Inequalities and bounds for some cumulative distribution functions

Javier Segura

Departamento de Matemáticas, Estadística y Computación.
Universidad de Cantabria, Spain

Cumulative distribution functions

$$F(x) = \int_{\alpha}^{x} f(t)dt,\ f(t) > 0,\ F(\beta) = 1.$$

$F(x)$: cumulative distribution function (CDF).
$f(t)$: probability density function (PDF).

The CDF gives the probability that a random variable X with PDF f will be found to have a value less than or equal to x

Given $0 < p < 1$, the inverse of $F(x) = p$ with respect to x is also an important funtion in statistics (quantile function).

An example of application is random number generation:

If U is a random uniform variable in $[0, 1]$ then $X = F^{-1}(U)$ is a random variable distributed according to the probability density function $f(t)$

A simple and well studied example is the gaussian distribution

$$F(x) = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

where erf is the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt$$

$erf(x)$ is a "nearly elementary" function for which efficient methods of computation and inversion exist.

Many other CDFs can be expressed, in some limit, in terms of error functions (at least asymptotically).

The error function is a particular case of the central gamma distribution.
$P_{1/2}(y) = \mathrm{erf}(\sqrt{y})$.

Cumulative γ and β distributions

The cumulative central gamma distribution is given by

$$P_\mu(y) = \frac{1}{\Gamma(\mu)} \int_0^y t^{\mu-1} e^{-t} dt, \; y \geq 0.$$

Observe that y plays the role of the variable previously denoted as x

The cumulative noncentral γ distribution can be defined as

$$P_\mu(x, y) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} P_\mu(y)$$

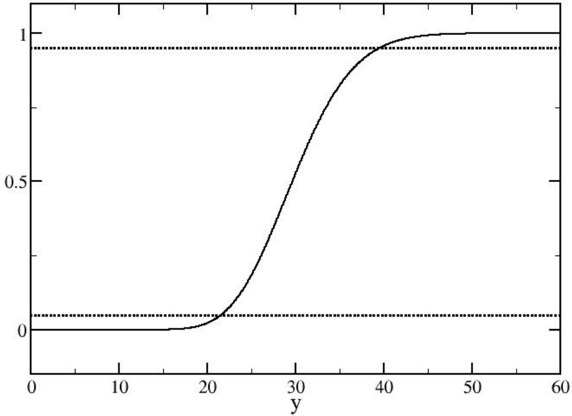
and it can also be written as

$$P_\mu(x, y) = x^{\frac{1}{2}(1-\mu)} \int_0^y t^{\frac{1}{2}(\mu-1)} e^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right).$$

The complementary distribution (also called upper tail distribution) is: $Q_\mu(x, y) = 1 - P_\mu(x, y)$.

$P_\mu(x, y)$ and $Q_\mu(x, y)$ are also called Marcum functions.

Graph of $P_\mu(y) = \frac{1}{\Gamma(\mu)} \int_0^y t^{\mu-1} e^{-t} dt, \ \mu = 30$



The cumulative central beta distribution is defined as

$$B_{a,b}(y) = \frac{1}{B(a,b)} \int_0^y t^{a-1}(1-t)^{b-1} dt,\; B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\; y \in [0,1]$$

(the standard notation in the NIST Handbook is $I_x(a,b)$)

The cumulative noncentral beta distribution can be defined as

$$B_{a,b}(x,y) = e^{-x/2} \sum_{n=0}^{\infty} \frac{(x/2)^n}{n!} B_{a,b}(y)$$

and it can also be written as

$$B_{a,b}(x,y) = \frac{e^{-x/2}}{B(a,b)} \int_0^y t^{a-1}(1-t)^{b-1} M\left(a+b, a, \frac{xt}{2}\right) dt.$$

These functions are hard to compute or invert, particularly the noncentral distributions.

Recent references on the computation and inversion of gamma distributions:

◦ Algorithm 939: Computation of the Marcum Q-function. A. Gil, JS, N. M. Temme. ACM Trans. Math. Softw. (2014)

◦ The asymptotic and numerical inversion of the Marcum Q-function. A. Gil, JS, N.M. Temme. Stud. Appl. Math.(2014)

◦ GammaCHI: a package for the inversion and computation of the gamma and chi-square cumulative distribution functions (central and noncentral). A. Gil, JS, N.M. Temme. Comput. Phys. Commun. (2015)

Less information and methods are available for the noncentral beta distribution. All methods of computation appear to be based on the application of the definition in terms of central distributions:

A note on the Noncentral Beta Distribution Function, R. Chattamvelly. Amer. Stat. 49 (1995) 231–234.

Information (sharp bounds) in terms of simpler functions is always useful, both for the direct computation and the inversion.

Bounds and L'Hôpital's rule

Theorem

Let

$$G_i(x) = \int_a^x g_i(t)dt,\ G_i(b) = 1,\ i = 1, 2,$$

$g_1(x)$ and $g_2(x)$ integrable in $[a, b]$ and continuous in (a, b), and $g_1(x) \neq 0$ in (a, b).

If $g_2(x)/g_1(x)$ is strictly monotonic in (a, b) then:

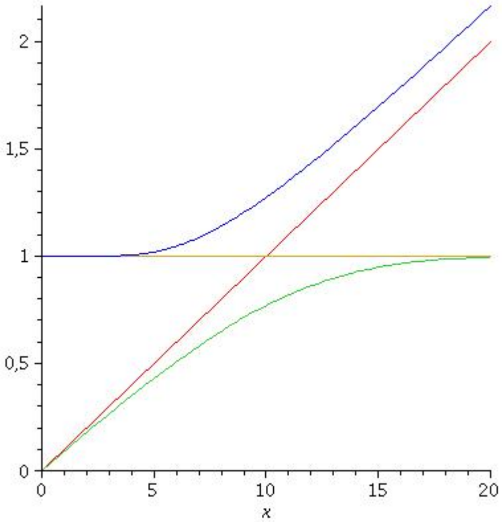
1. *There exists one and only one x_0 in (a, b) such that $g_1(x_0) = g_2(x_0)$*
2. *The following statements are equivalent in (a, b):*

   a) $G_2(x)/G_1(x) < 1$ *($G_2(x)/G_1(x) > 1$)*
   b) $g_2(x)/g_1(x)$ *is increasing (decreasing).*
   c) $G_2(x)/G_1(x) < g_2(x)/g_1(x)$ *($G_2(x)/G_1(x) > g_2(x)/g_1(x)$)*
   d) $G_2(x)/G_1(x)$ *is increasing (decreasing)*

   The same holds for $\bar{G}_2(x)/\bar{G}_1(x) = (1 - G_2(x))/(1 - G_1(x))$ but reversing the inequalities in (a) and (c).

The implication $b) \Rightarrow d)$ is called L'Hôpital's monotone rule [Anderson, Vamanamurthy, Vuorinen (1993)]

Representation of: $\frac{G_2(x)}{G_1(x)} = \frac{P_{11}(x)}{P_{10}(x)}$, $\frac{\bar{G}_2(x)}{\bar{G}_1(x)} = \frac{1-P_{11}(x)}{1-P_{10}(x)}$, $\frac{g_2(x)}{g_1(x)} = \frac{x}{10}$ and 1



The proof is elementary and is a consequence of Rolle's theorem and Cauchy's mean value theorem.

For instance, the implication b)$\Rightarrow$ c) goes as follows.

We are assuming that $g_2(x)/g_1(x)$ is strictly increasing (b)). We apply Cauchy's mean value theorem in $[a, x]$, $x \leq b$; then, there exists $c \in (a, x)$ such that

$$\frac{G_2(x)}{G_1(x)} = \frac{g_2(c)}{g_1(c)} < \frac{g_2(x)}{g_1(x)}$$

Ratios of integrals can be bounded in terms of monotonic ratios of integrands.

And bounds for one of the integrals can be obtained when:

a. one of the integrals is known.

b. there is some other relation.

If the difference $G_1(x) - G_2(x)$ is known (and assuming $G_2(x)/G_1(x) < 1$) we have:

$$G_1(x) < \frac{G_1(x) - G_2(x)}{1 - r(x)} \equiv U_1(x),\ G_2 < U_2(x) = r(x)U_1(x),\ x < x_0.$$

where $r(x) = g_2(x)/g_1(x)$ and x_0 is such that $r(x_0) = 1$. These two bounds tend to be sharper as $x \to a^+$.

In a similar way, for $\bar{G}_i(x) = 1 - G_i(x)$ we obtain

$$\bar{G}_1(x) < -U_1(x),\ \bar{G}_2(x) < -U_2(x),\ x > x_0,$$

and these two bounds tend to be sharper as $x \to b^-$.

Application to gamma distributions

Bounds using the recurrence

$$P_\mu(x,y) = \int_0^y g_\mu(x,t)dt,\ Q_\mu(x,y) = 1 - P_\mu(x,y)$$

with

$$g_\mu(x,y) = \left(\frac{y}{x}\right)^{(\mu-1)/2} e^{-x-y} I_{\mu-1}(2\sqrt{xy}).$$

Recurrences: $P_{\mu+1}(x,y) = P_\mu(x,y) - g_{\mu+1}(x,y),\ Q_{\mu+1}(x,y) = Q_\mu(x,y) + g_{\mu+1}(x,y)$

Remark:The variable y plays the role of the variable x in the main theorem

Taking $G_2 = P_{\mu+1}$ and $G_1 = P_\mu$ (same for Q), the difference $G_2 - G_1$ is known.

$g_{\mu+1}(x,y)/g_\mu(x,y)$ is increasing as a function of y and we have

(1) $$\frac{P_{\mu+1}(x,y)}{P_\mu(x,y)} < \frac{g_{\mu+1}(x,y)}{g_\mu(x,y)} = \sqrt{\frac{y}{x}}\frac{I_\mu(2\sqrt{xy})}{I_{\mu-1}(2\sqrt{xy})},\ \frac{Q_{\mu+1}(x,y)}{Q_\mu(x,y)} > \frac{g_{\mu+1}(x,y)}{g_\mu(x,y)}$$

From this, bounds for P_μ and Q_μ become available [JS, Appl. Math. Comput. 2014]

Bounds in terms of error functions

Now we take $G_2(y) = \int_0^y g_{\mu+\alpha}(x,t)dt$ and $G_1(y) = \int_0^y g_\mu(x,t)dt$, $\alpha \in \mathbb{R}^+$.

$$r(y) = \frac{g_{\mu+\alpha}(x,y)}{g_\mu(x,y)} = C(a,x)(\sqrt{y})^\alpha \frac{I_{\mu-1+\alpha}(2\sqrt{xy})}{I_{\mu-1}(2\sqrt{xy})}.$$

with $C(a,x)$ not depending on y.
$r(y)$ is strictly monotonic (increasing) as a function of y for $\mu \geq 1$:

Lemma

The function $f(t) = t^\alpha \frac{I_{\mu+\alpha}(t)}{I_\mu(t)}$ is increasing as a function of t for $\alpha, \mu > 0$.

Proof:

$$f'(t) = f(t)\left(\frac{I_{\mu+\alpha-1}(t)}{I_{\mu+\alpha}(t)} - \frac{I_{\mu-1}(t)}{I_\mu(t)}\right).$$

and $I_{\nu-1}(t)/I_\nu(t)$ is increasing as a function of $\nu \geq 0$ [JS, AMC. 2014: Lemma 2] □

We conjecture that $I_{\nu-1}(t)/I_\nu(t)$ is increasing as a function of $\nu \geq -1/2$

As a consequence of the monotonicity of $r(y)$ we have that if $\nu > \mu \geq 1/2$ then

$$(2) \qquad P_\nu(x,y) < \left(\frac{y}{x}\right)^{\frac{\nu-\mu}{2}} \frac{I_{\nu-1}(2\sqrt{xy})}{I_{\mu-1}(2\sqrt{xy})} P_\mu(x,y)$$

and

$$(3) \qquad Q_\nu(x,y) > \left(\frac{y}{x}\right)^{\frac{\nu-\mu}{2}} \frac{I_{\nu-1}(2\sqrt{xy})}{I_{\mu-1}(2\sqrt{xy})} Q_\mu(x,y)$$

Additionally, the trivial bounds $P_\nu(x,y) < P_\mu(x,y)$ and $Q_\nu(x,y) > Q_\mu(x,y)$ hold. Using that $P_{1/2}(x,y) = \frac{1}{2}\left(\mathrm{erf}(\sqrt{y}+\sqrt{x}) + \mathrm{erf}(\sqrt{y}-\sqrt{x})\right)$ bounds in terms of a Bessel function and two error functions are obtained.

In [A. Baricz, JMAA, 2009], similar bounds are given, for instance:

$$\tilde{Q}_\nu(a,b) \geq \frac{G_\nu(a,b)}{\sinh(ab)}\left(\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right) I_{\nu-1}(ab),\, b \geq a > 0\; \nu \geq 1,$$

where $\tilde{Q}_\nu(a,b) = Q_\nu(\frac{a^2}{2}, \frac{b^2}{2})$ and $G_\nu(a,b) = \dfrac{\sqrt{\pi}b^\nu}{2^{3/2}a^{\nu-1}}$.

Some of our bounds:

From the bound for Q_ν in the previous theorem we have, taking $\mu = 1/2$:

(4)
$$\tilde{Q}_\nu(a,b) > B_\nu^{[1]}(a,b)$$
$$B_\nu^{[1]}(a,b) = \frac{G_\nu(a,b)}{\cosh(ab)}\left(\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right) I_{\nu-1}(ab),$$

and from the bound for P_ν:

(5)
$$\tilde{Q}_\nu(a,b) > B_\nu^{[2]}(a,b)$$
$$B_\nu^{[2]}(a,b) = 1 - \frac{G_\nu(a,b)}{\cosh(ab)}\left(\operatorname{erf}\left(\frac{b-a}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{b+a}{\sqrt{2}}\right)\right) I_{\nu-1}(ab)$$

These bounds are valid for $\nu > 1/2$, $a, b > 0$.

And taking $\mu = 3/2$:

(6)
$$\tilde{Q}_\nu(a,b) > \frac{aB_\nu^{[1]}(a,b)}{b\tanh(ab)} + \left(\frac{b}{a}\right)^{\nu-1} e^{-(a^2+b^2)/2} I_{\nu-1}(ab)$$

and

(7)
$$\tilde{Q}_\nu(a,b) > 1 - \frac{aB_\nu^{[2]}(a,b)}{b\tanh(ab)} + \left(\frac{b}{a}\right)^{\nu-1} e^{-(a^2+b^2)/2} I_{\nu-1}(ab)$$

valid for $\nu > 3/2$, $a, b > 0$.

Bounds in terms of $\gamma_\nu(y)$, $\Gamma_\nu(y)$

We take $G_2(y)=\int_0^y g_\nu(\rho^2 x,t)dt$ and $G_1(y)=\int_0^y g_\nu(x,t)dt$, with $\rho>0$, $\rho\neq 1$, then

$$r(y)=\frac{g_\nu(\rho^2 x,y)}{g_\nu(x,y)}=C_\nu(\rho,x)\frac{I_{\nu-1}(2\rho\sqrt{xy})}{I_{\nu-1}(2\sqrt{xy})}$$

$r(y)$ is decreasing if $\rho<1$ and $\mu\geq 0$. Then, for $\rho<1$:

(8)
$$\begin{array}{l} P_\nu(x,y)<\rho^{\mu-1}e^{-x(1-\rho^2)}\dfrac{I_{\nu-1}(2\sqrt{xy})}{I_{\nu-1}(2\rho\sqrt{xy})}P_\nu(\rho^2x,y),\\ Q_\nu(x,y)>\rho^{\mu-1}e^{-x(1-\rho^2)}\dfrac{I_{\nu-1}(2\sqrt{xy})}{I_{\nu-1}(2\rho\sqrt{xy})}Q_\nu(\rho^2x,y). \end{array}$$

In addition we have the trivial bounds: $P_\nu(x,y)<P_\nu(\rho^2x,y)$, $Q_\nu(x,y)>Q_\nu(\rho^2x,y)$. Taking the limit $\rho\to 0$:

(9)
$$\begin{array}{l} P_\nu(x,y)<e^{-x}\left(\sqrt{xy}\right)^{1-\nu}I_{\nu-1}(2\sqrt{xy})\gamma_\nu(y)\\ Q_\nu(x,y)>e^{-x}\left(\sqrt{xy}\right)^{1-\nu}I_{\nu-1}(2\sqrt{xy})\Gamma_\nu(y) \end{array}$$

The second bound implies: $I_{\nu-1}(2\sqrt{xy})\Gamma_\nu(y)<e^{x}\left(\sqrt{xy}\right)^{\nu-1},\ \nu\geq 0,\ x,y>0$

UNIVERSIDAD DE CANTABRIA

Application to beta distributions

The cumulative noncentral beta distribution can be defined as

$$B_{a,b}(x,y) = e^{-x/2} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{2}\right)^j B_{a,b}(0,y)$$

where $B_{a,b}(0,y)$ (denoted as $I_y(a,b)$ in the NIST Handbook) is the central beta distribution

$$B_{a,b}(0,y) = \frac{1}{B(a,b)} \int_0^y t^{a-1}(1-t)^{b-1}dt,\ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

A different notation in the literature is $I_x(a,b,\lambda)$, where $B_{a,b}(x,y) = I_y(a,b,x)$.
We have

$$B_{a,b}(x,y) = \int_0^y g_{a,b}(x,t)dt$$

with

$$g_{a,b}(x,y) = \frac{e^{-x/2}}{B(a,b)} y^{a-1}(1-y)^{b-1} M\left(a+b, a, \frac{xy}{2}\right)$$

With the recurrences for the central beta distribution and using the definition we obtain:

$$B_{a,b}(x,y) = B_{a+1,b}(x,y) + e^{-x/2} \frac{y^a(1-y)^b}{aB(a,b)} M(a+b, a+1, xy/2).$$

Bounds in terms of the incomplete beta function

We have

$$\frac{g_{a,b}(\rho x, y)}{g_{a,b}(x, y)} = C(a, b, x)\frac{M(a+b, a, \rho z)}{M(a+b, a, z)}, \ z = xy/2.$$

$q(z) = \dfrac{M(a+b, a, \rho z)}{M(a+b, a, z)}$ is decreasing if $\rho < 1$ because

$$q'(z) = \frac{\alpha}{z}\frac{M(\alpha, \beta, \rho z)}{M(\alpha, \beta, z)}\left[\frac{M(\alpha+1, \beta, \rho z)}{M(\alpha, \beta, \rho z)} - \frac{M(\alpha+1, \beta, z)}{M(\alpha, \beta, z)}\right].$$

and $\dfrac{M(\alpha+1, \beta, z)}{M(\alpha, \beta, z)}$ is increasing as a function of z for $\alpha, \beta > 0$. With this

(10) $$\frac{B_{a,b}(\rho x, y)}{B_{a,b}(x, y)} > \frac{g_{a,b}(\rho x, y)}{g_{a,b}(x, y)}$$

and

(11) $$B_{a,b}(x, y) < e^{x(\rho-1)/2}\frac{M(a+b, a, xy/2)}{M(a+b, a, \rho xy/2)}B_{a,b}(\rho x, y)$$

and in particular

(12) $$B_{a,b}(x, y) < e^{-x/2}M(a+b, a, xy/2)B_{a,b}(0, y)$$

Bounds using the recurrence

We have

$$r(y)=\frac{g_{a+1,b}(x,y)}{g_{a,b}(x,y)}=\frac{a+b}{a}y\frac{M(a+b+1,a+1,z)}{M(a+b,a,z)},\ z=xy/2$$

which is increasing as a function of y. Then, invoking the main theorem:

(13) $$\frac{B_{a+1,b}(x,y)}{B_{a,b}(x,y)}<\frac{g_{a+1,b}(x,y)}{g_{a,b}(x,y)}=\frac{a+b}{a}y\frac{M(a+b+1,a+1,z)}{M(a+b,a,z)},\ z=xy/2$$

And using $B_{a,b}(x,y)=B_{a+1,b}(x,y)+e^{-x/2}\dfrac{y^a(1-y)^b}{aB(a,b)}M(a+b,a+1,xy/2)$:

(14) $$B_{a,b}(x,y)<\frac{C_{a,b}(x,y)M(a+b,a+1,xy/2)}{a\left(1-\dfrac{g_{a+1,b}}{g_{a,b}}\right)}$$

and particularizing for the central case $x=0$ we have:

(15) $$B_{a,b}(0,y)<\frac{y^a(1-y)^b}{B(a,b)(a-(a+b)y)},\ y<\frac{a}{a+b}$$

Bounds using only the recurrence

We consider the recurrence again:

$$B_{a,b}(x,y) = B_{a+1,b}(x,y) + e^{-x/2}\frac{y^a(1-y)^b}{aB(a,b)}M(a+b,a+1,xy/2)$$

Applying N times the recurrence we have

$$B_{a,b}(x,y) = B_{a+N+1,b} + \frac{C_{a,b}(x,y)}{a}\sum_{j=0}^{N}\frac{(a+b)_j}{(a+1)_j}y^j M(a+b+j,a+j+1,xy/2)$$

which gives the bound

$$B_{a,b}(x,y) > \frac{C_{a,b}(x,y)}{a}\sum_{j=0}^{N}\frac{(a+b)_j}{(a+1)_j}y^j M(a+b+j,a+j+1,xy/2) \tag{16}$$

This gives a convergent sequence of bounds (because $B_{a,b}(x,y)$ is minimal solution of the recurrence as $a\to+\infty$).

Application: inversion of the central distribution

The bounds, when they are sharp, can be used as estimations for starting inversion. For instance, we have

$$\frac{y^a(1-y)^b}{aB(a,b)}\left(1+\frac{a+b}{a+1}y\right) < B_{a,b}(0,y) < \frac{y^a(1-y)^b}{aB(a,b)}\left(1-\frac{a+b}{a}y\right)^{-1}$$

And the bounds are sharp as $y \to 0^+$.

The upper and lower bounds can be used to estimate the solution of $B_{a,b}(0, y_\beta) = \beta$.

The solution of $\frac{{y_l}^a(1-y_l)^b}{aB(a,b)}\left(1-\frac{a+b}{a}y_l\right)^{-1} = \beta$ gives a lower bound.

The solution of $\frac{{y_u}^a(1-y_u)^b}{aB(a,b)}\left(1+\frac{a+b}{a+1}y_u\right) = \beta$ gives an upper bound.

The seeked value y_β is such that $y_\beta \in (y_l, y_u)$.

With this, either taking y_l or y_u as starting value, the Schwartzian-Newton method [JS, 2015, submitted] converges with certainty to y_β.

A short diversion from the main topic:

The Schwarzian-Newton method (SNM) for solving $f(x)=0$ is a fourth order method with good non-local convergence properties (JS, 2015, arXiv:1505.01983). It is defined by the following fixed-point iteration:

$$x_{n+1}=g(x_n),\ g(x)=x-\operatorname{arctan}\left(\frac{1}{2}\{f,x\},\frac{f}{f'-\dfrac{f''}{2f'}f}\right),$$

where

$$\operatorname{arctan}(\lambda,x)=\begin{cases}\dfrac{1}{\sqrt{\lambda}}\arctan(\sqrt{\lambda}x) & , \quad \lambda>0,\\ x & , \quad \lambda=0\\ \dfrac{1}{\sqrt{-\lambda}}\operatorname{arctanh}(\sqrt{-\lambda}x) & , \quad \lambda<0\end{cases}$$

and $\{f,x\}$ is the Schwarzian derivative of f with respect to x: $\{f,x\}=\dfrac{f'''}{f'}-\frac{3}{2}\left(\dfrac{f''}{f'}\right)^2$.

The method is exact for functions with constant Schwarzian derivative. The functions with constant Schwarzian derivative are

$$h(x)=\frac{\tan(\lambda,x)+A}{B\tan(\lambda,x)+C},$$

with $\{h,x\}=2\lambda$.

The SNM has a geometrical interpretation similar to Newton's method but in terms of osculatory functions different from straight lines, and with higher order of osculation.

END of the diversion

For computing y_l and y_u we can iterate $y_{n+1} = h(y_n)$, $y_0 = 0$, with
$h(y) = (\beta B(a,b)(a-(a+b)y)(1-y)^{-b})^{1/a}$ for y_l, and
$h(y) = \left(1+\dfrac{a+b}{a+1}y\right)^{-1/a}(a\beta B(a,b)(1-y)^{-b})^{1/a}$ for y_u

Some numerical examples

a	b	β	y_l	y_u	y_β
5	10	10^{-1}	0.174545	0.197809	0.185134
5	10	10^{-2}	0.100007	0.103405	0.101928
5	10	10^{-4}	0.0364679	0.0366690	0.0366313
5	10	10^{-6}	0.0140692	0.0140938	0.0140905
10	10	10^{-1}	0.343314	0.398764	0.357930
10	10	10^{-2}	0.250254	0.262148	0.253953
10	10	10^{-4}	0.143187	0.144979	0.143852
10	10	10^{-6}	0.0859576	0.0863646	0.0861431
15	10	10^{-1}	0.458509	0.541477	0.473588
15	10	10^{-2}	0.366118	0.385811	0.370493
15	10	10^{-4}	0.246761	0.250977	0.247770
15	10	10^{-6}	0.172704	0.174033	0.173064

For more information and additional techniques for the inversion of cumulative distribution functions, don't miss tomorrow's talk by Amparo Gil:

10:30-10:55 Computation and Inversion of Certain Cumulative Distribution Functions (session MS36: Numerical Methods for Special Functions, Lecture Room B)

For more information and additional techniques for the inversion of cumulative distribution functions, don't miss tomorrow's talk by Amparo Gil:

10:30-10:55 Computation and Inversion of Certain Cumulative Distribution Functions (session MS36: Numerical Methods for Special Functions, Lecture Room B)

THANK YOU FOR YOUR ATTENTION