



Cristina Tîrnăucă | Timo Knuutila

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# Efficient Language Learning with Correction Queries

**Cristina Tîrnăucă**

Research Group on Mathematical Linguistics, Rovira i Virgili University  
Pl. Imperial Tàrraco 1, Tarragona 43005, Spain  
`cristina.bibire@estudiants.urv.es`

**Timo Knuutila**

Department of Information Technology, University of Turku  
Joukahaisenkatu 3-5 B, FI-20014 Turku, Finland  
`timo.knuutila@it.utu.fi`

## Abstract

We investigate two of the language classes intensively studied by the algorithmic learning theory community in the context of learning with correction queries. More precisely, we show that any pattern language can be inferred in polynomial time in length of the pattern by asking just a linear number of correction queries, and that  $k$ -reversible languages are efficiently learnable within this setting. Note that although the class of all pattern languages is learnable with membership queries, this cannot be done in polynomial time. Moreover, the class of  $k$ -reversible languages is not learnable at all using membership queries only. Furthermore, we present results on a newly introduced class of languages, namely the class of injective languages.

**Keywords:** correction query, pattern languages,  $k$ -reversible languages, injective languages, polynomial time algorithms

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# 1 Introduction

Without any doubt, there is no formal model that can capture all aspects of human learning. Nevertheless, the overall aim of researchers working in algorithmic learning theory has been to gain a better understanding of what learning really is. Actually, the field itself has been introduced as an attempt to construct a precise model for the notion of “being able to speak a language” [Gol67].

Among the most celebrated models (Gold’s model of *learning from examples* [Gol67], Angluin’s *query learning* model [Ang87], Valiant’s *PAC learning* model [Val84]), the best one for describing the child-adult interaction within the process of child acquiring his native language is the one proposed in [Ang87]. There, *the learner* receives information about a target concept by asking queries of a specific kind (depending on the chosen query model type) which will be truthfully answered by *the teacher*. After at most finitely many queries, the learner is required to return its hypothesis, and this should be the correct one.

The first query learning algorithm ( $L^*$ ) was able to identify any minimal complete deterministic finite automaton (DFA) in polynomial time, using membership queries (MQs) and equivalence queries (EQs) [Ang87]. Meanwhile, other types of queries have been introduced: subset, superset, disjointness and exhaustive queries [Ang88], structured MQs [Sak90], *etc.*, and also various target concepts have been investigated: non-deterministic finite automata [Yok94], context-free grammars [Sak90], two-tape automata [Yok96], regular tree languages [DH07, TT07], *etc.*

Still, none of the above mentioned queries reflects one important aspect of children language acquisition, namely that, although children are not explicitly provided negative information, they are corrected when they make mistakes. Following this idea, L. Becerra-Bonache, A.H. Dediu and C. Tîrnăuică introduced in [BBDT06] a new type of a query, the so-called *correction query* (CQ), and showed that DFAs are learnable in polynomial time using CQs and EQs. Continuing the investigation on CQs, C. Tîrnăuică and S. Kobayashi found necessary and sufficient conditions for an indexable class of recursive languages to be learnable with CQs only [TK07]. Also, they showed some relations existing between this model and other well-known (query and Gold-style) learning models.

In contrast with the approach in [TK07], where the learnability is studied regardless time complexity, we focus in this paper on algorithms for identifying language classes in polynomial time. Thus, the rest of the paper is structured as follows. Preliminary notions and results are presented in Section 2. Then, we give polynomial time algorithms for learning  $k$ -reversible languages (Section 3) and injective languages (Section 4) with CQs. Section 5 contains an algorithm for learning pattern languages, along with discussions about correctness, termination and time analysis. In Section 6 we present some other results on the learnability with MQs of the classes investigated in the previous sections. Concluding remarks and further research topics are presented in Section 7.

## 2 Preliminaries

Familiarity with standard recursion theoretic and language theoretic notions is assumed, see [HU79, MVMP04].

Let  $\Sigma$  be a finite alphabet of symbols. By  $\Sigma^*$  we denote the set of all finite strings of symbols from  $\Sigma$ . A *language* is any set of strings over  $\Sigma$ . The length of a string  $w$  is denoted by  $|w|$ , and the concatenation of two strings  $u$  and  $v$  by  $uv$  or  $u \cdot v$ . The empty string (i.e., the unique string of length 0) is denoted by  $\lambda$ . If  $w = uv$  for some  $u, v \in \Sigma^*$ , we say that  $u$  is a *prefix* of  $w$  and  $v$  is a *suffix* of  $w$ .

A set  $S$  is said to be *prefix-closed* if for all strings  $u$  in  $S$  and all  $v$  prefixes of  $u$ , the string  $v$  is also in  $S$ . The notion of *suffix-closed* set is defined similarly.

By  $\Sigma^{\leq k}$  we denote the set  $\{w \in \Sigma^* \mid |w| \leq k\}$ , by  $\text{Pref}(L)$  the set  $\{u \mid \exists v \in \Sigma^* \text{ such that } uv \in L\}$  of all prefixes of a language  $L \subseteq \Sigma^*$ , and by  $\text{Tail}_L(u) = \{v \mid uv \in L\}$  the left-quotient of  $L$  and  $u$ . Thus,  $\text{Tail}_L(u) \neq \emptyset$  if and only if  $u \in \text{Pref}(L)$ .

A *deterministic finite automaton* is a 5-tuple  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  where  $Q$  is a finite set of *states*,  $\Sigma$  is a finite alphabet,  $q_0 \in Q$  is the *initial state*,  $F \subseteq Q$  is the set of *final states*, and  $\delta$  is a partial function, called *transition function*, that maps  $Q \times \Sigma$  to  $Q$ . This function can be extended to strings by writing  $\delta(q, \lambda) = q$ , and  $\delta(q, u \cdot a) = \delta(\delta(q, u), a)$ , for all  $q \in Q$ ,  $u \in \Sigma^*$  and  $a \in \Sigma$ . A string  $u$  is *accepted* by  $\mathcal{A}$  if  $\delta(q_0, u) \in F$ . The set of strings accepted by  $\mathcal{A}$  is denoted by  $L(\mathcal{A})$  and is called a *regular language*. The number of states of an automaton  $\mathcal{A}$  is also called the *size* of  $\mathcal{A}$ . A DFA  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  is *complete* if for all  $q$  in  $Q$  and  $a$  in  $\Sigma$ ,  $\delta(q, a)$  is defined, i.e.,  $\delta$  is a total function. For any regular language  $L$ , there exists a minimum state DFA  $\mathcal{A}_L$  such that  $L(\mathcal{A}_L) = L$  (see [HU79], pp. 65-71).

A state  $q$  is called *reachable* if there exists  $u \in \Sigma^*$  such that  $\delta(q_0, u) = q$  and *co-reachable* if there exists  $u \in \Sigma^*$  such that  $\delta(q, u) \in F$ . A reachable state that is not co-reachable is called a *sink state*. Note that in a minimum DFA there is at most one sink state, and all states are reachable.

Given a language  $L$ , one can define the following relation on strings:  $u_1 \equiv_L u_2$  if and only if for all  $u$  in  $\Sigma^*$ ,  $u_1 \cdot u \in L \Leftrightarrow u_2 \cdot u \in L$ . It is easy to show that  $\equiv_L$  is an equivalence relation on strings, and thus it divides the set of all finite strings into one or more equivalence classes. We denote by  $[u]_L$  (or simply  $[u]$ , when there is no confusion) the equivalence class of the string  $u$  (i.e.,  $\{u' \mid u' \equiv_L u\}$ ), and by  $\Sigma^*/\equiv_L$  the set of all equivalence classes induced by  $\equiv_L$  on  $\Sigma^*$ .

The Myhill-Nerode Theorem states that the number of equivalence classes of  $\equiv_L$  (also called the *index* of  $L$ ) is equal to the number of states of  $\mathcal{A}_L$ . As a direct consequence, a language  $L$  is regular if and only if the index of  $L$  is finite.

Assume that  $\Sigma$  is a totally ordered set and let  $\prec_{lex}$  be the lexicographical order on  $\Sigma^*$ . Then, the *lex-length order*  $\prec$  on  $\Sigma^*$  is defined by:  $u \prec v$  if either  $|u| < |v|$ , or else  $|u| = |v|$  and  $u \prec_{lex} v$ . In other words, strings are compared first according to length and then lexicographically.

If  $f : A \rightarrow B$  is a function, by  $f(X)$  we denote the set  $\{f(x) \mid x \in X\}$ .

Moreover, we say that  $f$  and  $g$  are equal if they have the same domain  $A$ , and  $f(x) = g(x)$  for all  $x \in A$ .

## 2.1 Query Learning

Let  $\mathcal{C}$  be a class of recursive languages over  $\Sigma^*$ . We say that  $\mathcal{C}$  is an *indexable class* if there is an effective enumeration  $(L_i)_{i \geq 1}$  of all and only the languages in  $\mathcal{C}$  such that membership is uniformly decidable, i.e., there is a computable function that, for any  $w \in \Sigma^*$  and  $i \geq 1$ , returns 1 if  $w \in L_i$ , and 0 otherwise. Such an enumeration will subsequently be called an *indexing* of  $\mathcal{C}$ . In the sequel we might say that  $\mathcal{C} = (L_i)_{i \geq 1}$  is an indexable class and understand that  $\mathcal{C}$  is an indexable class and  $(L_i)_{i \geq 1}$  is an indexing of  $\mathcal{C}$ .

In the query learning model a learner has access to an oracle that truthfully answers queries of a specified kind. A *query learner*  $M$  is an algorithmic device that, depending on the reply on the previous queries, either computes a new query, or returns a hypothesis and halts.

More formally, let  $\mathcal{C} = (L_i)_{i \geq 1}$  be an indexable class,  $M$  a query learner, and let  $L \in \mathcal{C}$ . We say that  $M$  learns  $L$  using some type of queries if it eventually halts and its only hypothesis, say  $i$ , correctly describes  $L$ , i.e.,  $L_i = L$ . So,  $M$  returns its unique and correct guess  $i$  after only finitely many queries. Moreover,  $M$  learns  $\mathcal{C}$  using some type of queries if it learns every  $L \in \mathcal{C}$  using queries of the specified type. In the sequel we consider:

*Membership queries.* The input is a string  $w$  and the answer is ‘yes’ or ‘no’, depending on whether or not  $w$  belongs to the target language  $L$ .

*Correction queries.* The input is a string  $w$  and the answer is the smallest string (in lex-length order) of the set  $\text{Tail}_L(w)$  if  $w \in \text{Pref}(L)$ , and the special symbol  $\theta \notin \Sigma$  otherwise. We denote the correction of a string  $w$  with respect to the language  $L$  by  $C_L(w)$ .

The collections of all indexable classes  $\mathcal{C}$  for which there is a query learner  $M$  such that  $M$  learns  $\mathcal{C}$  using MQs (CQs) is denoted by  $\text{MemQ}$  ( $\text{CorQ}$ , respectively).

## 3 Learning $k$ -Reversible Languages with CQs

Angluin introduced the class of  $k$ -reversible languages (henceforth denoted by  $k\text{-Rev}$ ) in [Ang82], and showed that it is inferable from positive data in the limit. We study its learnability in the context of learning with CQs, and show that there is a polynomial time algorithm which identifies any language in the class  $k\text{-Rev}$  after asking a finite number of CQs.

Although the original definition of  $k$ -reversible languages uses the notion of  $k$ -reversible automata, we will give here only a purely language-theoretic characterization.

**Theorem 1** (Angluin, [Ang82]). *Let  $L$  be a regular language. Then  $L$  is  $k$ -reversible if and only if whenever  $u_1vw, u_2vw$  are in  $L$  and  $|v| = k$ ,  $Tail_L(u_1v) = Tail_L(u_2v)$ .*

Note that for the 0-reversible languages we have:

**Theorem 2** (Angluin, [Ang82]). *Let  $L$  be a regular language. Then  $L$  is 0-reversible if and only if whenever  $u_1v$  and  $u_2v$  are in  $L$ ,  $Tail_L(u_1) = Tail_L(u_2)$ .*

Let  $\Sigma$  be an alphabet, and  $L \subseteq \Sigma^*$  be the target  $k$ -reversible language. For any string  $u$  in  $\Sigma^*$ , we define the function  $row(u) : \Sigma^{\leq k} \rightarrow \Sigma^* \cup \{\theta\}$  by  $row(u)(v) = C_L(uv)$ . We show that each equivalence class in  $\Sigma^*/\equiv_L$  is uniquely identified by the values of function  $row$  on  $\Sigma^{\leq k}$ .

**Proposition 3.** *Let  $L$  be a  $k$ -reversible language. Then, for all  $u_1, u_2 \in \Sigma^*$ ,  $u_1 \equiv_L u_2$  if and only if  $row(u_1) = row(u_2)$ .*

*Proof.* Let us first notice that for all regular languages  $L$  and for any  $k \in \mathbb{N}$ ,  $u_1 \equiv_L u_2 \Rightarrow row(u_1) = row(u_2)$  (by the definition of function  $row$ ), so we just have to show that  $row(u_1) = row(u_2) \Rightarrow u_1 \equiv_L u_2$ .

Indeed, suppose there exist  $u_1, u_2 \in \Sigma^*$  such that  $row(u_1) = row(u_2)$  and  $u_1 \not\equiv_L u_2$ . Hence, there must exist  $w$  such that either

- $u_1w \in L$  and  $u_2w \notin L$ ; or
- $u_1w \notin L$  and  $u_2w \in L$ .

Let us assume the former case (the other one is similar).

- 1) If  $|w| \leq k$ , then  $w \in \Sigma^{\leq k}$ , and since  $row(u_1) = row(u_2)$  we get in particular  $row(u_1)(w) = row(u_2)(w)$ , that is  $C_L(u_1w) = C_L(u_2w)$ . But  $u_1w \in L$  implies  $C_L(u_1w) = \lambda$ , and so  $C_L(u_2w) = \lambda$  which is in contradiction with  $u_2w \notin L$ .
- 2) If  $|w| > k$ , then there must exist  $v, w' \in \Sigma^*$  such that  $w = vw'$  and  $|v| = k$ . Moreover, by assumption  $u_1vw' \in L$  and  $u_2vw' \notin L$ , so  $u_1v \not\equiv_L u_2v$ . On the other hand since  $row(u_1) = row(u_2)$  and  $v \in \Sigma^{\leq k}$ , we have  $row(u_1)(v) = row(u_2)(v)$ , that is  $C_L(u_1v) = C_L(u_2v) = v'$ . Because  $u_1v \cdot w' \in L$ ,  $Tail_L(u_1v) \neq \emptyset$  and hence  $C_L(u_1v) \in \Sigma^*$ . Since  $L \in k\text{-Rev}$ ,  $u_1vv' \in L$ ,  $u_2vv' \in L$  and  $|v| = k$ , we get  $Tail_L(u_1v) = Tail_L(u_2v)$  (cf. Theorem 1) which is in contradiction with  $u_1v \not\equiv_L u_2v$ .

□

This result tells us that if  $\mathcal{A}_L = (Q, \Sigma, \delta, q_0, F)$  is the minimal complete automaton for the  $k$ -reversible language  $L$ , then the values of function  $row(u)$  on  $\Sigma^{\leq k}$  uniquely identify the state  $\delta(q_0, u)$ .

### 3.1 The Algorithm

The algorithm follows the lines of  $L^*$ . We have an *observation table* (here and there denoted by  $(S, E, C)$ ) in which lines are indexed by the elements of a prefix-closed set  $S$ , columns are indexed by the elements of a suffix-closed set  $E$ , and the element of the table situated at the intersection of line  $u$  with column  $v$  is  $C_L(uv)$ . We start with  $S = \{\lambda\}$  and  $E = \Sigma^{\leq k}$ , and then increase the size of  $S$  by adding elements with distinct row values. An important difference between our algorithm and  $L^*$  is that in our case the set  $E$  is never modified during the run of the algorithm (in  $L^*$ ,  $E$  contains only one element in the beginning, and it is gradually enlarged when needed).

We say that the observation table  $(S, E, C)$  is *closed* if for all  $u \in S$  and  $a \in \Sigma$ , there exists  $u' \in S$  such that  $\text{row}(u') = \text{row}(ua)$ . Moreover,  $(S, E, C)$  is *consistent* if for all  $u_1, u_2 \in S$ ,  $\text{row}(u_1) \neq \text{row}(u_2)$  (which is equivalent to  $u_1 \not\equiv_L u_2$ , or to  $[u_1] \neq [u_2]$ ). It is clear that if the table  $(S, E, C)$  is consistent and  $S$  has exactly  $n$  elements (where  $n$  is the index of  $L$ ), then the strings in  $S$  are in bijection with the classes of  $\Sigma^*/\equiv_L$ .

For any closed and consistent table  $(S, E, C)$  we construct the automaton  $\mathcal{A}(S, E, C) = (Q, \Sigma, \delta, q_0, F)$  as follows:  $Q = \{[u] \mid u \in S\}$ ,  $q_0 = [\lambda]$ ,  $F = \{[u] \mid u \in S \text{ and } C_L(u) = \lambda\}$ , and  $\delta([u], a) = [ua]$  for all  $u \in S$  and  $a \in \Sigma$ .

To see that this is a well defined automaton, note that since  $S$  is a non-empty prefix-closed set, it must contain  $\lambda$ , so  $q_0$  is defined. Because  $S$  is consistent, there are no two elements  $u_1, u_2$  in  $S$  such that  $[u_1] = [u_2]$ . Thus,  $F$  is well defined. Since the observation table  $(S, E, C)$  is closed, for each  $u \in S$  and  $a \in \Sigma$ , there exists  $u'$  in  $S$  such that  $\text{row}(ua) = \text{row}(u')$  (hence  $[ua] = [u']$ ), and because it is consistent, this  $u'$  is unique. So  $\delta$  is well defined. It is easy to check that  $\delta(q_0, u) = [u]$  for all  $u$  in  $S \cup S\Sigma$ .

We present a polynomial time algorithm that learns any  $k$ -reversible language  $L$  after asking a finite number of CQs.

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**Algorithm 1** An algorithm for learning the class  $k\text{-Rev}$  with CQs

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1:  $S := \{\lambda\}$ ,  $E := \Sigma^{\leq k}$ 
2:  $\text{closed} := \text{TRUE}$ 
3: update the table by asking CQs for all strings in  $\{uv \mid u \in S \cup S\Sigma, v \in E\}$ 
4: repeat
5:   if  $(\exists u \in S \text{ and } a \in \Sigma \text{ such that } \text{row}(ua) \notin \text{row}(S))$  then
6:     add  $ua$  to  $S$ 
7:     update the table by asking CQs for all strings in  $\{uaa'v \mid a' \in \Sigma, v \in E\}$ 
8:      $\text{closed} := \text{FALSE}$ 
9:   end if
10: until  $\text{closed}$ 
11: output  $\mathcal{A}(S, E, C)$  and halt.
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Note that since the algorithm adds to  $S$  only elements with distinct row values, the table  $(S, E, C)$  is always consistent. We will see that as long as  $|S| < n$ , it is not closed.

**Lemma 4.** *If  $|S| < n$ , then  $(S, E, C)$  is not closed.*

*Proof.* Let us assume that there exists  $m < n$  such that  $|S| = m$  and the table  $(S, E, C)$  is closed. Let  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  be the automaton  $\mathcal{A}(S, E, C)$  and  $\mathcal{A}_L = (Q', \Sigma, \delta', q'_0, F')$  the minimal complete automaton accepting  $L$ .

We define the function  $\varphi : Q \rightarrow Q'$  by  $\varphi([u]) := \delta'(q'_0, u)$ . Note that  $\varphi$  is well defined because there are no two strings  $u_1, u_2$  in  $S$  such that  $[u_1] = [u_2]$ . Moreover, it is injective since  $\varphi([u_1]) = \varphi([u_2])$  implies  $\delta'(q'_0, u_1) = \delta'(q'_0, u_2)$  which is equivalent to  $[u_1] = [u_2]$ . We show that  $\varphi$  is a morphism of automata from  $\mathcal{A}(S, E, C)$  to  $\mathcal{A}_L$ , that is:  $\varphi(q_0) = q'_0$ ,  $\varphi(F) \subseteq F'$ , and  $\varphi(\delta([u], a)) = \delta'(\varphi([u]), a)$  for all  $u \in S$  and  $a \in \Sigma$ .

Clearly,  $\varphi(q_0) = \varphi([\lambda]) = \delta'(q'_0, \lambda) = q'_0$ . Let us now take  $[u]$  in  $F$ . Since  $\varphi([u]) = \delta'(q'_0, u)$  and  $C_L(u) = \lambda$  (that is,  $u \in L$ ), it follows that  $\varphi([u]) \in F'$ . Finally,  $\varphi(\delta([u], a)) = \varphi([ua]) = \varphi([v])$  for some  $v$  in  $S$  such that  $[ua] = [v]$  (the table is closed), and  $\delta'(\varphi([u]), a) = \delta'(\delta'(q'_0, u), a) = \delta'(q'_0, ua)$ . It is enough to see that  $\varphi([v]) = \delta'(q'_0, v) = \delta'(q'_0, ua)$  (because  $[v] = [ua]$  and  $\mathcal{A}_L$  is the minimal automaton accepting  $L$ ) to conclude the proof.

We have constructed an injective morphism from  $\mathcal{A}$  to  $\mathcal{A}_L$  such that  $|Q| = m < n = |Q'|$ . Since both  $\mathcal{A}$  and  $\mathcal{A}_L$  are complete automata, this leads us to a contradiction.  $\square$

In the following sections we show that the algorithm runs in polynomial time, and that it terminates with the minimal automaton for the target language as its output.

### 3.2 Correctness and Termination

We have seen that as long as  $|S| < n$ , the table is not closed, so there will always be an  $u$  in  $S$  and a symbol  $a$  in  $\Sigma$  such that  $\text{row}(ua) \notin \text{row}(S)$ . Since the cardinality of the set  $S$  is initially 1, and increases by 1 with each “repeat-until” loop (lines 4-10), it will eventually be  $n$ , and hence the algorithm is guaranteed to terminate.

We claim that when  $|S| = n$ , the observation table  $(S, E, C)$  is closed and consistent, and  $\mathcal{A}(S, E, C)$  is isomorphic to  $\mathcal{A}_L$ . Indeed, if  $|S| = n$  then the set  $\{[u] \mid u \in S\}$  has cardinality  $n$ , since the elements of  $S$  have distinct row values. Thus for all  $u \in S$  and  $a \in \Sigma$ ,  $\text{row}(ua) \in \text{row}(S)$  (otherwise  $[ua]$  would be the  $(n+1)^{\text{th}}$  equivalence class of  $\Sigma^*/\equiv_L$ ), and hence the table is closed.

To see that  $\mathcal{A}(S, E, C)$  and  $\mathcal{A}_L$  are isomorphic, let us take  $\mathcal{A}(S, E, C) = (Q, \Sigma, \delta, q_0, F)$ ,  $\mathcal{A}_L = (Q', \Sigma, \delta', q'_0, F')$ , and the function  $\varphi : Q \rightarrow Q'$  defined by  $\varphi([u]) := \delta'(q'_0, u)$  for all  $u \in S$ . As in the proof of Lemma 4, it can be shown

that  $\varphi$  is a well-defined and injective automata morphism. Since the two automata have the same number of states,  $\varphi$  is also surjective, and hence bijective. Let us now show that  $\varphi(F) = F'$ . Indeed, take  $q \in F'$ . Because  $\varphi$  is bijective, there exists  $u$  in  $S$  such that  $\varphi([u]) = q$ . It follows immediately that  $\delta'(q'_0, u) \in F'$ , and hence  $u \in L$ . Thus,  $C_L(u) = \lambda$  and  $[u] \in F$ . Clearly,  $\varphi([u]) = q \in \varphi(F)$ . So,  $F' \subseteq \varphi(F)$ , and since  $\varphi(F) \subseteq F'$ ,  $\varphi(F) = F'$  which concludes the proof.

### 3.3 Time Analysis and Query Complexity

Let us now discuss the time complexity of the algorithm. While the cardinality of  $S$  is smaller than  $n$ , the algorithm searches for a string  $u$  in  $S$  and a symbol  $a$  in  $\Sigma$  such that  $\text{row}(ua)$  is distinct from all  $\text{row}(v)$  with  $v \in S$ . This can be done using at most  $|S|^2 \cdot |\Sigma| \cdot |E|$  operations: there are  $|S|$  possibilities for choosing  $u$  (and the same number for  $v$ ),  $|\Sigma|$  for choosing  $a$ , and  $|E|$  operations to compare  $\text{row}(ua)$  with  $\text{row}(v)$ . If we take  $|\Sigma| = l$ , the total running time of the “repeat-until” loop can be bounded by  $(1^2 + 2^2 + \dots + (n-1)^2) \cdot l \cdot (1 + l + l^2 + \dots + l^k)$ . Note that by “operations” we mean string comparisons, since they are generally acknowledged as being the most costly tasks.

On the other hand, to construct  $\mathcal{A}(S, E, C)$  we need  $n$  comparisons for determining the final states, and another  $n^2 \cdot |\Sigma| \cdot |E|$  operations for constructing the transition function. This means that the total running time of the algorithm is bounded by  $n + l \cdot \frac{l^{k+1}-1}{l-1} \cdot \frac{n(n+1)(2n+1)}{6}$ , that is  $O(n^3)$ .

As for the number of queries asked by the algorithm, it can be bounded by  $|S \cup S\Sigma| \cdot |E|$  (i.e., by the size of the final observation table), so the query complexity of the algorithm is  $O(n)$ .

## 4 Injective Regular Languages

Another example of languages efficiently learnable with CQs is the so-called class of *injective languages*. We say that a regular language  $L \subseteq \Sigma^*$  is *injective* if for all  $u_1, u_2 \in \Sigma^*$ ,  $u_1 \equiv_L u_2$  if and only if  $C_L(u_1) = C_L(u_2)$ . We present an algorithm which learns any injective language  $L$  after asking a finite number of CQs.

### 4.1 The Algorithm

Let  $\mathcal{Inj}$  be the class of all injective regular languages over the fixed alphabet  $\Sigma$ , and let  $L$  be the target language. We use again an observation table (here and there denoted by  $(S, C)$ ), with the only difference that the set  $E$  has only one element ( $E = \{\lambda\}$ ), so we do not have to mention it explicitly. Moreover, instead of the function  $\text{row}(u)$  we can just use its value in  $\lambda$ , namely  $C_L(u)$ .

The notions of closed and consistent observation table are defined accordingly: a table  $(S, C)$  is *closed* if for all  $u \in S$  and  $a \in \Sigma$ ,  $C_L(ua) \in C_L(S)$ , and it

is *consistent* if for all  $u_1, u_2 \in S$ ,  $C_L(u_1) \neq C_L(u_2)$ . When  $(S, C)$  is closed and consistent, we can construct the corresponding automaton  $\mathcal{A}(S, C)$  as in the previous section. The following algorithm correctly identifies  $L$  after asking a finite number of CQs.

---

**Algorithm 2** An algorithm for learning the class  $\mathcal{I}nj$  with CQs

---

```

1:  $S := \{\lambda\}$ 
2:  $closed := \text{TRUE}$ 
3: update the table by asking CQs for all strings in  $S \cup S\Sigma$ 
4: repeat
5:   if  $(\exists u \in S, a \in \Sigma \text{ such that } C_L(ua) \notin C_L(S))$  then
6:     add  $ua$  to  $S$ 
7:     update the table by asking CQs for all strings in  $\{uaa' \mid a' \in \Sigma\}$ 
8:      $closed := \text{FALSE}$ 
9:   end if
10: until  $closed$ 
11: output  $\mathcal{A}(S, C)$  and halt.
```

---

Notice that the algorithm is essentially the same as the one for learning  $k$ -reversible languages, and its correction is based on the fact that for all injective languages, the correcting string for  $u$  uniquely identifies the state  $\delta(q_0, u)$ , where  $\mathcal{A}_L = (Q, \Sigma, \delta, q_0, F)$ .

## 4.2 Time Analysis and Query Complexity

The worst case complexity of Algorithm 2 is again in  $O(n^3)$ , since the main loop requires at most  $(1^2 + 2^2 + \dots + (n-1)^2) \cdot l$  operations, and the automaton can be constructed using at most  $n + n^2l$  comparisons (that is, in total, at most  $n + \frac{n(n+1)(2n+1)}{6} \cdot l$  operations).

The number of CQs asked by the algorithm is bounded by  $n(1 + l)$ , which means that the query complexity is linear in the size of the minimal complete automaton accepting the target language.

## 4.3 On the Relation of $\mathcal{I}nj$ with $k\text{-Rev}$

Since for learning the class  $\mathcal{I}nj$  we used a simpler version of the Algorithm 1 for learning  $k\text{-Rev}$ , one may think that the class  $\mathcal{I}nj$  is included in  $k\text{-Rev}$ . We show that this is not the case.

**Proposition 5.** *For any  $k > 0$ , the classes  $k\text{-Rev}$  and  $\mathcal{I}nj$  are incomparable.*

*Proof.* We first show that for all  $k > 0$ , there exists an injective language which is not  $k$ -reversible. For this, consider the finite language  $L_k^1 = \{ab^k b, b^{k+2}, ab^k a\}$ . The automaton  $\mathcal{A}_{L_k^1}$  is represented in Figure 1.

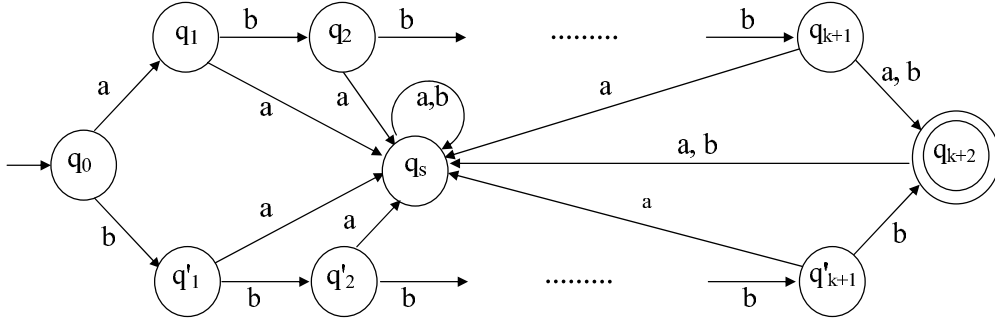


Figure 1: The minimal automaton for  $L_k^1$

We have:  $C_{L_k^1}(ab^i) = b^{k-i}a$  and  $C_{L_k^1}(bb^i) = b^{k-i}b$  for all  $i \in \{0, \dots, k\}$ ,  $C_{L_k^1}(\lambda) = ab^k a$ ,  $C_{L_k^1}(ab^k a) = C_{L_k^1}(ab^k b) = C_{L_k^1}(b^{k+2}) = \lambda$ , and  $C_{L_k^1}(u) = \theta$  for all other strings  $u$ . So, there are no two non-equivalent strings with the same correction. Thus,  $L_k^1$  is in  $\mathcal{Inj}$ .

On the other hand, if we assume that  $L_k^1$  is in  $k\text{-Rev}$ , then by Theorem 1, whenever  $u_1 v w, u_2 v w$  are in  $L_k^1$  and  $|v| = k$ ,  $\text{Tail}_{L_k^1}(u_1 v) = \text{Tail}_{L_k^1}(u_2 v)$ . But  $a \cdot b^k \cdot b, b \cdot b^k \cdot b$  are in  $L_k^1$  and  $|b^k| = k$ , so  $\text{Tail}_{L_k^1}(a \cdot b^k) = \text{Tail}_{L_k^1}(b \cdot b^k)$ , that is  $\{a, b\} = \{b\}$ , a contradiction.

Now let us take the language  $L_k^2 = \{a^{k+1}, ba^k, ab^{k+1}\}$  (for any  $k > 0$ ). The automaton  $\mathcal{A}_{L_k^2}$  is represented in Figure 2.

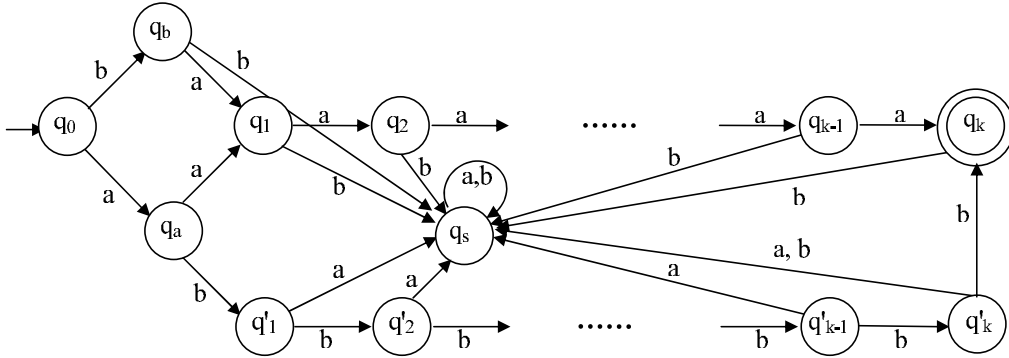


Figure 2: The minimal automaton for  $L_k^2$

The language  $L_k^2$  is not injective, since there exist two strings  $u_1, u_2$  such that  $C_{L_k^2}(u_1) = C_{L_k^2}(u_2)$  and  $u_1 \not\equiv_{L_k^2} u_2$  (if we take  $u_1 = a, u_2 = b$ , then  $\text{Tail}_{L_k^2}(a) = \{a^k, b^{k+1}\} \neq \{a^k\} = \text{Tail}_{L_k^2}(b)$ , so  $a \not\equiv_{L_k^2} b$  and  $C_{L_k^2}(a) = C_{L_k^2}(b) = a^k$ ).

Because only  $a^{k+1}$  and  $ba^k$  have a common suffix of length at least  $k$ , checking the  $k$ -reversibility of  $L_k^2$  reduces to showing that  $\text{Tail}_{L_k^2}(a^{k+1}) = \text{Tail}_{L_k^2}(ba^k)$  (which is obviously true, since both equal  $\{\lambda\}$ ).  $\square$

So, for any  $k > 0$ , the classes  $k\text{-Rev}$  and  $\mathcal{Inj}$  are incomparable. We next show that  $0\text{-Rev}$  is a proper subclass of  $\mathcal{Inj}$ .

**Proposition 6.** *The class  $0\text{-Rev}$  is strictly included in  $\mathcal{Inj}$ .*

*Proof.* Let  $L$  be a 0-reversible language, and assume that there exist  $u_1, u_2$  in  $\Sigma^*$  such that  $u_1 \not\equiv_L u_2$  and  $C_L(u_1) = C_L(u_2) = u$ . Clearly,  $u$  is in  $\Sigma^*$  (if  $u = \theta$  then  $\text{Tail}_L(u_1) = \text{Tail}_L(u_2) = \emptyset$  which contradicts  $u_1 \not\equiv_L u_2$ ). Because  $u_1u$  and  $u_2u$  are both in  $L$ , by Theorem 2,  $\text{Tail}_L(u_1) = \text{Tail}_L(u_2)$ . This implies that  $u_1 \equiv_L u_2$  which contradicts our assumption. So,  $L$  is injective.

Now let us consider the language finite  $L_0 = \{a^2, ab^2, b^3\}$  with the minimal automaton  $\mathcal{A}_{L_0}$  represented in Figure 3.

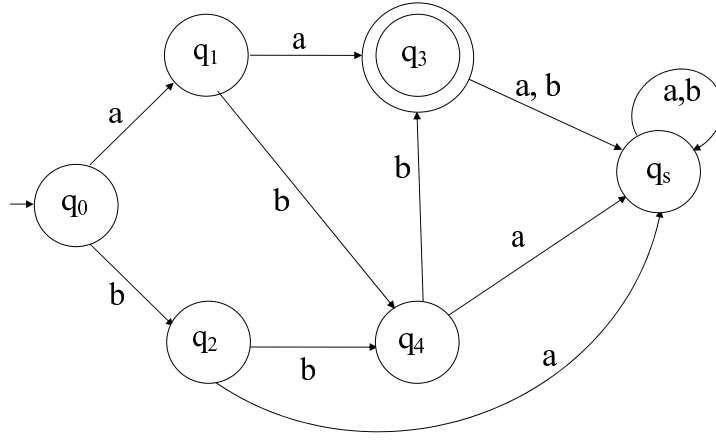


Figure 3: The minimal automaton for  $L_0$

The language  $L_0$  is injective since any two non-equivalent strings have different corrections ( $C_{L_0}(\lambda) = a^2, C_{L_0}(a) = a, C_{L_0}(b) = b^2, C_{L_0}(a^2) = \lambda, C_{L_0}(b^2) = b, C_{L_0}(ba) = \theta$ ), and not 0-reversible because  $\text{Tail}_{L_0}(a) \neq \text{Tail}_{L_0}(b)$  although  $a \cdot b^2$  and  $b \cdot b^2$  are both in  $L_0$ .  $\square$

## 5 Pattern Languages

Initially introduced by Angluin (in [Ang79]) to show that there are non-trivial classes of languages learnable from text, in the limit, the class of pattern languages has been intensively studied in the context of language learning ever since.

We assume a finite alphabet  $\Sigma$  such that  $|\Sigma| \geq 2$ , and a countable, infinite set of variables  $X = \{x, y, z, x_1, y_1, z_1, \dots\}$ . A *pattern*  $\pi$  is any non-empty string over  $\Sigma \cup X$ . The *pattern language*  $L(\pi)$  consists of all the words obtained by replacing the variables in  $\pi$  with arbitrary strings in  $\Sigma^+$ . Let us denote by  $\mathcal{P}$  the set of all pattern languages over a fixed alphabet  $\Sigma$ .

We say that the pattern  $\pi$  is in normal form if the variables occurring in  $\pi$  are precisely  $x_1, \dots, x_k$  and for every  $j$  with  $1 \leq j < k$ , the leftmost occurrence of  $x_j$  in  $\pi$  is left to the leftmost occurrence of  $x_{j+1}$ .

## 5.1 The Algorithm

Suppose that the target language is a pattern language  $L(\pi)$ , where  $\pi$  is in normal form. Then the following algorithm outputs the pattern  $\pi$  after asking a finite number of CQs.

---

**Algorithm 3** An algorithm for learning the class  $\mathcal{P}$  with CQs

---

```

1:  $w := C_L(\lambda), n := |w|$ 
2: for  $i := 1$  to  $n$  do
3:    $\pi[i] := null$ 
4: end for
5: for  $i := 1$  to  $n$  do
6:   if  $(\pi[i] = null)$  then
7:     choose  $a \in \Sigma \setminus \{w[i]\}$  arbitrarily
8:      $v := C_L(w[1 \dots i-1]a), m := |v|$ 
9:     if  $(|v| = |w[i+1, \dots, n]|)$  then
10:       $\pi[i] := x_i$ 
11:      for all  $j \in \{1, \dots, m\}$  for which  $v[j] \neq w[i+j]$  do
12:         $\pi[i+j] := x_i$ 
13:      end for
14:    else
15:       $\pi[i] := w[i]$ 
16:    end if
17:  end if
18: end for
19: output  $\pi$ 

```

---

## 5.2 Correctness and Termination

The correctness of the algorithm is based on the following observation. If  $w$  is the smallest string (in lex-length order) in  $L(\pi)$  and  $n = |w|$ , then for all  $i$  in  $\{1, \dots, n\}$  we have:

- if  $\pi[i]$  is a variable  $x$  such that  $i$  is the position of the leftmost occurrence of  $x$  in  $\pi$ , then  $|C_L(w[1, \dots, i-1]a)| = |w[i+1, \dots, n]|$  for any symbol  $a \in \Sigma$ ; moreover, we can detect the other occurrences of the variable  $x$  in  $\pi$  by just checking the positions where the strings  $C_L(w[1, \dots, i-1]a)$  and  $w[i+1, \dots, n]$  do not coincide, where  $a$  is any symbol in  $\Sigma \setminus \{w[i]\}$

- if  $\pi[i] = a$  for some  $a$  in  $\Sigma$ , then  $C_L(w[1, \dots, i-1]b)$  is either  $\theta$ , or longer than  $w[i+1, \dots, n]$  for all  $b \in \Sigma \setminus \{a\}$ .

Obviously, the algorithm terminates in finite steps.

### 5.3 Time Analysis and Query Complexity

For each symbol in the pattern, the algorithm makes at most  $n + 1$  comparisons, where  $n$  is the length of the pattern. This implies that the total running time of the algorithm is bounded by  $n(n + 1)$ , that is  $O(n^2)$ .

It is easy to see that the query complexity is linear in the length of the pattern, since the algorithm does not ask more than  $n + 1$  CQs.

## 6 Learning with CQs versus Learning with MQs

The notion of CQ appeared as an extension of the well-known and intensively studied MQ. The inspiration for introducing them comes from a real life setting (which is the case for MQs also): when children make mistakes, the adults do not reply by a simple 'yes' or 'no' (the agreement is actually implicit), but they also provide them with a corrected word. Clearly, CQs can be thought as some more informative MQs. So, it is only natural to compare the two learning settings (learning with CQs *vs.* learning with MQs), and to analyze their expressive power.

The first step in this direction has already been done: Tîrnăucă and Kobayashi showed in [TK07] that learning with CQs is strictly more powerful than learning with MQs, when we neglect the time complexity.

In this section we make a step further towards understanding the differences and similarities between these two learning models by taking into consideration the efficiency of the learning algorithms, that is, the time complexity. For this, we need further terminology.

Let  $\mathcal{C} = (L_i)_{i \geq 1}$  be an indexable class. We say that  $\mathcal{C}$  is *polynomially learnable with MQs* (or *with CQs*) if there exists a polynomial time algorithm which learns  $\mathcal{C}$  using MQs (CQs, respectively). We denote the collection of all indexable classes  $\mathcal{C}$  which are polynomially learnable with MQs by  $PolMemQ$  ( $PolCorQ$  is defined similarly).

Recall that if the correction for a given string  $u$  is  $\lambda$  then the string is in the language, and the oracle's answer would be 'yes'; in all other cases, the string is not in the language, and the answer would be 'no'. So, since the answer to any CQ gives us also the answer to the corresponding MQ, it follows immediately that the class  $PolMemQ$  is included in  $PolCorQ$ . We show that the inclusion is strict using pattern languages as the separating case.

**Theorem 7.** *The class  $\mathcal{P}$  is in  $PolCorQ \setminus PolMemQ$ .*

*Proof.* It is clear that  $\mathcal{P}$  is in  $PolCorQ$  since Algorithm 3 is a polynomial time algorithm which identifies any pattern language using COs (see Section 5).

Assume now that  $\mathcal{P}$  is in  $PolMemQ$ , and consider the class of singletons  $\mathcal{S}$  of fixed length  $n$  over the alphabet  $\Sigma$ . Because every language  $L = \{w\}$  in  $\mathcal{S}$  can be written as a pattern language ( $L = L(w)$ , where  $w$  is a pattern without any variables),  $\mathcal{S}$  is also in  $PolMemQ$ . But Angluin shows that, if  $k$  is the cardinality of the alphabet, then any algorithm which learns  $\mathcal{S}$  using MQs needs to ask at least  $k^n - 1$  MQs ([Ang81]), which leads to a contradiction.  $\square$

Note that although  $\mathcal{P}$  is not polynomially learnable with MQs, it is in  $MemQ$  (see [Muk92], page 266). However, there are classes of languages in  $PolCorQ$  which cannot be learned at all (polynomially or not) using MQs, as we will see in the sequel.

**Theorem 8.** *The class  $k\text{-Rev}$  is in  $PolCorQ \setminus MemQ$ .*

*Proof.* Since Algorithm 1 learns any  $k$ -reversible language using CQs in polynomial time (see Section 3), it follows immediately that  $k\text{-Rev}$  is in  $PolCorQ$ .

To show that  $k\text{-Rev}$  is not in  $MemQ$ , we use Mukouchi's characterization of the class  $MemQ$  in terms of pairs of definite finite tell-tales. A pair  $\langle T, F \rangle$  is said to be a *pair of definite finite tell-tales of  $L_i$*  if:

- (1)  $T_i$  is a finite subset of  $L_i$ ,  $F_i$  is a finite subset of  $\Sigma^* \setminus L_i$ , and
- (2) for all  $j \geq 1$ , if  $L_j$  is *consistent with the pair  $\langle T, F \rangle$*  (that is,  $T \subseteq L_j$  and  $F \subseteq \Sigma^* \setminus L_j$ ), then  $L_j = L_i$ .

Mukouchi proves in [Muk92] that an indexable class  $\mathcal{C} = (L_i)_{i \geq 1}$  belongs to  $MemQ$  if and only if a pair of definite finite tell-tales of  $L_i$  is uniformly computable for any index  $i$ .

So, let us assume that  $k\text{-Rev}$  is in  $MemQ$ . Consider the alphabet  $\Sigma$  such that  $\{a, b\} \subseteq \Sigma$ , and the language  $L = \{a\}$ . Clearly,  $L$  is in  $k\text{-Rev}$  for all  $k \geq 0$  and hence a pair of definite finite tell-tales  $\langle T, F \rangle$  is computable for  $L$ . This means that  $T \subseteq L$  and  $F$  is a finite set included in  $\Sigma^* \setminus \{a\}$ . Let us take  $m = \max\{|w| \mid w \in F\}$  and the language  $L' = \{a, ba^m b\}$ . It is clear that  $L'$  is in  $k\text{-Rev}$  for all  $k \geq 0$ , and that it is consistent with  $\langle T, F \rangle$ . Moreover,  $L' \neq L$ , which leads to a contradiction.  $\square$

Note that the same kind of proof can be used to show that  $\mathcal{I}nj$  is also in  $PolCorQ \setminus MemQ$ , and in this case the query complexity is actually linear. On the other hand, very simple classes of languages cannot be learned in polynomial time using CQs. For example, if we take  $\mathcal{S}$  to be  $\mathcal{S} = (L_w)_{w \in \Sigma^*}$ , where  $L_w = \Sigma^* \setminus \{w\}$ , then any algorithm would require at least  $1 + k + k^2 + \dots + k^n$  CQs in order to learn  $L_w$ , where  $n = |w|$  and  $k = |\Sigma|$ .

## 7 Concluding Remarks

We have investigated the learnability of some well-known language classes in the query learning setting. A synthesis of the results obtained is presented in Figure 4.

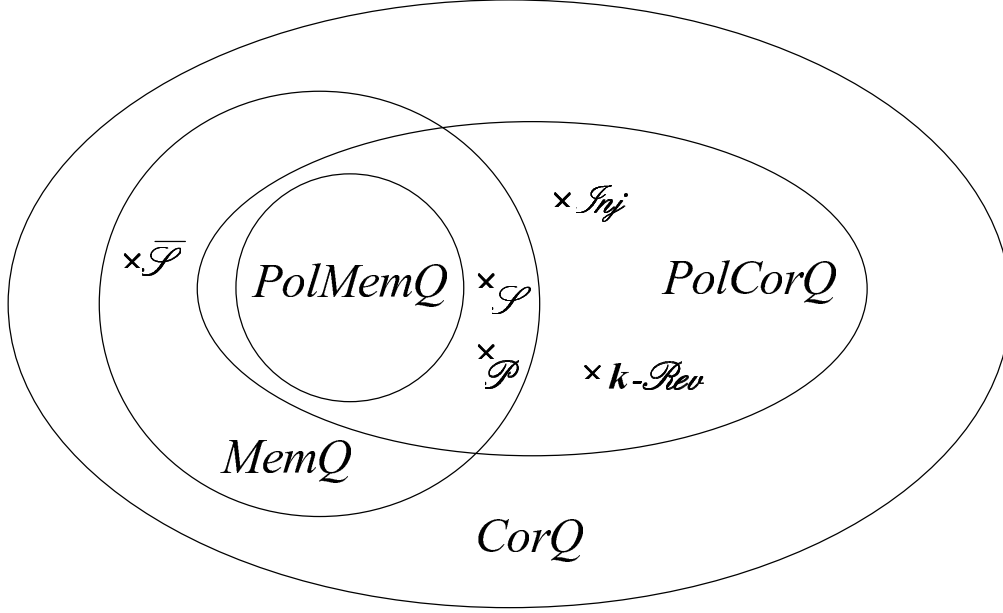


Figure 4: CQ learning vs MQ learning

The class of pattern languages was known to be learnable with MQs. We gave a polynomial time algorithm for learning  $\mathcal{P}$  using CQs, and showed that they cannot be efficiently learned with MQs. Moreover, we proved that  $k$ -reversible languages and injective languages are efficiently learnable with CQs, and not learnable (at all) with MQs.

For the future, we would like to see what happens with the learnability results obtained so far when we change the correcting string. A possible direction could be to choose as correction the closest string in the edit distance.

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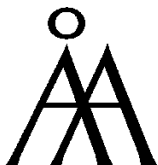
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