mathbb{R}^{\#\Lambda}$, we consider any $f \in A^n(\Omega, \mathbb{R})$ such that

$$\partial^\lambda f(h(y)) = a_\lambda$$

for every $\lambda \in \Lambda$. Then we define

$$S_y(a_{\lambda_1}, \ldots, a_{\lambda_{\#\Lambda}}) := (Tf\mathbf{e})(y).$$

The map S_y is linear and, as we have seen above, does not depend on the function f we choose. This implies that it is well defined.

Then it is easy to see that there exist functions α_λ from Ω' into F, $\lambda \in \Lambda$, such that for every $y \in \Omega'$ and every $f \in A^n(\Omega, \mathbb{R})$,

$$(Tf\mathbf{e})(y) = \sum_{\lambda \in \Lambda} \alpha_\lambda(y) \partial^\lambda f(h(y)). \tag{4.1}$$

From now on we consider $i \in \{1, 2, \ldots, p\}$ fixed.

Next we define some functions

$$\alpha_i^0, \alpha_i^1, \ldots, \alpha_i^{n+1}$$

from $\Omega' \times \mathbb{R}$ into F. For every $y \in \Omega'$ and $t \in \mathbb{R}$,

$$
\begin{aligned}
\alpha_i^0(y,t) &:= (T\widehat{\mathbf{e}})(y),\\
\alpha_i^1(y,t) &:= (Tx_i\mathbf{e})(y) - \alpha_i^0(y,t)t,\\
2!\,\alpha_i^2(y,t) &:= (Tx_i^2\mathbf{e})(y) - \alpha_i^0(y,t)t^2 - 2\alpha_i^1(y,t)t,\\
3!\,\alpha_i^3(y,t) &:= (Tx_i^3\mathbf{e})(y) - \alpha_i^0(y,t)t^3 - 3\alpha_i^1(y,t)t^2 - 6\alpha_i^2(y,t)t,
\end{aligned}
$$

and in general, for $k \in \{1,2,\ldots,n,n+1\}$

$$
\begin{aligned}
k!\,\alpha_i^k(y,t) := {} & (Tx_i^k\mathbf{e})(y) - \alpha_i^0(y,t)t^k - k\alpha_i^1(y,t)t^{k-1} - k(k-1)\alpha_i^2(y,t)t^{k-2}\\
& - \ldots - k!\,\alpha_i^{k-1}(y,t)t.
\end{aligned}
$$

Claim 4.1 *For $l \in \{0,1,\ldots,n+1\}$, $l!\,\alpha_i^l$ is a polynomial in t whose coefficients are a linear combination of $T\widehat{\mathbf{e}}, Tx_i\mathbf{e},\ldots,Tx_i^l\mathbf{e}$. Moreover, for $y \in \Omega'$ fixed, the degree of the polynomial $l!\alpha_i^l(y,t)$ is at most l. If we also assume that $(T\widehat{\mathbf{e}})(y) \neq 0$, then the degree of $l!\alpha_i^l(y,t)$ is l and its leading coefficient is equal to $(-1)^l\alpha_i^0(y,t)$ (notice that this term does not depend on t).*

We are going to prove it by applying induction on l. It is clear that this is true for $l = 0$. Suppose that this relation also holds for $l \in \{0,1,2,\ldots,k\}$ for some $k \leq n$. We are going to see that it holds for $l = k+1$. We have that

$$
\begin{aligned}
(k+1)!\,\alpha_i^{k+1}(y,t) := {} & (Tx_i^{k+1}\mathbf{e})(y) - \alpha_i^0(y,t)t^{k+1}\\
& - (k+1)\alpha_i^1(y,t)t^k - (k+1)k\alpha_i^2(y,t)t^{k-1}\\
& - (k+1)k(k-1)\alpha_i^3(y,t)t^{k-2}\\
& - \ldots\\
& - (k+1)!\,\alpha_i^k(y,t)t,
\end{aligned}
$$

which implies that it is a polynomial in t and, for fixed $y \in \Omega$, its coefficient for the term t^{k+1} is $\alpha_i^0(y,t)(-1+(k+1)-\binom{k+1}{2}+\binom{k+1}{3}-\ldots-(k+1)(-1)^k)$, which is equal to $\alpha_i^0(y,t)(-(1-1)^{k+1}+\binom{k+1}{k+1}(-1)^{k+1})$, that is, to $(-1)^{k+1}\alpha_i^0(y,t)$.

Thus the claim is proved.

Now, by Lemma 3.1, we have that for every $k \in \{0,1,\ldots,n+1\}$, α_i^k belongs to $C^m(\Omega' \times \mathbb{R}, F)$.

Next define $\alpha_0 := T\widehat{\mathbf{e}}$, and for $k \in \{1,2,\ldots,n\}$, $\alpha_k := \alpha_\lambda : \Omega' \to F$, where

$$
\lambda = (\underbrace{0,0,\ldots,0,k,}_{i}0,\ldots,0).
$$

Also, let h_i stand for the i-th coordinate function of h.

Claim 4.2 *For every $k \in \{0, 1, \dots, n\}$, and for every $y \in \Omega'$,*

$$\alpha_k(y) = \alpha_i^k(y, h_i(y)).$$

First we have from Equation 4.1 that for $k \in \{0, 1, \dots, n\}$ and $y \in \Omega'$,

$$(Tx_i^k \mathbf{e})(y) = \sum_{\lambda \in \Lambda} \alpha_\lambda(y) \partial^\lambda x_i^k(h(y)),$$

which can be written as

$$\begin{aligned}(Tx_i^k \mathbf{e})(y) = &\alpha_0(y) h_i^k(y) \\ &+ k\alpha_1(y) h_i^{k-1}(y) \\ &+ \dots \\ &+ k!\alpha_{k-1}(y) h_i(y) \\ &+ k!\alpha_k(y),\end{aligned} \tag{4.2}$$

because

$$\partial^\lambda x_i^k(h(y)) = 0$$

whenever $\lambda \in \Lambda$, $\lambda \neq (\underbrace{0, 0, \dots, j,}_{i} 0, \dots, 0)$, $j \in \{0, 1, \dots, k\}$.

On the other hand, it is clear that $\alpha_0(y) = (T\hat{\mathbf{e}})(y) = \alpha_i^0(y, h_i(y))$ for every $y \in \Omega'$. Also suppose that $k < n$ and that $\alpha_j(y) = \alpha_i^j(y, h_i(y))$ for every $j \in \{0, 1, \dots, k\}$ and every $y \in \Omega'$. Then, by Equation 4.2, for $y \in \Omega'$

$$\begin{aligned}(k+1)!\alpha_{k+1}(y) = &(Tx_i^{k+1}\mathbf{e})(y) \\ &- \alpha_0(y) h_i^{k+1}(y) \\ &- (k+1)\alpha_1(y) h_i^k(y) \\ &- \dots \\ &- (k+1)!\alpha_k(y) h_i(y),\end{aligned}$$

which coincides with $(k+1)!\alpha_i^{k+1}(y, h_i(y))$, and the claim is proved.

On the other hand, notice that in the same way as we obtain Equation 4.2, we have

$$\begin{aligned}(Tx_i^{n+1}\mathbf{e})(y) = &\alpha_0(y) h_i^{n+1}(y) \\ &+ (n+1)\alpha_1(y) h_i^n(y)\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&+ \dots \\
&+ (n+1)!\alpha_n(y)h_i(y)
\end{aligned}
$$

for every $y \in \Omega'$.

Claim 4.3 *Suppose that $y_0 \in \Omega'$ satisfies $\alpha_0(y_0) \neq 0$. Then for every open neighborhood U of y_0, there exists a nonempty open subset W_i' of U where h_i is of class C^m.*

First we define $F_i^0 : \Omega' \times \mathbb{R} \to F$ as $F_i^0 := (n+1)!\alpha_i^{n+1}$, that is,

$$
\begin{aligned}
F_i^0(y,t) = &(Tx_i^{n+1}\mathbf{e})(y) \\
&- \alpha_i^0(y,t)t^{n+1} \\
&- (n+1)\alpha_i^1(y,t)t^n \\
&- (n+1)n\alpha_i^2(y,t)t^{n-1} \\
&- \dots \\
&- (n+1)!\,\alpha_i^n(y,t)t,
\end{aligned}
$$

for every $y \in \Omega'$, $t \in \mathbb{R}$.

Then, if $j \in \{1, 2, \dots, n+1\}$, we define $F_i^j : \Omega' \times \mathbb{R} \to F$ as

$$
F_i^j(y,t) = \frac{\partial^j F_i^0}{\partial t^j}(y,t),
$$

for all $y \in \Omega'$, $t \in \mathbb{R}$.

Notice that from the definition of F_i^0, Claim 4.2 and Equation 4.3, we deduce that

$$
F_i^0(y, h_i(y)) = 0
$$

for every $y \in \Omega'$. Also, as we stated in Claim 4.1, the coefficients of F_i^0 as a polynomial of degree $n+1$ in t are linear combinations of

$$
(T\hat{\mathbf{e}})(y), (Tx_i\mathbf{e})(y), \dots, (Tx_i^{n+1}\mathbf{e})(y),
$$

and consequently, by Lemma 3.1, for $k \in \{1, 2, \dots, n\}$, F_i^k belongs to $C^m(\Omega' \times \mathbb{R}, F)$.

Taking into account that $F_i^{n+1}(y,t) = (n+1)!(-1)^{n+1}\alpha_0(y)$ for every $(y,t) \in \Omega' \times \mathbb{R}$, and the fact that $\alpha_0(y_0) \neq 0$, there exists $k_0 \in \{0, 1, \dots, n\}$ such that

$F_i^{k_0}(y, h_i(y)) = 0$ for every y in a neighborhood of y_0 and $F_i^{k_0+1}(y, h_i(y))$ takes a value different from 0 for some y in every neighborhood of y_0. Suppose then that U is an open neighborhood of y_0 such that $F_i^{k_0}(y, h_i(y)) = 0$ for every $y \in U$ and that $y_1 \in U$ satisfies $F_i^{k_0+1}(y_1, h_i(y_1)) = \mathbf{f} \in F$, $\mathbf{f} \neq 0$. Now take $\mathbf{f}'$ in the dual space F' (where F is wiewed as a *real* Banach space) such that $\mathbf{f}'(\mathbf{f}) \neq 0$. According to the Implicit Function Theorem ([14, p.148]), there exist a neighborhood V of $(y_1, h_i(y_1))$, an open neighborhood W of y_1, and a function $\phi : W \to \mathbb{R}$ of class C^m such that $\phi(y_1) = h_i(y_1)$ and

$$\{(y, t) \in V : \mathbf{f}' \circ F_i^{k_0}(y, t) = 0\} = \{(y, \phi(y)) : y \in W\}.$$

It is easy to prove that this implies that $\phi \equiv h_i$ on a neighborhood W_i' of y_1, that is, for every open neighborhood U of y_0, there exists an open subset W_i' of U where h_i is of class C^m. The claim is proved.

Since both T and T^{-1} are biseparating, we can assume from now on, without loss of generality, that $n \leq m$.

Claim 4.4 *Suppose that U is a nonempty open subset of Ω'. Then there exists a nonempty open subset W' of U such that the restriction of h to W' is a diffeomorphism of class C^m.*

Notice first that the the open set $\{y \in \Omega' : \alpha_0(y) \neq 0\}$ is dense in Ω'. Otherwise we could find $g \in A^m(\Omega', F)$, $g \neq 0$, such that $c(T\widehat{\mathbf{e}}) \cap c(g) = c(\alpha_0) \cap c(g) = \emptyset$. Since T^{-1} is separating, this would give us $\Omega \cap c(T^{-1}g) = c(\widehat{\mathbf{e}}) \cap c(T^{-1}g) = \emptyset$, which is impossible.

So far we have considered $i \in \{1, 2, \ldots, p\}$ fixed. Of course a similar process can be done for every $i \in \{1, 2, \ldots, p\}$. In particular, taking into account the above paragraph, by Claim 4.3, there exists an open subset W_1' of U such that h_1 is of class C^m in W_1'. For the same reason we can find an open subset W_2' of W_1' where h_2 is of class C^m. Following this process we construct (nonempty) open sets $W_1', \ldots, W_p'$ with $W_1' \supset W_2' \supset \ldots \supset W_p'$ such that h is of class C^m in W_p'. It is clear that a similar reasoning shows that the map h^{-1} is of class C^n in an open subset V of $h(W_p')$. Then our situation is as follows: h^{-1} is of class C^n in V and h is of class C^m in $h^{-1}(V)$. It is well known that, since $m \geq n \geq 1$, this implies that h is a diffeomorphism of class C^m in $W' := h^{-1}(V)$, and we are done.

Claim 4.5 *Let W' be as in Claim 4.4. For every $k \in \{0, 1, \ldots, n\}$, the map α_k belongs to $C^m(W', F)$.*

First we have that $\alpha_0 = T\hat{\mathbf{e}}$ belongs to $A^m(\Omega', F)$. It is also clear that if $k \in \{0, 1, \ldots, n-1\}$, then as given in Equation 4.2,

$$(k+1)!\,\alpha_{k+1} = Tx_i^{k+1}\mathbf{e} - \alpha_0 h_i^{k+1} - (k+1)\alpha_1 h_i^k - \ldots - (k+1)!\,\alpha_k h_i$$

on Ω'.

Consequently, since $h_i \in C^m(W')$, if $\alpha_0, \alpha_1, \ldots, \alpha_k$ belong to $C^m(W', F)$, α_{k+1} also belongs to $C^m(W', F)$ and then we are done.

Claim 4.6 *For every $k \in \{1, 2, \ldots, n\}$, $\alpha_k \equiv 0$ in W'.*

Suppose that $y_0 \in W'$ and $\alpha_n(y_0) \neq 0$. Since by Claim 4.5 α_n is continuous in W', there exists an open neighborhood $U(y_0)$ of y_0 such that $U(y_0) \subset W'$ and $\alpha_n(y) \neq 0$ for every $y \in U(y_0)$. Then take $g \in C^n(\mathbb{R}, \mathbb{R})$ such that $g^{(n)}$ is not derivable at the point $h_i(y_0)$. We define $f \in A^n(\Omega, \mathbb{R})$ as

$$f(x) := g(x_i)$$

for every $x = (x_1, x_2, \ldots, x_p) \in \Omega$. In this way we have that

$$\frac{\partial^{n+1} f}{\partial x_i^{n+1}}(h(y_0))$$

does not exist. Consequently, using a reasoning similar to that giving Equation 4.2, we have that Equation 4.1 applied to f yields

$$(Tf\mathbf{e})(y) = \alpha_0(y) f(h(y)) + \alpha_1(y)\frac{\partial f}{\partial x_i}(h(y)) + \ldots + \alpha_n(y)\frac{\partial^n f}{\partial x_i^n}(h(y))$$

for every $y \in U(y_0)$. Now we analyze the terms in the above equation, taking into account that we are assuming $1 \leq n \leq m$, and that by Claim 4.4, h is a diffeomorphism of class C^m in W'. First $Tf\mathbf{e}$ is of class C^1 in $U(y_0)$. Also, by Claim 4.5, for $k \in \{0, 1, \ldots, n\}$, each α_k is of class C^1 in $U(y_0)$. Finally, f and all of its partial derivatives up to order $n-1$ are of class C^1 in Ω.

Thus we deduce from the above equation that

$$\alpha_n \frac{\partial^n f}{\partial x_i^n} \circ h$$

is of class C^1 in $U(y_0)$, and by Lemma 3.2 the same applies to the function

$$\frac{\partial^n f}{\partial x_i^n} \circ h.$$

But, as we said before, h is a diffeomorphism of class C^m in W', and consequently

$$\frac{\partial^n f}{\partial x_i^n}$$

admits a partial derivative with respect to the i-th coordinate at the point $h(y_0)$, which is a contradiction. This implies that $\alpha_n \equiv 0$ in W'. In a similar way we can see that $\alpha_k \equiv 0$ in W' for $k \geq 1$, that is, $\alpha_\lambda \equiv 0$ in W' whenever $\lambda \in \Lambda$ is of the form

$$\lambda = (\underbrace{0, 0, \ldots, k,}_{i} 0, \ldots, 0).$$

Claim 4.7 *For every $\lambda \in \Lambda - \{(0, 0, \ldots, 0)\}$, $\alpha_\lambda \equiv 0$ in W'.*

Here our reasoning will be similar to the one given in Claim 4.6. In this way, if $i, j \in \{1, 2, \ldots, p\}$, $i \neq j$, again by Equation 4.1, for every $y \in W'$,

$$(T(x_i x_j \mathbf{e}))(y) = \alpha_0(y) h_i(y) h_j(y) + \alpha_{\lambda_0^1}(y)$$

where $\lambda_0^1 := (\lambda_{01}^1, \lambda_{02}^1, \ldots, \lambda_{0p}^1)$, $\lambda_{0i}^1 = 1 = \lambda_{0j}^1$ and $\lambda_{0k}^1 = 0$, whenever $k \neq i, j$. Taking into account that α_0, h_i and h_j are of class C^m in W', we easily deduce that $\alpha_{\lambda_0^1}$ is of class C^m in W'. Likewise, we can inductively prove that $\alpha_{\lambda_0^l}$ is of class C^m in W', where $\lambda_0^l := (\lambda_{01}^l, \lambda_{02}^l, \ldots, \lambda_{0p}^l)$, $\lambda_{0i}^l = l$, $\lambda_{0j}^l = 1$ and $\lambda_{0k}^l = 0$, whenever $k \neq i, j$, $l \in \{1, 2, \ldots, n-1\}$. Suppose that $\alpha_{\lambda_0^{n-1}}(y_0) \neq 0$ for some $y_0 \in W'$. Then, as in the proof of Claim 4.6, we take an open neighborhood $U(y_0)$ of y_0 such that $U(y_0) \subset W'$ and $\alpha_{\lambda_0^{n-1}}(y) \neq 0$ for every $y \in U(y_0)$.

Also, we take $f(x) = g(x_i)$ for every $x = (x_1, x_2, \ldots, x_p) \in \Omega$, where these functions meet the same requirements as in the proof of Claim 4.6, and define

$$d(x) := x_j f(x),$$

for every $x \in \Omega$. Clearly d just depends on the i-th and j-th coordinates, which implies that its only partial derivatives which possibly are not zero at $h(y) \in h(U(y_0))$ are maybe those

$$\partial^\lambda d$$

for

$$\lambda = (\underbrace{0, 0, \ldots, 1,}_{j} 0, \ldots, 0),$$

$$\lambda = (\underbrace{0, 0, \ldots, k,}_{i} 0, \ldots, 0),$$

$k = 1, 2, \ldots, n$, or

$$\lambda = \lambda_0^l,$$

$l \in \{1, 2, \ldots, n-1\}$. Taking into account that $\alpha_\lambda \equiv 0$ on W' for

$$\lambda = (\underbrace{0, 0, \ldots, 1,}_{j} 0, \ldots, 0)$$

and

$$\lambda = (\underbrace{0, 0, \ldots, k,}_{i} 0, \ldots, 0),$$

Equation 4.1 gives us, for every $y \in U(y_0)$,

$$(Td\mathbf{e})(y) = \alpha_0(y)d(h(y)) + \alpha_{\lambda_0^1}(y)\frac{\partial f}{\partial x_i}(h(y)) + \ldots + \alpha_{\lambda_0^{n-1}}(y)\frac{\partial^{n-1} f}{\partial x_i^{n-1}}(h(y)).$$

We deduce as in the proof of Claim 4.6 that

$$\frac{\partial^{n-1} f}{\partial x_i^{n-1}}(h(y))$$

admits a second partial derivative with respect to x_i at the point $h(y_0)$, which is a contradiction. This implies that $\alpha_{\lambda_0^{n-1}} \equiv 0$ in W'. In the same way we deduce that $\alpha_{\lambda_0^l} \equiv 0$ in W', for $l \in \{1, 2, \ldots, n-2\}$.

A similar pattern of proof leads us to the fact that $\alpha_\lambda \equiv 0$ in W' for every $\lambda \neq (0, 0, \ldots, 0)$, $\lambda \in \Lambda$.

Claim 4.8 *For every $\lambda \in \Lambda - \{(0, 0, \ldots, 0)\}$, $\alpha_\lambda \equiv 0$ in Ω'.*

Notice that given an open subset U of Ω', in Claim 4.4 we obtain a subset W' of U. Notice also that this process can be done for any open subset of Ω'

because, as we saw in the proof of Claim 4.4, $c(\alpha_0)$ is dense in Ω'. Also in Claim 4.7 we proved that, for $\lambda \neq (0, 0, \ldots, 0)$, $\alpha_\lambda \equiv 0$ on all the subsets W' obtained in this way. This implies clearly that all these functions α_λ are equal to 0 on a dense subset of Ω'. Consequently, to prove Claim 4.8, it is enough to show that all these functions are continuous.

We are going to prove it using induction on $|\lambda|$. First, for $|\lambda| = 0$, we have that $\alpha_{(0,\ldots,0)} = T\widehat{\mathbf{e}}$ belongs to $A^m(\Omega', F)$ and, consequently, it is continuous.

Now assume that $k \leq n - 1$, and whenever $|\lambda| \leq k$, then α_λ is a continuous function. Then fix $\lambda = (\lambda_1, \ldots, \lambda_p) \in \Lambda$ with $|\lambda| = k + 1$.

Next define $f \in A^n(\Omega, \mathbb{R})$ as

$$f := x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_p^{\lambda_p}.$$

It is clear that given $\mu = (\mu_1, \ldots, \mu_p) \in \Lambda$, if $\mu_i > \lambda_i$ for some $i \in \{1, \ldots, p\}$, then

$$\partial^\mu f(h(y)) = 0$$

for every $y \in \Omega'$. This implies that in our situation, Equation 4.1 can be written as

$$(Tf\mathbf{e})(y) = \sum_{\mu << \lambda} \alpha_\mu(y) \partial^\mu f(h(y)),$$

where $\mu << \lambda$ means $\mu_i \leq \lambda_i$ for every $i \in \{1, \ldots, p\}$.

As a consequence, for every $y \in \Omega'$,

$$\lambda_1! \ldots \lambda_p! \alpha_\lambda(y) = \alpha_\lambda(y) \partial^\lambda f(h(y)) = (Tf\mathbf{e})(y) - \sum_{\mu << \lambda, \mu \neq \lambda} \alpha_\mu(y) \partial^\mu f(h(y)).$$

On the other hand, if $\mu << \lambda$ and $\mu \neq \lambda$, then $|\mu| \leq k$, and consequently, taking into account that h is continuous and the hypothesis of induction, we deduce that α_λ is continuous, and the claim is proved.

Recall that all the process developed so far concerns functions of the form $f\mathbf{e} \in A^n(\Omega, E)$, where $\mathbf{e} \in E - \{0\}$ and $f \in A^n(\Omega, \mathbb{R})$. For this $\mathbf{e}$, we define

$$a_{\mathbf{e}} := \alpha_0 = T\widehat{\mathbf{e}}.$$

Notice that by Claim 4.8, we have

$$(Tf\mathbf{e})(y) = a_{\mathbf{e}}(y)f(h(y)) \tag{4.4}$$

for every $y \in \Omega'$ and every $f \in A^n(\Omega, \mathbb{R})$.

Next we define a map $J : \Omega' \to L'(E, F)$ as $(Jy)(0) = 0$, and

$$(Jy)(\mathbf{e}) := a_{\mathbf{e}}(y)$$

for each $y \in \Omega'$ and $\mathbf{e} \in E - \{0\}$.

Claim 4.9 *For every $f \in A^n(\Omega, E)$ and $y \in \Omega'$,*

$$(Tf)(y) = (Jy)(f(h(y))).$$

Fix $y \in \Omega'$. Suppose that

$$\partial^\lambda f(h(y)) = \mathbf{e}_\lambda \in E$$

for each $\lambda \in \Lambda$.

Let $\Lambda_* := \{\lambda \in \Lambda : \mathbf{e}_\lambda \neq 0\}$. Next, for each $\lambda \in \Lambda_*$, take a function $f_\lambda \in A^n(\Omega, \mathbb{R})$ such that

$$\partial^\lambda f_\lambda(h(y)) = 1$$

and

$$\partial^\mu f_\lambda(h(y)) = 0$$

for every $\mu \neq \lambda$, $\mu \in \Lambda$.

It is easy to see that, for every $\mu \in \Lambda$,

$$\partial^\mu \sum_{\lambda \in \Lambda_*} f_\lambda \mathbf{e}_\lambda(h(y)) = \mathbf{e}_\mu,$$

if $\mu \in \Lambda_*$, and

$$\partial^\mu \sum_{\lambda \in \Lambda_*} f_\lambda \mathbf{e}_\lambda(h(y)) = 0$$

if $\mu \notin \Lambda_*$.

According to Lemma 4.1, this implies that

$$(Tf)(y) = \left(T \sum_{\lambda \in \Lambda_*} f_\lambda \mathbf{e}_\lambda \right)(y).$$

Consequently, by Equation 4.4,

$$(Tf)(y) = \sum_{\lambda \in \Lambda_*} a_{\mathbf{e}_\lambda}(y) f_\lambda(h(y)).$$

But by the way we have constructed the functions f_λ, we have that $f_\lambda(h(y)) = 0$ if $\lambda \neq (0, 0, \ldots, 0)$.

On the other hand, let us denote by 0 the multiindex $(0, 0, \ldots, 0)$. If $0 \notin \Lambda_*$, that is, if $\mathbf{e}_0 = 0$, we conclude from the above equality that $(Tf)(y) = 0 = a_{\mathbf{e}_0}(y) = (Jy)(0)$. Finally, if $0 \in \Lambda_*$, taking into account that $f_0(h(y)) = 1$, we deduce that

$$(Tf)(y) = a_{\mathbf{e}_0}(y) f_0(h(y)) = a_{\mathbf{e}_0}(y) = (Jy)(\mathbf{e}_0),$$

and we are done.

Claim 4.10 *Given $y \in \Omega'$, there exists $\mathbf{e} \in E$ such that $a_{\mathbf{e}}(y) \neq 0$.*

Notice that T is bijective, so if $\mathbf{f} \in F$, $\mathbf{f} \neq 0$, there exists $g \in A^n(\Omega, E)$ with $Tg = \widehat{\mathbf{f}}$. In particular, by Claim 4.9, we have that $(Jy)(g(h(y))) = \mathbf{f}$. In other words, if we take $y \in c(g \circ h)$ and define $\mathbf{e} := g(h(y))$, we have that $a_{\mathbf{e}}(y) = (Jy)(\mathbf{e}) = \mathbf{f} \neq 0$.

Claim 4.11 *h is a function of class C^m.*

Fix $y_0 \in \Omega'$. By Claim 4.10, we can take $\mathbf{e} \in E$ such that $a_{\mathbf{e}}(y) \neq 0$. Since $a_{\mathbf{e}} = T\widehat{\mathbf{e}}$, it is a continuous function, and we deduce that for some neighborhood V of y_0, $a_{\mathbf{e}}(y) \neq 0$ for every $y \in V$.

Now recall that Equation 4.4,

$$(Tf\mathbf{e})(y) = a_{\mathbf{e}}(y) f(h(y)),$$

holds in particular for every $y \in V$ and every $f \in A^n(\Omega, \mathbb{R})$.

Consequently, for $i \in \{1, 2, \ldots, p\}$,

$$(Tx_i\mathbf{e})(y) = a_{\mathbf{e}}(y)h_i(y)$$

for every $y \in V$. Since $a_{\mathbf{e}}(y) \neq 0$ for every $y \in V$, applying Lemma 3.2, we have that h_i is of class C^m in V. Clearly this implies that h is of class C^m, and we are done.

Claim 4.12 *$n = m$ and h is a diffeomorphism of class C^m.*

Recall that we are assuming that $n \leq m$. Now, we have that by Claim 4.4, for every nonempty open set $V \subset \Omega'$, there is a nonempty open set $V' \subset V$ such that the restriction of h to V' is a diffeomorphism of class C^m. Take $y_0 \in \Omega'$. By Claim 4.10, there exists $\mathbf{e} \in E$ and an open neighborhood V of y_0 such that $a_{\mathbf{e}}(y) \neq 0$ for every $y \in V$. Now, as we mentioned above, there exists an open set V', $V' \subset V$, where the restriction of h is a diffeomorphism of class C^m. Assume now that $n < m$ and take $g \in A^n(\Omega', \mathbb{R}) - C^m(\Omega', \mathbb{R})$ such that $c(g) \subset V'$. Next define $f : \Omega \to E$, as

$$f(x) := g(h^{-1}(x))\mathbf{e},$$

for each $x \in \Omega$. We are going to prove that g is of class C^m, obtaining a contradiction.

It is immediate that $f \in A^n(\Omega, E)$, and applying Equation 4.4, we get

$$(Tf)(y) = a_{\mathbf{e}}(y)g(h^{-1}(h(y))),$$

that is,

$$(Tf)(y) = a_{\mathbf{e}}(y)g(y),$$

for every $y \in \Omega'$.

Now we have that $a_{\mathbf{e}}(y) \neq 0$ for every $y \in V$. Finally, by Lemma 3.2, g is of class C^m in V, and so is in Ω', which contradicts our assumption. This implies that $m \leq n$, and since we are assuming that $n \leq m$, the claim is proved.

Claim 4.13 *For every $y \in \Omega'$, $Jy \in L'(E, F)$ is bijective.*

Since $m = n$, all claims above also hold for T^{-1}, and this means that there exists $K : \Omega \to L'(F, E)$ such that for every $g \in A^m(\Omega', F)$ and $x \in \Omega$, $(T^{-1}g)(x) = (Kx)(g(h^{-1}(x)))$.

Fix $y \in \Omega'$ and $\mathbf{f} \in F - \{0\}$. Let $x = h(y)$. Now take $g \in A^m(\Omega', F)$ with $g(y) = \mathbf{f}$. Then its is clear that $\mathbf{f} = g(y) = (T(T^{-1}g))(y)$, that is,

$$\begin{aligned}\mathbf{f} &= (Jy)((T^{-1}g)(x)) \\ &= (Jy)((Kx)(g(h^{-1}(x)))) \\ &= (Jy)((Kx)(g(y))) \\ &= (Jy)((Kx)(\mathbf{f})).\end{aligned}$$

This implies that $(Jy)(Kx)$ is the identity map on F. In the same way we can prove that $(Kx)(Jy)$ is the identity map on E. Consequently, Jy is bijective.

This ends the proof of the proposition. □

5 A result on automatic continuity

In this section we see that, if we endow the spaces with some natural topologies, then we obtain the continuity as a consequence. Notice that, according to Proposition 4.2, we can assume in particular that $n = m$ and $p = q$.

Theorem 5.1 *Assume that $A^n(\Omega, E)$ and $A^n(\Omega', F)$ are endowed with any topologies which are compatible with the pointwise convergence. Suppose that $T : A^n(\Omega, E) \to A^n(\Omega', F)$ is a $\mathbb{K}$-linear biseparating map. Then T is continuous.*

Proof. In our proof we will take advantage of the description of T given in Proposition 4.2. For this reason we will use the notation given there. We start proving the following claim.

Claim 5.1 *Let U be a (nonempty) bounded open subset of Ω' with $\mathrm{cl}_{\mathbb{R}^p} U \subset \Omega'$. Then the set*

$$A := \{y \in U : Jy \in I'(E, F)\, \text{is not continuous}\}$$

is finite.

Suppose that this is not the case, but there exist infinitely many $y \in U$ such that Jy is not continuous. We are going to construct inductively a sequence of points in A, a sequence (U_n) of pairwise disjoint open subsets of U, a sequence of functions (f_n) in $C_c^n(\Omega, \mathbb{R})$, and a sequence $(\mathbf{e}_n)$ of norm-one elements of E, satisfying the following properties:

- 1. $h(y_n) \in c(f_n) \subset h(U_n)$ for every $n \in \mathbb{N}$.
- 2. $\|f_n\| := \max_{\lambda \in \Lambda} \sup_{x \in \Omega} \left|\partial^\lambda f_n(x)\right| = 1/2^n$ for every $n \in \mathbb{N}$.
- 3. $\|(Jy_n)(\mathbf{e}_n)\| \geq n / |f_n(h(y_n))|$ for every $n \in \mathbb{N}$.

Take any point $y_1 \in A$ such that there are accumulation points of A in $\Omega' - \{y_1\}$. Then consider an open subset U_1 of U in such a way that $y_1 \in U_1$, and there are infinitely many points of A outside $\mathrm{cl}_{\mathbb{R}^p} U_1$. Next take $f_1 \in C_c^n(\Omega, \mathbb{R})$ such that $\|f_1\| = 1$, and such that $h(y_1) \in c(f_1) \subset h(U_1)$. Since Jy_1 is not continuous, there exists $\mathbf{e} \in E$, $\|\mathbf{e}\| = 1$, with

$$\|(Jy_1)(\mathbf{e})\| \geq \frac{1}{|f_1(h(y_1))|}.$$

Next assume that we have $\{y_1, y_2, \ldots, y_n\} \subset A$, $U_1, U_2, \ldots, U_n \subset U$ open and pairwise disjoint such that there are infinitely many points of A outside $\mathrm{cl}_{\mathbb{R}^p} U_1 \cup \mathrm{cl}_{\mathbb{R}^p} U_2 \cup \ldots \cup \mathrm{cl}_{\mathbb{R}^p} U_n$, $\{f_1, f_2, \ldots, f_n\} \subset C_c^n(\Omega, \mathbb{R})$ with $h(y_i) \in c(f_i) \subset h(U_i)$ and $\|f_i\| = 1/2^i$, for $i \in \{1, 2, \ldots, n\}$, and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in E$ all of them with norm 1, and such that $\|(Jy_i)(\mathbf{e}_i)\| \geq i / |f_i(h(y_i))|$ for $i = 1, 2, \ldots, n$.

Now it is easy to see how to take y_{n+1}, U_{n+1}, f_{n+1}, and $\mathbf{e}_{n+1}$ so that Properties 1, 2 and 3 above hold.

Since $\|f_n\| = 1/2^n$ for every $n \in \mathbb{N}$, we deduce that the map

$$g := \sum_{n=1}^{\infty} f_n \mathbf{e}_n$$

belongs to $A^n(\Omega, E)$. Consequently, Tg should belong to $A^n(\Omega', F)$. But we know by Proposition 4.2 that $(Tg)(y_n) = (Jy_n)(g(h(y_n)))$ for every $n \in \mathbb{N}$. This implies, by Property 3 above,

$$\begin{aligned}\|(Tg)(y_n)\| &= \|(Jy_n)(f_n(h(y_n))\mathbf{e}_n)\| \\ &= |f_n(h(y_n))| \, \|(Jy_n)(\mathbf{e}_n)\| \\ &\geq n.\end{aligned}$$

As a consequence Tg is unbounded in U. Since this is not possible, we conclude that the claim is correct.

Next, it is clear that to prove that T is continuous it is enough to show that it is closed, because we are dealing with Fréchet spaces. To prove it, let us consider a sequence (f_n) in $A^n(\Omega, E)$ convergent to zero, and assume that (Tf_n) converges to $g \in A^n(\Omega', F)$. We are going to prove that $g = 0$.

Take a bounded open subset U of Ω' with $\mathrm{cl}_{\mathbb{R}^p} U \subset \Omega'$. By Claim 5.1 above, we have that the subset A of points $y \in U$ such that Jy is not continuous is finite. So, if $y \in U - A$, Jy belongs to $I(E, F)$. Consequently, since $f_n(h(y))$ goes to zero (because the topology in $A^n(\Omega, E)$ is compatible with the pointwise convergence), then we have that $(Tf_n)(y) = (Jy)(f_n(h(y)))$ also goes to zero, that is, $g(y)$ must be zero. But taking into account that $U - A$ is dense in U, we deduce that $g \equiv 0$ on U. The conclusion follows now easily and T is continuous. □

6 Biseparating maps and functions of class s$-C^n$

Our aim in this section is to give a final description of biseparating maps between spaces of vector-valued differentiable functions taking into account that we know that they must be continuous when the spaces are endowed with some natural topologies. Of course these topologies will be compatible with the pointwise convergence. Namely, it is well known that by means of the seminorms p_K defined as

$$p_K(f) := \max_{\lambda \in \Lambda} \max_{x \in K} \left\| \partial^\lambda f(x) \right\|$$

for $f \in C^n(\Omega, E)$, where K runs through the compact subsets of Ω, $C^n(\Omega, E)$ becomes a locally convex space. In fact it is a Fréchet space. In the same way, in $C^n(\bar{\Omega}, E)$ we can consider the norm $\|\cdot\|$ defined as

$$\|f\| := \max_{\lambda \in \Lambda} \sup_{x \in \Omega} \left\| \partial^\lambda f(x) \right\|$$

for $f \in C^n(\bar{\Omega}, E)$. With this norm, our space $C^n(\bar{\Omega}, E)$ is also complete. We assume that $A^n(\Omega, E)$ and $A^n(\Omega', F)$ are endowed with the above topologies. Remark also that, as it follows easily from the Closed Graph Theorem, the topologies compatible with the pointwise convergence in our spaces coincide with these topologies.

Next proposition states that when $J : \Omega' \to L(E, F)$ is of class $\mathrm{s} - C^n$, we can define maps through J from $A^n(\Omega, E) \to A^n(\Omega', F)$ in a natural way.

Proposition 6.1 *Suppose that $n = m$, $p = q$. Let $J : \Omega' \to L(E, F)$ be a map of class $\mathrm{s} - C^n$, and let h be a diffeomorfism of class C^n from Ω' onto Ω. If,* *for $f \in C^n(\Omega, E)$, we define $(Tf)(y) := (Jy)(f(h(y)))$ for every $y \in \Omega'$, then $Tf \in C^n(\Omega', F)$.*

Proof. We consider first the map $\Phi : L(E,F) \times E \to F$ defined as $\Phi(A, \mathbf{e}) := A\mathbf{e}$ for each $(A, \mathbf{e}) \in L(E,F) \times E$. This is clearly bilinear and continuous when we consider in $L(E,F)$ the topology of the norm.

Suppose next that $L(E,F)$ is endowed again with the topology of the norm, and that $K : \Omega' \to L(E,F)$ is a continuous map. Then, given $g : \Omega' \to E$ continuous, the map $S_K^g : \Omega' \to L(E,F) \times E$ sending each $y \in \Omega'$ into $(Ky, g(y))$ is continuous.

On the other hand, if we suppose that $L(E,F)$ is endowed with the topology of the norm and that $f \in C^n(\Omega, E)$, then $S_J^{f \circ h}$ is of class C^{n-1} because both maps J and $f \circ h$ are. Consequently the composition map $\Phi \circ S_J^{f \circ h} : \Omega' \to F$, mapping each $y \in \Omega'$ into $(Jy)(f(h(y))) \in F$ is of class C^{n-1}.

Now we check the form of its first partial derivatives. We just see the partial derivative with respect to the first coordinate x_1. It is easy to check that

$$\begin{aligned}
\frac{\partial}{\partial x_1}(\Phi \circ S_J^{f \circ h})(y) &= \frac{\partial}{\partial X}\Phi(S_J^{f \circ h}(y)) \circ \frac{\partial}{\partial x_1} Jy + \frac{\partial}{\partial Y}\Phi(S_J^{f \circ h}(y)) \circ \frac{\partial}{\partial x_1}(f \circ h)(y) \\
&= \Phi\left(\frac{\partial}{\partial x_1} Jy, f(h(y))\right) + \Phi\left(Jy, \frac{\partial}{\partial x_1}(f \circ h)(y)\right) \\
&= \left(\Phi \circ S^{f \circ h}_{\frac{\partial}{\partial x_1} J}\right)(y) + \left(\Phi \circ S_J^{\frac{\partial}{\partial x_1} f \circ h}\right)(y).
\end{aligned}$$

By an inductive reasoning, we see that the partial derivatives of order $n-1$ of $\Phi \circ S_J^{f \circ h}$ are just a sum of terms of the form $\Phi \circ S^{\partial^\mu (f \circ h)}_{\partial^\lambda J}$, where $|\lambda|, |\mu| \leq n-1$. Consequently, to prove that $Tf \in C^n(\Omega', F)$, we just have to show that each one of the terms $\Phi \circ S^{\partial^\mu (f \circ h)}_{\partial^\lambda J}$ ($|\lambda|, |\mu| \leq n-1$) is of class C^1. We suppose that $S_K^g : \Omega' \to L(E,F) \times E$ is one of the above $S^{\partial^\mu (f \circ h)}_{\partial^\lambda J}$ (that is, let us denote $g = \partial^\mu (f \circ h) \in C^1(\Omega', E)$ and $K = \partial^\lambda J$), which is continuous when $L(E,F)$ is endowed with the topology of the norm, as we stated above.

Now we check the form of its first partial derivatives. Take $i \in \{1, \ldots, p\}$, and assume without loss of generality that $i = 1$. It is easy to check that

$$\lim_{k \to 0} \frac{(\Phi \circ S_K^g)(y + (k, 0, \ldots, 0)) - (\Phi \circ S_K^g)(y)}{k},$$

that is,

$$\lim_{k\to 0} \frac{(K(y+(k,0,\ldots,0)))(g(y+(k,0,\ldots,0))) - (Ky)(g(y))}{k}$$

is equal to

$$\lim_{k\to 0} \frac{(K(y+(k,0,\ldots,0)))(g(y+(k,0,\ldots,0))) - (K(y+(k,0,\ldots,0)))(g(y))}{k} +$$
$$+\lim_{k\to 0} \frac{(K(y+(k,0,\ldots,0)))(g(y)) - (Ky)(g(y))}{k},$$

that is, it is equal to

$$(Ky)\left(\frac{\partial}{\partial x_1} g\,(y)\right) + \left(\frac{\partial_{\mathrm{s}}}{\partial x_1} Ky\right)(g(y)),$$

by the definition of $\frac{\partial_{\mathrm{s}}}{\partial x_1} Ky$ and the fact that K is continuous for $L(E,F)$ endowed with the topology of the norm.

Applied to our context, we have that

$$\frac{\partial}{\partial x_1}(\Phi \circ S^{\partial^\mu (f\circ h)}_{\partial^\lambda J})(y) = \left(\Phi \circ S^{\frac{\partial}{\partial x_1}\partial^\mu (f\circ h)}_{\partial^\lambda J}\right)(y) + \left(\Phi \circ S^{\partial^\mu (f\circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda J}\right)(y).$$

Now, as we noted above, $S^{\frac{\partial}{\partial x_1}\partial^\mu (f\circ h)}_{\partial^\lambda J}$ is continuous when considering in $L(E,F)$ the topology of the norm. As a consequence, to obtain the continuity of all partial derivatives of order n of Tf, it is enough to see that $\Phi \circ S^{\partial^\mu (f\circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda J}$ is continuous.

In order to prove this, notice first that, since J is of class $\mathrm{s} - C^{n-1}$, then for the above λ the map $\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda J$ is continuous when $L(E,F)$ is endowed with the strong operator topology. This means that, given $\mathbf{e} \in E$, the map $\frac{\partial_{\mathrm{s}}}{\partial x_1}(\partial^\lambda J)_{\mathbf{e}} : \Omega' \to F$ sending each $y \in \Omega'$ into $(\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda Jy)(\mathbf{e})$ is continuous. Now take $\epsilon > 0$ and $y_0 \in \Omega'$. We are going to show that there exists $\delta > 0$ such that if $|y - y_0| < \delta$, $y \in \Omega'$, then

$$\left\| \Phi \circ S^{\partial^\mu (f\circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda J}(y) - \Phi \circ S^{\partial^\mu (f\circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial x_1}\partial^\lambda J}(y_0) \right\| < \epsilon.$$

Let $\mathbf{e}_0 := \partial^\mu (f\circ h)(y_0)$, and take $\delta_1 > 0$ such that the closed ball $\bar{B}(y_0, \delta_1)$ is contained in Ω'. Since $\frac{\partial_{\mathrm{s}}}{\partial x_1}(\partial^\lambda J)_{\mathbf{e}_0}$ is continuous, there exists an upper bound

M for this function on the compact set $\bar{B}(y_0, \delta_1)$. Also, there exists $\delta_2 > 0$, $\delta_2 < \delta_1$, such that if $|y - y_0| < \delta_2$, then

$$\left\| \left(\frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J y - \frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J y_0 \right) (\mathbf{e}_0) \right\| < \frac{\epsilon}{2}.$$

On the other hand, since $\partial^\mu (f \circ h)$ is continuous, there exists $\delta > 0$, $\delta < \delta_2$, such that if $y \in B(y_0, \delta)$, then $\|\partial^\mu (f \circ h)(y) - \mathbf{e}_0\| < \epsilon M/2$.

Consequently, if $|y - y_0| < \delta$, we have

$$\left\| \Phi \circ S^{\partial^\mu (f \circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J} (y) - \Phi \circ S^{\partial^\mu (f \circ h)}_{\frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J} (y_0) \right\|$$

is less than or equal to

$$\left\| \left(\frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J y \right) (\partial^\mu (f \circ h)(y) - \mathbf{e}_0) \right\| + \left\| \left(\frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J y - \frac{\partial_{\mathrm{s}}}{\partial_{x_1}} \partial^\lambda J y_0 \right) (\mathbf{e}_0) \right\|,$$

which is easily strictly less than ϵ. This proves the continuity of our functions, as it was to see. □

The following result is a direct consequence of Proposition 4.2 and Theorem 5.1. Roughly speaking, it says that Proposition 6.1 provides the only way to construct linear biseparating maps from $A^n(\Omega, E)$ onto $A^n(\Omega', F)$. We state the result in its complete form.

Theorem 6.2 *Suppose that $T : A^n(\Omega, E) \to A^m(\Omega', F)$ is a $\mathbb{K}$-linear biseparating map. Then $p = q$, $n = m$, and there exist a diffeomorphism h of class C^n from Ω' onto Ω and a map $J : \Omega' \to L(E, F)$ of class $\mathrm{s} - C^n$ such that for every $y \in \Omega'$ and every $f \in A^n(\Omega, E)$,*

$$(Tf)(y) = (Jy)(f(h(y))).$$

Moreover, $Jy \in I(E, F)$ for every $y \in \Omega'$.

Proof. We will follow the same notation as in Proposition 4.2, which provided a first description of linear biseparating maps, in particular everything related to the definition of J and h. Also, by Proposition 4.2, $p = q$ and $n = m$.

First, for each $\lambda \in \Lambda$, we define a map $J_\lambda : \Omega' \to L'(E, F)$ as

$$(J_\lambda y)(\mathbf{e}) := \partial^\lambda (T\widehat{\mathbf{e}})(y),$$

for $y \in \Omega'$ and $\mathbf{e} \in E$. J_λ is clearly well defined.

The rest of the proof will apply just for the case when we are in Situation 1, but it is easy to see that slight changes in it allow to prove the theorem when we are in Situation 2. We will prove it through several claims.

Claim 6.1 *For every $y \in \Omega'$ and every $\lambda \in \Lambda$, $J_\lambda y$ belongs to $L(E, F)$.*

Take $y \in \Omega'$ and a sequence $(\mathbf{e}_n)$ in E converging to zero. We will see that $((J_\lambda y)(\mathbf{e}_n))$ goes to zero. First we have that, since T is continuous by Theorem 5.1, the sequence of functions $(T\widehat{\mathbf{e}_n})$ converges to zero, which implies in particular that $((\partial^\lambda T\widehat{\mathbf{e}_n})(y))$ goes to zero. But this last sequence is precisely $((J_\lambda y)(\mathbf{e}_n))$, so the claim is proved.

Claim 6.2 *For each $\lambda \in \Lambda$, $|\lambda| \leq n - 1$, the map $J_\lambda : \Omega' \to L(E, F)$ is continuous when $L(E, F)$ is endowed with the topology of the norm.*

We will show that if $y_0 \in \Omega'$, then J_λ is continuous at y_0. Since T is continuous, we have that, for $r > 0$ such that the closed ball $\bar{B}(y_0, r)$ is contained in Ω', there exists $M > 0$ such that $p_{\bar{B}(y_0,r)}(T\widehat{\mathbf{e}}) < M$ holds for every $\mathbf{e} \in E$ with $\|\mathbf{e}\| \leq 1$. This implies that, for these $\mathbf{e}$, if $|y - y_0| < r$, then $\left\|(D\partial^\lambda T\widehat{\mathbf{e}})(y)\right\| \leq pM$. Consequently, as it can be seen for instance in [8, Theorem 3.3.2], we have that

$$\left\|(\partial^\lambda T\widehat{\mathbf{e}})(y) - \partial^\lambda (T\widehat{\mathbf{e}})(y_0)\right\| \leq pM\,|y - y_0|\,,$$

for every $\mathbf{e} \in E$ with $\|\mathbf{e}\| \leq 1$. Now, taking into account that $(J_\lambda y)(\mathbf{e}) = (\partial^\lambda T\widehat{\mathbf{e}})(y)$ for every $y \in \Omega'$, the result follows, and the claim is proved.

Claim 6.3 *For each $\lambda \in \Lambda$, $|\lambda| \leq n - 1$, $\partial^\lambda J = J_\lambda$.*

Of course, the result is clear if $n = 1$, so we suppose that $n \geq 2$. We will just prove the claim in the particular case when $\lambda = \lambda_1 := (1, 0, \ldots, 0)$. The proof for all other $\lambda \in \Lambda$ is similar and can be achieved inductively.

Take $y_0 \in \Omega'$ and $r > 0$ such that the closed ball $\bar{B}(y_0, r) \subset \Omega'$. It is clear that if $h \in \mathbb{R} - \{0\}$, $|h| < r$, and if $\mathbf{e}$ is in the closed unit ball of E, then

$$\left\|\frac{J(y_0 + (h, 0, \ldots, 0))(\mathbf{e}) - (Jy_0)(\mathbf{e})}{h} - (J_{\lambda_1} y_0)(\mathbf{e})\right\|$$

is equal to

$$\left\| \frac{(T\widehat{\mathbf{e}})(y_0 + (h, 0, \ldots, 0)) - (T\widehat{\mathbf{e}})(y_0)}{h} - (\partial^{\lambda_1} T\widehat{\mathbf{e}})(y_0) \right\|,$$

which, by [23, Corollary XIII.4.4], is less than

$$\sup_{|y-y_0|<2h} \left\| (\partial^{\lambda_1} T\widehat{\mathbf{e}})(y_0) - (\partial^{\lambda_1} T\widehat{\mathbf{e}})(y) \right\|,$$

that is, less than

$$\sup_{|y-y_0|<2h} \left\| J_{\lambda_1} y_0 - J_{\lambda_1} y \right\|.$$

Clearly this implies that

$$\lim_{h \to 0} \left\| \frac{J(y + (h, 0, \ldots, 0)) - Jy}{h} - J_{\lambda_1} y \right\| = 0.$$

Consequently, the partial derivative of J with respect to the first coordinate exists at each $y_0 \in \Omega'$ and is equal to $J_{\lambda_1} y_0$.

Claim 6.4 *Take $\lambda = (n_1, n_2, \ldots, n_i, \ldots, n_p) \in \Lambda$ with $|\lambda| = n - 1$. Then J_λ is of class $\mathrm{s} - C^1$. Moreover, if for each $i \in \{1, \ldots, n\}$, $\mu_i = (n_1, n_2, \ldots, n_i + 1, \ldots, n_p)$, then*

$$\frac{\partial_{\mathrm{s}}}{\partial x_i} J_\lambda = J_{\mu_i}.$$

The proof that $\frac{\partial_{\mathrm{s}}}{\partial x_i} J_\lambda = J_{\mu_i}$ is similar to the proof of Claim 6.3 we have done, taking into account that J_{μ_i} is perhaps no longer continuous when $L(E, F)$ is endowed with its norm but $\partial^{\mu_i} T\widehat{\mathbf{e}}$ is continuous for every $\mathbf{e} \in E$. Consequently, to finish we just have to show that J_μ, $|\mu| = n$, is continuous when considering in $L(E, F)$ the strong operator topology. We have to prove that if $\mu \in \Lambda$, and (y_n) is a sequence in Ω' converging to $y \in \Omega'$, then $(J_\mu y_n)(\mathbf{e})$ converges to $(J_\mu y)(\mathbf{e})$ for each $\mathbf{e} \in E$. But this is immediate from the definition of J_μ. $\Box$

Remark 6.3 Notice that, in the case when we are in Situation 2, in Theorem 6.2 the map J and its partial (s-)derivatives up to order n can also be extended continuously to the boundary of Ω' in a natural way, when considering in $L(E, F)$ the strong operator topology.

In the special case when $E = \mathbb{K} = F$, we immediately deduce the following result.

Corollary 6.4 *Suppose that $T : A^n(\Omega, \mathbb{K}) \to A^m(\Omega', \mathbb{K})$ is a $\mathbb{K}$-linear biseparating map. Then $p = q$, $n = m$, and there exist a diffeomorphism h of class C^n from Ω' onto Ω and a map $a : \Omega' \to \mathbb{K}$ of class C^n which does not vanish at any point of Ω', such that for every $y \in \Omega'$ and every $f \in A^m(\Omega, \mathbb{K})$,*

$$(Tf)(y) = a(y) f(h(y)).$$

We finish with a corollary whose proof is easy from Theorem 6.2.

Corollary 6.5 *If $A^n(\Omega, E)$ and $A^n(\Omega', F)$ are endowed with the topology of the pointwise convergence, then every linear biseparating map $T : A^n(\Omega, E) \to A^n(\Omega', F)$ is continuous.*

7 Final Remark

Even if in previous sections we consider two possible situations for our spaces of vector-valued functions, Proposition 4.2 and Theorems 5.1 and 6.2 can be given in a broader context.

Let $\Omega \subset \mathbb{R}^p$ and $\Omega' \subset \mathbb{R}^q$ be (nonempty) open sets, and consider any of the following contexts.

- Case i) $\Omega_1 := \Omega$ and $\Omega'_1 := \Omega'$.
- Case ii) Ω and Ω' are bounded, $\mathrm{int}_{\mathbb{R}^p} \mathrm{cl}_{\mathbb{R}^p} \Omega = \Omega$, $\mathrm{int}_{\mathbb{R}^q} \mathrm{cl}_{\mathbb{R}^q} \Omega' = \Omega'$, and $\Omega_1 := \mathrm{cl}_{\mathbb{R}^p} \Omega$ and $\Omega_1 := \mathrm{cl}_{\mathbb{R}^q} \Omega$.

Suppose now that $A \subset C^n(\Omega, \mathbb{R}) \cap C(\Omega_1, \mathbb{R})$ and $B \subset C^m(\Omega', \mathbb{R}) \cap C(\Omega'_1, \mathbb{R})$ are strongly regular rings (see [2] for an appropriate description). Take now $A(\Omega, E) \subset C^n(\Omega, E) \cap C(\Omega_1, E)$ and $B(\Omega', F) \subset C^m(\Omega', F) \cap C(\Omega'_1, F)$. Namely, to construct the support map $h : \Omega'_1 \to \Omega_1$, we need that subspaces $A(\Omega, E) \subset C^n(\Omega, E) \cap C(\Omega_1, E)$ and $B(\Omega', F) \subset C^m(\Omega', F) \cap C(\Omega'_1, F)$ are a compatible A-module and a compatible B-module (see [2]). In both cases h will be a homeomorphism from Ω' onto Ω.

On the other hand, apart from these necessary conditions for constructing h, a careful reading of the proofs shows that the essential requirements that linear subspaces $A(\Omega, E)$ and $B(\Omega', F)$ must meet so as to satisfy Lemma 4.1 and Proposition 4.2 are: 1) $A(\Omega, E)$ and $B(\Omega', F)$ contain the constant functions; 2) $A(\Omega, E)$ (respectively, $B(\Omega', F)$) contains all functions in $C^n(\Omega, E)$ (respectively, $C^m(\Omega', F)$) with compact support; and 3) $C^n_c(\Omega, \mathbb{R}) \subset A$ and

$C_c^m(\Omega', \mathbb{R}) \subset B$. In the case when Ω and Ω' are bounded and the coordinate projections x_i belong to A and B, then the proof of Proposition 4.2 can be followed step by step with no changes. Otherwise (even if Ω and Ω' are not bounded), changes are few and natural.

Finally, if we want to obtain results similar to Theorems 5.1 and 6.2 (with the same proofs) for our spaces $A(\Omega, E)$ and $B(\Omega', F)$, besides all the above conditions, they must be endowed with a *suitable* norm or family of seminorms providing a topology compatible with the pointwise convergence. This will be the case, for instance, of the spaces of functions with bounded derivatives.

So we study the case of the spaces $C_*^n(\Omega, E) \subset C^n(\Omega, E)$ and $C_*^m(\Omega', F) \subset C^m(\Omega', F)$ consisting of all functions such that all partial derivatives up to orders n and m, respectively, are *bounded*. The space $C_*^n(\Omega, E)$ (and similarly $C_*^m(\Omega', F)$) becomes a Banach space with the norm defined for each $f \in C_*^n(\Omega, E)$ as

$$\|f\| := \max_{\lambda \in \Lambda} \sup_{x \in \Omega} \left\| \partial^\lambda f(x) \right\|.$$

This is a suitable norm in the above sense because the proof of Theorem 6.2 can be followed easily for these spaces equipped with such norm.

First, notice that, since we are assuming that $n, m \geq 1$, then in particular when Ω is convex all functions in $C_*^n(\Omega, E)$ admit a continuous extension to the closure of Ω in $\mathbb{R}^p$: suppose that (x_n) is a sequence in Ω converging to x_0 in the boundary of Ω, and that $f \in C_*^n(\Omega, E)$; then since the differential Df is bounded on the whole Ω by an $M > 0$, we have that, by [8, Theorem 3.3.2],

$$\|f(x_n) - f(x_m)\| \leq M \, |x_n - x_m| \, ,$$

which implies that $(f(x_n))$ is a Cauchy sequence. In this way we would define the extension $f(x)$ as the limit of this sequence. It is straightforward to see that the new extended function is continuous in the closure of Ω.

On the other hand, when Ω and Ω' are *bounded* and *convex*, then it is easy to see that $C_*^n(\Omega, E)$ is a $C^n(\bar{\Omega}, \mathbb{R})$-module, and a similar statement is also valid for $C_*^m(\Omega', F)$. As a consequence, by the comments given above, Proposition 4.2 and Theorems 5.1 and 6.2 can also be stated in this new situation (for Ω and Ω' bounded and convex). Furthermore, in this case, as in Remark 6.3, it is also possible to say that partial derivatives up to order $n-1$ of J admit a continuous extension (when $L(E, F)$ is equipped with the strong operator topology) to the boundary of Ω'. As for the partial s-derivatives of all partial derivatives of order $n-1$ of J, an elementary application of the Uniform Boundedness Theorem shows that they are bounded on Ω'.

What happens if Ω is for instance not bounded? One might be tempted to follow a similar pattern as indicated above when trying to describe linear biseparating maps defined between $C^n_*(\Omega, E)$ and $C^m_*(\Omega', F)$. But, in that case, we have that $C^n_*(\Omega, E)$ is no longer a $C^n(\Omega, \mathbb{R})$-module (as $A^n(\Omega, E)$ in previous sections was). Anyway, it is a $C^n_*(\Omega, \mathbb{R})$-module, and we could try to follow the proof of Proposition 4.2 to get a similar description of linear biseparating maps, but even if we could manage to adapt the proof step by step (with some changes), there is a major problem from the beginning: in general the space $C^n_*(\Omega, \mathbb{R})$ is not a strongly regular ring, so our results cannot be applied, and in particular the existence of the support map h is not clear. Let us see an example where $C^n_*(\Omega, \mathbb{R})$ is not a strongly regular ring.

Example 7.1 Suppose that $\Omega \subset \mathbb{R}$ is the open set defined as $\Omega := \bigcup_{k=1}^{\infty}(k - 1/k, k + 1/k)$. Let us define K as the closure of $\mathbb{N}$ in $\beta\Omega$ (the Stone-Čech compactification of Ω). Now take $x_0 \in \beta\Omega - (K \cup \mathbb{R})$. It is clear that if a derivable function $f : \Omega \to \mathbb{R}$ satisfies $f^{\beta\Omega} \equiv 0$ on K and $f^{\beta\Omega} \equiv 1$ on a neighborhood of x_0 (where $f^{\beta\Omega}$ stands for its extension to $\beta\Omega$), then for some sequence (x_n) in Ω going to infinity, the sequence $(f(x_n))$ converges to 1. As a consequence from the Mean Value Theorem, we conclude that the derivative of f cannot be bounded on Ω, that is, $f \notin C^1_*(\Omega, \mathbb{R})$, as we wanted to show.

We end the paper with some related questions, concerning special cases where our techniques cannot be applied.

Problem 1. Assume that there exists a biseparating map $T : A^n(\Omega, E) \to A^m(\Omega', F)$ which is not linear. Can we deduce that the support map $h : \Omega' \to \Omega$ is a diffeomorphism of class n? Remark that the assumption of linearity in Proposition 4.2 is necessary for its proof.

Problem 2. Suppose that $C^\infty(\Omega, E)$ is the space of E-valued functions which are of class C^∞ in Ω, and that $C^\infty(\Omega', F)$ is defined in a similar way. Describe the linear biseparating maps from $C^\infty(\Omega, E)$ onto $C^\infty(\Omega', F)$. Must such a map be continuous? Notice that by the comments given in the Final Remark above, the construction of the support map h is possible, but the proof of Proposition 4.2 is no longer valid.

Problem 3. Let Ω and Ω' be unbounded open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Describe the linear biseparating maps from $C^n_*(\Omega, E)$ onto $C^m_*(\Omega', F)$.

Problem 4. Determine all subspaces $A(\Omega, E) \subset A^n(\Omega, E)$ and $B(\Omega', F) \subset A^m(\Omega', F)$ such that the existence of a (linear) biseparating map from $A(\Omega, E)$ onto $B(\Omega', F)$ implies that E and F are isomorphic as Banach spaces.

Acknowledgements

The author wishes to thank three anonymous referees for their valuable suggestions and, also, Professor R. Aron for drawing his attention to some papers which appear in the references.

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