Validated evaluation of Special mathematical functions

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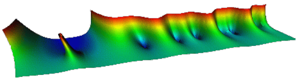
Motivation

Related to large-scale projects:

- `dlmf.nist.gov`
  “the NBS handbook rewritten with regard to the needs of today”



NIST Digital Library of Mathematical Functions

Project News

2012-03-23 *DLMF Update; Version 1.0.4*
2011-10-19 *Digital Library of Mathematical Functions Team Cited for IT Innovation*

- *More news*

Companion to the NIST Handbook of Mathematical Functions

dlmf.nist.gov

Motivation

Related to large-scale projects:

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- `functions.wolfram.com`

  "the world´s largest collection of formulas and graphics available on the web"


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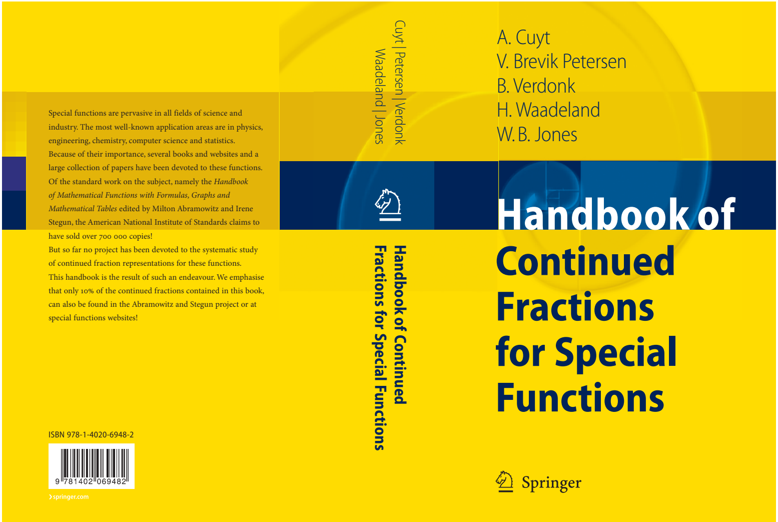
- Askey-Bateman project

  "will be an encyclopedia of special functions (Ismail / Van Assche)"


Special functions are pervasive in all fields of science and industry. The most well-known application areas are in physics, engineering, chemistry, computer science and statistics. Because of their importance, several books and websites and a large collection of papers have been devoted to these functions. Of the standard work on the subject, namely the *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* edited by Milton Abramowitz and Irene Stegun, the American National Institute of Standards claims to have sold over 700 000 copies!
But so far no project has been devoted to the systematic study of continued fraction representations for these functions.
This handbook is the result of such an endeavour. We emphasise that only 10% of the continued fractions contained in this book, can also be found in the Abramowitz and Stegun project or at special functions websites!
ISBN 978-1-4020-6948-2
9 781402 069482
springer.com
Cuyt | Petersen | Verdonk
Waadeland | Jones
Handbook of Continued
Fractions for Special Functions
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V. Brevik Petersen
B. Verdonk
H. Waadeland
W. B. Jones
Handbook of
Continued
Fractions
for Special
Functions
Springer

Project

Deliverables:

- Book: *formulas* with *tables* and *graphs*
- Maple library: all *series* developments and continued *fractions*
- C++ library: all *functions*
- Web version to explore: all of the above (`www.cfsf.ua.ac.be`)

A lot of well-known constants in mathematics, physics and engineering, as well as elementary and special functions enjoy very nice and rapidly converging series or continued fraction representations.

"Algorithms with strict bounds on truncation and rounding errors are not generally available for special functions " (Dan Lozier)

Only 15% of the CF representations in CFSF handbook are also found in the NBS handbook or at the Wolfram site!

special function: $f(z)$

continued fraction: $b_0(z) + \mathrm{K}_{m=1}^{\infty} \frac{a_m(z)}{b_m(z)}$

$$f(z) = b_0(z) + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \cfrac{a_3(z)}{b_3(z) + \cdots}}}$$

$$= b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots$$

- SF: limit-periodic representation
- CF: larger convergence domain

Example:

$$\mathrm{Atan}(z) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} z^{2m+1},$$

$$|z| < 1$$

$$= \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{(m-1)^2 z^2}{2m-1},$$

$$iz \notin (-\infty, -1) \cup (1, +\infty)$$

tail: $t_n(z) = \mathrm{K}_{m=n+1}^{\infty} \dfrac{a_m(z)}{b_m(z)}$

Example:

$$\frac{\sqrt{1+4x}-1}{2} = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{x}{1}, \qquad x \geqslant -\frac{1}{4}, \qquad t_n(x) = f(x)$$

$$\sqrt{2}-1 = \frac{1}{1} \underset{+}{} \frac{2}{1} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{2}{1} \underset{+}{} \dots, \qquad t_{2n} \to \sqrt{2}-1$$

$$t_{2n+1} \to \sqrt{2}$$

$$1 = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{m(m+2)}{1}, \qquad t_n \to \infty$$

approximant:

$$f_n(z;0) = b_0(z) + \mathop{\mathrm{K}}_{m=1}^{n} \frac{a_m(z)}{b_m(z)}$$

modified approximant:

$$f_n(z;w_n) = b_0(z) + \mathop{\mathrm{K}}_{m=1}^{n-1} \frac{a_m(z)}{b_m(z)} + \frac{a_n(z)}{b_n(z) + w_n}$$

Part I: Basic theory

- Basics
- Continued fraction representation of functions
- Convergence criteria
- Padé approximants
- Moment theory and orthogonal polynomials

Part II: Numerics

- Continued fraction construction
- Truncation error bounds
- Continued fraction evaluation

Part III: Special functions

- Elementary functions
- Incomplete gamma and related
- Error function and related
- Exponential integrals
- Hypergeometric ${}_2F_1(a, n; c; z)$, $n \in \mathbb{Z}$
- Legendre functions
- Confluent hypergeometric ${}_1F_1(n; b; z)$, $n \in \mathbb{Z}$
- Parabolic cylinder functions
- Coulomb wave functions
- Bessel functions of integer and fractional order
- Zeta function and related

Formulas

$$\mathrm{Ln}(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m z}{1}\right), \qquad |\operatorname{Arg}(1+z)| < \pi, \qquad (11.2.2)$$

$$a_{2k} = \frac{k}{2(2k-1)}, \quad a_{2k+1} = \frac{k}{2(2k+1)}$$

$$= \frac{2z}{2+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)}\right), \qquad (11.2.3)$$

$$|\operatorname{Arg}\left(1 - z^2/(2+z)^2\right)| < \pi$$

$$\mathrm{Ln}\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z^2}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi, \qquad (11.2.4)$$

Tables

TABLE 11.2.3: Relative error of 20^{th} partial sum and 20^{th} approximants.

x	$\text{Ln}(1+x)$	(11.2.1)	(11.2.2)	(11.2.3)	(6.8.8)
-0.9	$-2.302585e+00$	$1.5e-02$	$2.8e-06$	$5.8e-12$	$1.5e-06$
-0.4	$-5.108256e-01$	$2.5e-10$	$1.8e-18$	$2.2e-36$	$2.4e-19$
0.1	$9.531018e-02$	$4.4e-23$	$5.3e-33$	$1.9e-65$	$1.3e-34$
0.5	$4.054651e-01$	$1.8e-08$	$1.9e-20$	$2.3e-40$	$2.0e-21$
1.1	$7.419373e-01$	$2.4e-01$	$2.8e-15$	$5.3e-30$	$5.5e-16$
5	$1.791759e+00$	$1.0e+13$	$4.2e-08$	$1.3e-15$	$2.0e-08$
10	$2.397895e+00$	$1.8e+19$	$5.3e-06$	$2.1e-11$	$3.3e-06$
100	$4.615121e+00$	$1.0e+40$	$1.8e-02$	$3.6e-04$	$2.5e-02$

Modification:

since in (11.2.2), $\lim_{m\to\infty} a_m z = z/4$ and

$$\lim_{m\to\infty} \frac{a_{m+1} - \frac{1}{4}}{a_m - \frac{1}{4}} = -1,$$

we find

$$w(z) = \frac{-1 + \sqrt{1+z}}{2}$$

and

$$\begin{cases} w_{2k}^{(1)}(z) = w(z) + \frac{kz}{2(2k+1)} - \frac{z}{4} \\ w_{2k+1}^{(1)}(z) = w(z) + \frac{(k+1)z}{2(2k+1)} - \frac{z}{4} \end{cases}$$

Table 11.2.4: Relative error of 20^{th} (modified) approximants.

x	$\text{Ln}(1+x)$	(11.2.2)	(11.2.2)	(11.2.2)
-0.9	$-2.302585e{+}00$	$2.8e{-}06$	$1.9e{-}07$	$2.9e{-}10$
-0.4	$-5.108256e{-}01$	$1.8e{-}18$	$9.3e{-}20$	$4.3e{-}22$
0.1	$9.531018e{-}02$	$5.3e{-}33$	$2.7e{-}34$	$3.2e{-}37$
0.5	$4.054651e{-}01$	$1.9e{-}20$	$9.5e{-}22$	$5.5e{-}24$
1.1	$7.419373e{-}01$	$2.8e{-}15$	$1.5e{-}16$	$1.8e{-}18$
5	$1.791759e{+}00$	$4.2e{-}08$	$2.6e{-}09$	$1.1e{-}10$
10	$2.397895e{+}00$	$5.3e{-}06$	$3.7e{-}07$	$2.8e{-}08$
100	$4.615121e{+}00$	$1.8e{-}02$	$2.9e{-}03$	$9.2e{-}04$

Graphs

FIGURE 11.2.1: Complex region where $f_8(z; 0)$ of (11.2.2) guarantees k significant digits for $\mathrm{Ln}(1 + z)$ (from light to dark $k = 6, 7, 8, 9$).

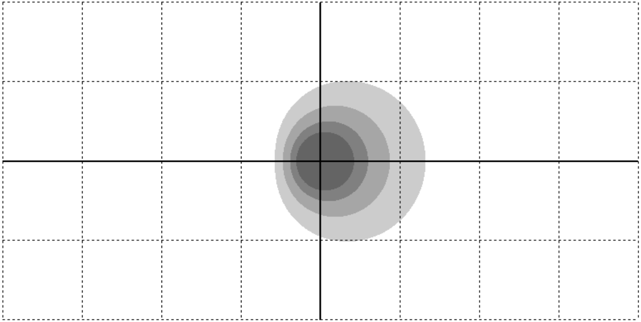
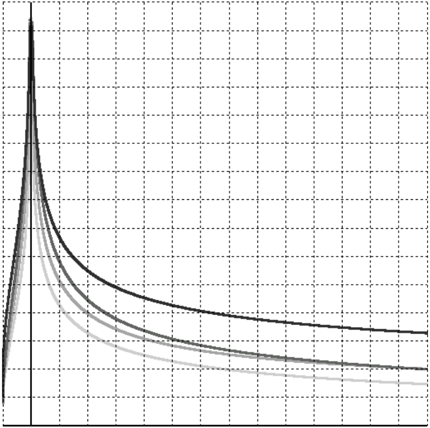


FIGURE 11.2.2: Number of significant digits guaranteed by the n^{th} classical approximant of (11.2.2) (from light to dark $n = 5, 6, 7$) and the 5^{th} modified approximant evaluated with $w_5^{(1)}(z)$ (darkest).



Web interface

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1}{2\sqrt{z}} \operatorname{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) $$

- series representation
- continued fraction representations:
  - C-fraction (15.3.7)
  - M-fraction (15.3.12)
  - Nörlund fraction (15.3.17)

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a,b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^- . \qquad (15.1.4) $$

$$ z\ {}_2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C} \setminus [1, +\infty) \qquad (15.3.7a) $$

with

$$ c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \ge 2. \qquad (15.3.7b) $$

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \ldots, \quad |z| < 1, \qquad (15.3.12) $$

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1}{1-z} + \frac{z(1-z)}{3/2 - 5/2 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{m(m-1/2)z(1-z)}{(m+1/2) - (2m+1/2)z}\right), \quad \Re z < 1/2. \qquad (15.3.17) $$

TABLE 15.3.1: Relative error of the 5th (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	1.9e−08	1.4e−05	6.1e−06
0.2	1.076022e+00	7.9e−07	4.4e−04	3.4e−04
0.3	1.123054e+00	8.0e−06	3.1e−03	4.9e−03
0.4	1.178736e+00	4.7e−05	1.2e−02	4.3e−02

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	1.9e−08	2.0e−10	1.6e−12
0.2	1.076022e+00	7.9e−07	8.7e−09	1.5e−10
0.3	1.123054e+00	8.0e−06	9.4e−08	2.7e−09
0.4	1.178736e+00	4.7e−05	5.9e−07	2.5e−08

TABLE 15.3.2: Relative error of the 20$^{\text{th}}$ (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	$1.035488e{+}00$	$4.0e{-}32$	$1.5e{-}20$	$1.5e{-}20$
0.2	$1.076022e{+}00$	$1.3e{-}25$	$1.5e{-}14$	$1.7e{-}13$
0.3	$1.123054e{+}00$	$1.4e{-}21$	$4.8e{-}11$	$7.6e{-}09$
0.4	$1.178736e{+}00$	$1.8e{-}18$	$1.4e{-}08$	$5.0e{-}05$

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	$1.035488e{+}00$	$4.0e{-}32$	$2.6e{-}35$	$6.5e{-}38$
0.2	$1.076022e{+}00$	$1.3e{-}25$	$8.8e{-}29$	$4.8e{-}31$
0.3	$1.123054e{+}00$	$1.4e{-}21$	$1.1e{-}24$	$9.6e{-}27$
0.4	$1.178736e{+}00$	$1.8e{-}18$	$1.4e{-}21$	$1.9e{-}23$

Special functions :
continued fraction and series representations

Handbook: A. Cuyt, S. Becuwe, V. Petersen, B. Verdonk, H. Waadeland, W. B. Jones

Software: F. Backeljauw, S. Becuwe, M. Colman, A. Cuyt, J. Van Deun

Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
- Exponential integrals and related functions
- Hypergeometric functions
  - 2F1(a,b;c;z)
  - 2F1(a,n;c;z)
  - 2F1(a,1;c;z)
  - 2F1(a,b;c;z) /
  - 2F1(a,b+1;c+1;z)
  - 2F1(a,b;c;z) /
  - 2F1(a+1,b+1;c+1;z)
  - 2F1(1/2,1;3/2;z)
  - 2F1(2a,1;a+1;1/2)
- Confluent hypergeometric functions
- Bessel functions

Hypergeometric functions » 2F1(1/2,1;3/2;z) » tabulate

Representation

☑	(HY.1.4)	${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a,b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-$
☑	(HY.3.7)	$z\,{}_2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C} \setminus [1, +\infty)$ $c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \geq 2$
☑	(HY.3.7)	with modification (limit-periodic)
☑	(HY.3.7)	with improved modification (limit-periodic)
☑	(HY.3.12)	${}_2F_1(1/2, 1; 3/2; z) = \frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \dots, \qquad \lvert z \rvert < 1$
☑	(HY.3.12)	with modification (limit-periodic)
☑	(HY.3.12)	with improved modification (limit-periodic)
☑	(HY.3.17)	${}_2F_1(1/2, 1; 3/2; z) = \frac{1}{1-z} + \frac{z(1-z)}{3/2 - 5/2 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{m(m-1/2)z(1-z)}{(m+1/2) - (2m+1/2)z}\right),$ $\Re z < 1/2$
☑	(HY.3.17)	with modification (limit-periodic)
☑	(HY.3.17)	with improved modification (limit-periodic)

Continue

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Input and computation

function parameters	none
base	10
digits	500 (fixed)
approximant	10 (1 ≤ approximant ≤ 999)
z	.1,.15,.2,.25,.3,.35,.4,.45
output	◉ relative error ○ absolute error

Output by Maple

10th approximant (relative error)

z	$_2F_1(1/2,1;3/2;z)$	(HY.1.4)	(HY.3.7)	(HY.3.7)	(HY.3.7)	(HY.3.12)	(HY.3.12)	(HY.3.12)	(HY.3.17)	(HY.3.17)	(HY.3.17)
0.1	1.035488e+00	4.62e-13	2.44e-16	6.39e-19	2.96e-21	1.47e-10	4.46e-13	8.45e-15	7.52e-11	3.88e-12	5.36e-14
0.15	1.055046e+00	4.14e-11	1.84e-14	4.96e-17	3.60e-19	8.31e-09	2.79e-11	8.36e-13	7.55e-09	4.01e-10	8.61e-12
0.2	1.076022e+00	1.01e-09	4.34e-13	1.20e-15	1.22e-17	1.44e-07	5.40e-10	2.28e-11	2.42e-07	1.34e-08	4.00e-10
0.25	1.098612e+00	1.23e-08	5.43e-12	1.55e-14	2.05e-16	1.31e-06	5.51e-09	3.09e-10	4.21e-06	2.47e-07	9.77e-09
0.3	1.123054e+00	9.48e-08	4.58e-11	1.36e-13	2.26e-15	7.90e-06	3.76e-08	2.69e-09	5.10e-05	3.28e-06	1.67e-07
0.35	1.149640e+00	5.39e-07	2.97e-10	9.12e-13	1.87e-14	3.58e-05	1.95e-07	1.74e-08	4.89e-04	3.62e-05	2.40e-06
0.4	1.178736e+00	2.45e-06	1.59e-09	5.09e-12	1.26e-13	1.32e-04	8.27e-07	9.09e-08	4.05e-03	3.79e-04	3.42e-05
0.45	1.210806e+00	9.42e-06	7.45e-09	2.49e-11	7.37e-13	4.14e-04	3.02e-06	4.05e-07	3.06e-02	4.59e-03	6.69e-04

Input and computation

function parameters	none
base	10
digits	500 (fixed)
approximant	5,10,15,20,25 (1 ≤ approximant ≤ 999)
z	.25
output	relative error / absolute error

Output by Maple

z = 1/4 (relative error)

n	${}_2F_1(1/2,1;3/2;z)$	(HY.1.4)	(HY.3.7)	(HY.3.7)	(HY.3.7)	(HY.3.12)	(HY.3.12)	(HY.3.12)	(HY.3.17)	(HY.3.17)	(HY.3.17)
5	1.098612e+00	2.19e-05	2.77e-06	3.15e-08	7.26e-10	1.29e-03	2.09e-05	2.04e-06	1.41e-03	1.73e-04	1.24e-05
10	1.098612e+00	1.23e-08	5.43e-12	1.55e-14	2.05e-16	1.31e-06	5.51e-09	3.09e-10	4.21e-06	2.47e-07	9.77e-09
15	1.098612e+00	8.41e-12	1.04e-17	1.33e-20	1.24e-22	1.30e-09	2.46e-12	9.70e-14	1.43e-08	5.51e-10	1.50e-11
20	1.098612e+00	6.33e-15	2.00e-23	1.44e-26	1.03e-28	1.27e-12	1.37e-15	4.17e-17	5.11e-11	1.47e-12	3.05e-14
25	1.098612e+00	5.02e-18	3.83e-29	1.77e-32	1.02e-34	1.25e-15	8.67e-19	2.14e-20	1.89e-13	4.32e-15	7.24e-17

Maple library

symbolic:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

$$f(z) = b_0(z) + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \cfrac{a_3(z)}{b_3(z) + \cdots}}}$$

Web interface

Special functions : continued fraction and series representations

Handbook: A. Cuyt, S. Becuwe, V. Petersen, B. Verdonk, H. Waadeland, W. B. Jones

Software: F. Backeljauw, S. Becuwe, M. Colman, A. Cuyt, J. Van Deun

Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
  - erf
  - dawson
  - erfc
  - Fresnel C
  - Fresnel S
- Exponential integrals and related functions
- Hypergeometric functions
- Confluent hypergeometric functions
- Bessel functions

Error function and related integrals » erfc » approximate

Representation

○	(ER.2.9)	$\operatorname{erfc}(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\frac{k}{2}+1)}, \quad z \in \mathbb{C}$
○	(ER.2.11)	$\sqrt{\pi} z e^{z^2} \operatorname{erfc}(z) \approx {}_2F_0\left(1, 1/2; -z^{-2}\right), \quad z \to \infty, \quad \lvert \arg z \rvert < 3\pi/4$
○	(ER.2.20)	$\operatorname{erfc}(z) = \frac{z}{\sqrt{\pi}} e^{-z^2} \left(\frac{a_1}{z^2} + \frac{a_2}{1} + \frac{a_3}{z^2} + \frac{a_4}{1} + \dots \right), \quad \Re z > 0,$ $a_1 = 1, \quad a_m = \frac{m-1}{2}, \quad m \geq 2$
◉	(ER.2.23)	$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \quad \Re z > 0$

Continue

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Special functions :
continued fraction and series representations

Handbook
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H. Waadeland
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Software
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S. Becuwe
M. Colman
A. Cuyt
J. Van Deun

Function categories

Error function and related integrals » erfc » approximate

Representation

◉ (ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	◉ none ○ standard ○ improved ○ user defined (simregular form)

Continue

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Function categories

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathbf{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	◉ none ○ standard ○ improved ○ user defined (simregular form)

Output by Maple

13th approximant

z	(ER.2.23a)	rel. error	abs. error	info
6.5	3.84214832712064746987580452585280852276470121e-20	4.66e-27	1.79e-46	i

tail: 0

Continue

Special functions : continued fraction and series representations

Handbook
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Software
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Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
  - erf
  - dawson
  - erfc
  - Fresnel C
  - Fresnel S
- Exponential integrals and related functions
- Hypergeometric functions
- Confluent hypergeometric functions
- Bessel functions

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \quad \Re z > 0$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none, standard (selected), improved; user defined (simregular form)

Continue

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Function categories

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	○ none ● standard ○ improved ○ user defined (simregular form)

Output by Maple

13th approximant

z	(ER.2.23a)	rel. error	abs. error	info
6.5	3.84214832712064746987580500914386200233212568e-20	1.21e-25	4.65e-45	

tail: -6.675e+01

Continue

$$\left|\frac{f(6.5) - f_n(6.5; w_n)}{f(6.5)}\right| \leqslant 1 \times 10^{-39}$$

- $n = 24$ and $w_n = 0$
- $n = 19$ and $w_n = -9.2861$
- $n = 14$ and $w_n = -5.5909501809$

C++ library

numeric:

$$\tilde{f}(x) = \pm d_0.d_1 \ldots d_{p-1} \times \beta^{e}$$

$$u(p) = \frac{1}{2}\beta^{-p+1}, \qquad \left|\frac{f(x) - \tilde{f}(x)}{f(x)}\right| \leqslant 2\kappa\, u(p), \qquad f(x) > 0$$

$$\Downarrow$$

$$\frac{\tilde{f}(x)}{1 + 2\kappa u(p)} \leqslant f(x) \leqslant \frac{\tilde{f}(x)}{1 - 2\kappa u(p)}$$

$$f(x) \approx F(x) \approx \mathtt{F}(x) = \tilde{f}(x)$$

truncation error:

$$f(x) \approx F(x)$$

$$\frac{|f(x) - F(x)|}{|f(x)|} \leqslant ?$$

round-off error:

$$\begin{array}{ccc} F(x) & \longrightarrow & \mathtt{F}(x) \\ +, -, \times, \div & & \oplus, \ominus, \otimes, \oslash \end{array}$$

$$\frac{|F(x) - \mathtt{F}(x)|}{|f(x)|} \leqslant ?$$

usually: $x \to y_x$, $x \in \mathbb{F}(\beta, p, L, U)$

- accumulate errors
- distribute thresholds

$$f(x) = f_1(x) * \cdots * f_k(x), \qquad * \in \{\times, \div, +, -\}$$

$$\tilde{f}(x) = \bigcirc_p(\tilde{f}_1(x) \circledast \cdots \circledast \tilde{f}_k(x)), \qquad \circledast \text{ in } \mathbb{F}(\beta, \hat{p}, L, U)$$

$$\tilde{f} = f(1 + \eta_k), \qquad |\eta_k| \leqslant 2\kappa u(p)$$

Example:

$$I_k(x) = I_0(x) \frac{I_1(x)}{I_0(x)} \cdots \frac{I_k(x)}{I_{k-1}(x)}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

take $* \in \{\times, \div\}$:

1. $\tilde{f}_i = f_i(1+\varepsilon_i), \quad i = 1, \ldots, k$
2. $\tilde{f}_i \circledast \tilde{f}_{i+1} = (\tilde{f}_i * \tilde{f}_{i+1})(1+\delta_i), \quad |\delta_i| \leqslant u(\hat{p})$
3. $\tilde{f} = \bigcirc_p(\tilde{f}_1 \circledast \cdots \circledast \tilde{f}_k) = (\tilde{f}_1 \circledast \cdots \circledast \tilde{f}_k)(1+\delta_k),$
   $|\delta_k| \leqslant u(p)$
4. $\left| \prod_{i=1}^{k} (1+\varepsilon_i)^{\sigma_i} \prod_{i=1}^{k} (1+\delta_i)^{\rho_i} \right| \leqslant 1 + 2\kappa u(p),$
   $\sigma_i, \rho_i = \pm 1$

accumulate errors:

$$|\delta_i| \leqslant \nu_i \delta, \qquad 1 \leqslant i \leqslant k, \qquad \sum_{i=1}^{k} \nu_i = 1$$

then accumulated error of the form

$$\left| \frac{\prod_{i=1}^{\ell} (1+\delta_i)}{\prod_{i=\ell+1}^{k} (1+\delta_i)} \right|$$

is bounded by

$$1 + \frac{\delta}{1-\delta}$$

distribute thresholds:

$$1 + \varepsilon_0 := (1 + \delta_k) \prod_{i=1}^{k-1} (1 + \delta_i)^{\rho_i}$$

relative error of the form

$$\left| \frac{\prod_{i=0}^{h} (1 + \varepsilon_i)}{\prod_{i=h+1}^{k} (1 + \varepsilon_i)} \right|$$

is bounded by $1 + \varepsilon$ if for $0 \leqslant i \leqslant k$

$$|\varepsilon_i| \leqslant \frac{\mu_i \varepsilon}{1 + \varepsilon}, \qquad \sum_{i=0}^{k} \mu_i = 1$$

Example:

$$\mathrm{erf}(x) = f_1(x) \times f_2(x) = \frac{2}{\sqrt{\pi}} \times \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)m!}$$

$$\widetilde{\mathrm{erf}}(x) = \bigcirc_p(\tilde{f}_1(x) \otimes \tilde{f}_2(x)),$$

$$\kappa = 1, \quad n(2n+1) \in \mathbb{F}(2, 53, L, U)$$

x	p	n	$\hat{p}$
0.125	125	15	136
	250	27	262
	500	49	512
0.250	125	17	136
	250	33	262
	500	59	513
0.375	125	21	136
	250	37	262
	500	67	513
0.500	125	23	136
	250	41	262
	500	73	513

x	p	n	$\hat{p}$
0.625	125	25	136
	250	45	262
	500	79	513
0.750	125	29	136
	250	49	262
	500	85	513
0.875	125	31	137
	250	53	262
	500	91	513
1.000	125	33	137
	250	55	262
	500	95	513

Table: For $\beta = 2$, given p and x, the n-th partial sum of $f_2(x)$ evaluated in precision $\hat{p}$, guarantees a total relative error of at most $2u(p)$ for $\left|\mathrm{erf}(x) - \widetilde{\mathrm{erf}}(x)\right| / |\mathrm{erf}(x)|$.

Example:

$$\operatorname{erfc}(x) = \frac{f_1(x)}{f_2(x)} \times f_3(x)$$

$$= \frac{e^{-x^2}}{\sqrt{\pi}} \times \left(\frac{2x/(2x^2+1)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{a_m(x)}{1} \right),$$

$$a_m(x) = \frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)},$$

$$4n - 3 \in \mathbb{F}(2, 53, L, U)$$

x	p	n	$\hat{p}$	w_n
1.750	125	80	135	−4.046410294629230e−01
	250	465	262	−4.597581472171540e−01
	500	2164	513	−4.812548482513949e−01
2.500	125	42	135	−3.183992613381111e−01
	250	234	261	−4.194909433431120e−01
	500	1074	512	−4.620433411164643e−01
3.250	125	28	134	−2.304003280854811e−01
	250	145	260	−3.691109207053306e−01
	500	649	511	−4.367537272669681e−01
4.000	125	21	134	−1.550574657036899e−01
	250	101	260	−3.124674318348620e−01
	500	441	510	−4.062340065351776e−01
4.750	125	17	134	−9.975138806260882e−02
	250	77	259	−2.557918680360318e−01
	500	323	510	−3.714271494689381e−01
5.500	125	14	134	−6.066417723702091e−02
	250	62	259	−2.027698945552964e−01
	500	251	510	−3.340681553755215e−01
6.250	125	12	134	−3.676750550585265e−02
	250	52	259	−1.566149475490524e−01
	500	203	510	−2.953698289199146e−01
7.000	125	11	134	−2.388988109112390e−02
	250	45	259	−1.187574015007353e−01
	500	169	509	−2.566479739506696e−01

Table: For $\beta = 2$, given p and x, the n-th approximant of $f_3(x)$ evaluated in precision $\hat{p}$ with double precision tail estimate w_n, guarantees a total relative error of at most $2u(p)$ for $\left|\mathrm{erfc}(x) - \widetilde{\mathrm{erfc}}(x)\right| / |\mathrm{erfc}(x)|$.

Example:

$$J_{5/2}(x) = J_{1/2}(x) \frac{J_{3/2}(x)}{J_{1/2}(x)} \frac{J_{5/2}(x)}{J_{3/2}(x)}$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = \frac{x/(2\nu+2)}{1} + \operatorname*{K}_{m=2}^{\infty} \frac{(ix)^2/(4(\nu+m-1)(\nu+m))}{1},$$

$$x \in \mathbb{R}, \quad \nu \geqslant 0$$

$$a_m(x) \nearrow 0$$

special function	series	continued fraction
$\gamma(a,x)$		$a > 0, x \neq 0$
$\Gamma(a,x)$		$a \in \mathbb{R}, x \geqslant 0$
$\mathrm{erf}(x)$	$\lvert x\rvert \leqslant 1$	identity via $\mathrm{erfc}(x)$
$\mathrm{erfc}(x)$	identity via $\mathrm{erf}(x)$	$\lvert x\rvert > 1$
$\mathrm{dawson}(x)$	$\lvert x\rvert \leqslant 1$	$\lvert x\rvert > 1$
Fresnel $S(x)$	$x \in \mathbb{R}$	
Fresnel $C(x)$	$x \in \mathbb{R}$	
$E_n(x),\ n > 0$		$n \in \mathbb{N}, x > 0$
${}_2F_1(a,n;c;x)$		$a \in \mathbb{R}, n \in \mathbb{Z}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-, x < 1$
${}_1F_1(n;c;x)$		$n \in \mathbb{Z}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-, x \in \mathbb{R}$
$I_n(x)$	$n = 0, x \in \mathbb{R}$	$n \in \mathbb{N}, x \in \mathbb{R}$
$J_n(x)$	$n = 0, x \in \mathbb{R}$	$n \in \mathbb{N}, x \in \mathbb{R}$
$I_{n+1/2}(x)$	$n = 0, x \in \mathbb{R}$	$n \in \mathbb{N}, x \in \mathbb{R}$
$J_{n+1/2}(x)$	$n = 0, x \in \mathbb{R}$	$n \in \mathbb{N}, x \in \mathbb{R}$

Web interface

Special functions : continued fraction and series representations

Handbook
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Function categories

- **Elementary functions**
- **Gamma function and related functions**
- **Error function and related integrals**
  - **erf**
  - **dawson**
  - **erfc**
  - **Fresnel C**
  - **Fresnel S**
- **Exponential integrals and related functions**
- **Hypergeometric functions**
- **Confluent hypergeometric functions**
- **Bessel functions**

Error function and related integrals » erfc » evaluate

Representation

$$\mathrm{erfc}(0) = 1, \quad \mathrm{erfc}(\infty) = 0, \quad \mathrm{erfc}(-\infty) = 2,$$
$$\mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x), \quad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x), \qquad 0 \leqslant x \leqslant 1$$
$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{2x/(2x^2+1)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1} \right), \qquad x > 1$$

Input and computation

function parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
x	6.5
verbose	yes ◉ no ○

Continue

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Validated evaluation of Special mathematical functions

Project

Book
- Part I
- Part II
- Part III
- Web

Maple
- Web

C++
- Web

Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
  - erf
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  - Fresnel S
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- Bessel functions

Error function and related integrals » erfc » evaluate

Representation

$$\operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0, \quad \operatorname{erfc}(-\infty) = 2,$$
$$\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x), \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \qquad 0 \leqslant x \leqslant 1$$
$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{2x/(2x^2+1)}{1} + \mathop{\mathbf{K}}_{m=2}^{\infty} \frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1} \right), \qquad x > 1$$

Input and computation

function parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
x	6.5
verbose	yes ◉ no ○

Output by MpIeee

x	value	rel. error	abs. error	info
6.5	3.842148327120647469875804543768776621449e-20	7.41e-41	2.85e-60	i

```
prec> CF: 45
approx> CF: 13
tail> CF: -3.691114343068676e-02
```

Continue

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Special functions : continued fraction and series representations

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Error function and related integrals » erfc » evaluate

Representation

$$\operatorname{erfc}(0)=1,\quad \operatorname{erfc}(\infty)=0,\quad \operatorname{erfc}(-\infty)=2,$$
$$\operatorname{erfc}(-x)=2-\operatorname{erfc}(x),\quad \operatorname{erfc}(x)=1-\operatorname{erf}(x)$$

$$\operatorname{erfc}(x)=1-\operatorname{erf}(x),\qquad 0\leqslant x\leqslant 1$$
$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{2x/(2x^2+1)}{1}+\mathop{\mathrm{K}}_{m=2}^{\infty}\frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1}\right),\qquad x>1$$

Input and computation

function parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
x	6.5
verbose	yes ○ no ◉

Continue

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Function categories

Elementary functions
Gamma function and related functions
Error function and related integrals
- **erf**
- **dawson**
- **erfc**
- **Fresnel C**
- **Fresnel S**

Exponential integrals and related functions
Hypergeometric functions
Confluent hypergeometric functions
Bessel functions

Error function and related integrals » erfc » evaluate

Representation

$$\operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0, \quad \operatorname{erfc}(-\infty) = 2,$$
$$\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x), \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \qquad 0 \leqslant x \leqslant 1$$
$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{2x/(2x^2+1)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1} \right), \qquad x > 1$$

Input and computation

function parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
x	6.5
verbose	yes ○ no ◉

Output by MpIeee

x	value	rel. error	abs. error
6.5	3.84214832712064746987580454376877662144929±04 e-20	1.38e-42	5.30e-62

Continue

Special functions : continued fraction and series representations

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Function categories

- Elementary functions
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- Error function and related integrals
  - erf
  - dawson
  - erfc
  - Fresnel C
  - Fresnel S
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- Bessel functions

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	○ none ○ standard ○ improved ◉ user defined (simregular form) −3.691114343068676e−02

Continue

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Function categories

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2+} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input and computation

function parameters	none
base	10
digits	45 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	○ none ○ standard ○ improved ◉ user defined (simregular form) −3.691114343068676e−02

Output by Maple

13th approximant

z	(ER.2.23a)	rel. error	abs. error	info
6.5	3.84214832712064746987580454376877662144928108e−20	9.42e−43	3.62e−62	i

tail: −3.691114343068676e−02

Continue

Function categories

Elementary functions
Gamma function and related functions
Error function and related integrals
Exponential integrals and related functions
Hypergeometric functions
Confluent hypergeometric functions
Bessel functions
- J(nu, z)
- J(n, z)
- J(nu+1, z) / J(nu, z)
- j(n, z)
- j(n+1, z) / j(n, z)
- I(nu, z)
- I(n, z)
- I(nu+1, z) / I(nu, z)
- i(n, z)
- i(n+1, z) / i(n, z)

Bessel functions » I(n, z) » evaluate

Representation

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{x}{2}\right)^{2k}, \qquad x \in \mathbb{R}$$

$$\frac{I_{n+1}(x)}{I_n(x)} = \frac{x/(2n+2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{a_m(n)x^2}{1}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

$$a_m(n) = \frac{1}{4(n+m-1)(n+m)}$$

$$I_n(x) = I_0(x) \prod_{k=1}^{n} \frac{I_k(x)}{I_{k-1}(x)}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

Input and computation

function parameters	n
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	50 (5 ≤ digits ≤ 999)
x	4.5
n	4
verbose	yes ○ no ◉

Continue

Function categories

Bessel functions » I(n, z) » evaluate

Representation

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!k!}\left(\frac{x}{2}\right)^{2k}, \qquad x \in \mathbb{R}$$

$$\frac{I_{n+1}(x)}{I_n(x)} = \frac{x/(2n+2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{a_m(n)x^2}{1}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

$$a_m(n) = \frac{1}{4(n+m-1)(n+m)}$$

$$I_n(x) = I_0(x) \prod_{k=1}^{n} \frac{I_k(x)}{I_{k-1}(x)}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

Input and computation

function parameters	n
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	50 (5 ≤ digits ≤ 999)
x	4.5
n	4
verbose	yes ○ no ◉

Output by MpIeee

x	value	rel. error	abs. error
4.5	2.7347222766930378559470450618354311463035025612656 54±03	1.10e-53	3.01e-53

Continue

Special functions :
continued fraction and series representations

Handbook
A. Cuyt
S. Becuwe
V. Petersen
B. Verdonk
H. Waadeland
W. B. Jones

Software
F. Backeljauw
S. Becuwe
M. Colman
A. Cuyt
J. Van Deun

Function categories

Bessel functions » I(n, z) » evaluate

Representation

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Continue

contact@cfhblive.ua.ac.be

Validated evaluation of Special mathematical functions

Project

Book
- Part I
- Part II
- Part III
- Web

Maple
- Web

C++
- Web

Function categories

- **Elementary functions**
- **Gamma function and related functions**
- **Error function and related integrals**
- **Exponential integrals and related functions**
- **Hypergeometric functions**
- **Confluent hypergeometric functions**
- **Bessel functions**
  - J(nu, z)
  - J(n, z)
  - J(nu+1, z) / J(nu, z)
  - j(n, z)
  - j(n+1, z) / j(n, z)
  - I(nu, z)
  - I(n, z)
  - I(nu+1, z) / I(nu, z)
  - i(n, z)
  - i(n+1, z) / i(n, z)

Bessel functions » I(n, z) » evaluate

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digits	50 (5 ≤ digits ≤ 999)
x	4.5
n	4
verbose	yes ◉ no ○

Output by MpIeee

x	value	rel. error	abs. error	info
4.5	2.7347222766930378559470450618354311463035025612657	1.68e-50	4.60e-50	i

Continue

```
prec> SE: 56; CF: 57, 57, 56, 57;
approx> SE: 33; CF: 25, 24, 23, 23;
tail> CF: 7.733062944939854e-03, 7.733062944939854e-03,
7.733062944939854e-03, 7.163861708630549e-03;
```

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