Remarks on the paper: “Bounds for functions involving ratios of modified Bessel functions”

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Abstract

A recent paper by C.G. Kokologiannaki [2] gives some properties for ratios of Bessel functions and, in particular, some bounds. These bounds are said to improve the range of some inequalities in [1] or to be sharper. Unfortunately, Kokologiannaki made some mistakes in the comparison and no improvement or extension is made over the results in [1].

Keywords: Special functions, Bessel function ratios, Bounds
2000 MSC: 33C10

The paper [2] makes a series of comparisons with reference [1], claiming that some improvements are found. Our aim is to explain the errors in these comments and to show that the bounds given in [2] are already contained in [1].

These errors are found in three different places:

1. The first error is probably a citation error, and the less important from a mathematical point of view. In the introduction of [2] we read the following: *“In the literature many inequalities and monotonicity properties for the functions $I_\nu(x)$ and for functions involving them, have been proved by many authors (see the review paper [2,3,12])”*. Reference [12] in the previous quotation corresponds to [1], which is not a review paper (as the quotation seems to suggest). In the same paragraph, it is not clear which is the meaning of the sentence “All of these are given in Baricz’s paper”. Reference [1] is certainly not contained in Baricz’s paper [3].

2. Reference [1] is mentioned twice in Remark 2.1 of [2], and both mentions are incorrect. First, C.G. Kokologiannaki says that the lower bound in [2, Eq. (2.1)], namely

$$f_\nu(x) = \frac{1}{x}\frac{I_{\nu+1}(x)}{I_\nu(x)} > -\frac{\nu+1}{x^2} + \sqrt{\frac{(\nu+1)^2}{x^2} + \frac{1}{x^2}}, \nu > -1, \qquad (1)$$

extends the range of validity of the bound in [1, Eq. (51)] and [4, Eq. (1.12)], which reads

$$r_\nu(x) = \frac{1}{x}\frac{I_\nu(x)}{I_{\nu-1}(x)} > (\nu + \sqrt{\nu^2 + x^2})^{-1},\ \nu \geq 0. \qquad (2)$$

But this is not true; the bound is the same and also the range of validity (see the different definitions of $f_\nu(x)$ and $r_\nu(x)$). In fact, in Eq. (1) the range can be written as $\nu \geq -1$ (as (2) shows). The bound (2) first appeared in [4], and with the same range of validity.
In Remark 2.1 it is also said that the iterative upper bounds given in [1] have a more limited range of validity than the upper bound in [2, Eq. (2.1)] which is not true by the same reason as before.

3. Reference [1] is also cited in Remark 2.5, where it is said that the bound in [2, Eq. (2.20)] given by

$$g_\nu(x) = \frac{I'_\nu(x)}{I_\nu(x)} > -\frac{1}{x} + \frac{\sqrt{x^2 + (\nu+1)^2}}{x}, \nu > -1 \qquad (3)$$

improves the bound in [1, Eq. (48)], which is true. But this improvement already appears in [1, Eq. (72)], namely (see also definition in [1, Eq. (65)]):

$$C(I_\nu(x)) = x\frac{I'_\nu(x)}{I_\nu(x)} > \sqrt{(\nu+1)^2 + x^2} - 1, \nu \geq -1. \qquad (4)$$

We notice that further improvements are also discussed in [1].

References

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