Computation of asymptotic expansions of turning point problems via Cauchy's integral formula

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OPSFA14, 2017

Introduction

In this talk we are dealing with asymptotic expansions for large u for the solutions of second order differential equations

$$\frac{d^2 w}{dz^2} = (u^2 f(z) + g(z))w,$$

in two cases:

1. In regions where $f(z)$ does not have a zero **(type I)**,
2. Around a simple zero of $f(z)$ **(type II)**,

to which we add a third **intermediate case**.

The first type of expansions are also called **Liouville-Green** expansions and the second one uniform **Airy-type** expansions. We will see how to link these two separates cases through the intermediate case of **LG Airy-type** expansions.

We will show applications to the computation of Bessel functions and Laguerre polynomials in the complex plane.

We say that an expansion of the form

$$h(u) \sim \sum_{n=0}^{\infty} a_n u^{-n},$$

is an **asymptotic expansion** as $u \to \infty$, if

$$u^N \left(h(u) - \sum_{n=0}^{N-1} a_n u^{-n} \right), \quad N = 0, 1, 2, \ldots,$$

is a bounded function for large values of u, with limit a_N as $u \to \infty$, for any N. This can also be written as

$$h(u) = \sum_{n=0}^{N-1} a_n u^{-n} + \mathcal{O}\left(u^{-N}\right), \quad u \to \infty.$$

Liouville-Green expansions

We start with the expansions of type I (Liouville-Green) for solutions of

$$\frac{d^2w}{dz^2} = (u^2 f(z) + g(z))w$$

for large u, and z away from turning points (zeros of $f(z)$).

Change of variable: $\left(\frac{d\xi}{dz}\right)^2 = f(z)$. Change of function: $W = f^{1/4}w$.

$$d^2W/d\xi^2 = \left\{u^2 + \phi(\xi)\right\} W,\ \phi(\xi) = \frac{4f(z)f''(z) - 5f'^2(z)}{16f^3(z)} + \frac{g(z)}{f(z)}.$$

This gives formal solutions

$$W_{\pm} \sim \exp\{\pm u\xi\} \sum_{s=0}^{\infty} (\pm 1)^s A_s(\xi) u^{-s}.$$

with coefficients that can be determined recursively by

$$A_{s+1}(\xi) = -\frac{1}{2}A_s'(\xi) + \frac{1}{2}\int \phi(\xi) A_s(\xi) d\xi$$

Symbolic differentiation is needed. Nested integration occurs.

An alternative: expansion in exponential form.

Formal solutions:

$$W_{\pm} \sim \exp\left\{ \pm u\xi + \sum_{s=1} (\pm)^s \frac{E_s(\xi)}{u^s} \right\}$$

In these, the coefficients are given by

$$E_s(\xi) = \int F_s(\xi) d\xi \quad (s = 1, 2, 3, \cdots),$$

where

$$F_1(\xi) = \frac{1}{2}\phi(\xi), \quad F_2(\xi) = -\frac{1}{4}\phi'(\xi),$$

and

$$F_{s+1}(\xi) = -\frac{1}{2}F_s'(\xi) - \frac{1}{2}\sum_{j=1}^{s-1} F_j(\xi)F_{s-j}(\xi) \quad (s = 2, 3, \cdots).$$

See F. W. J. Olver, Asymptotics and Special Functions, Chap. 10, Ex 2.1.

Symbolic differentiation is needed. No nested integration occurs.

Bessel equation: $t^2y''(t)+ty'(t)+(\nu^2-t^2)y(t)=0$. $w(z)=\sqrt{z}y(\nu z)$ satisfies:

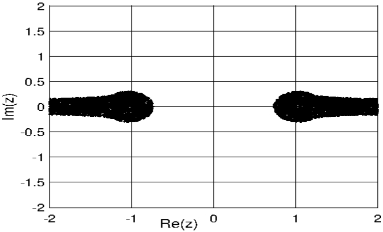
$$w''(z)=\left\{\nu^2 f(z)+g(z)\right\}w,\ f(z)=-1+z^{-2},\ g(z)=1/(4z^2)$$

Formal solutions:

$$W\sim \exp\left\{\pm\nu\xi+\sum_{s=1}^{\infty}(\pm 1)^s\frac{E_s(\xi)}{\nu^s}\right\},\ \xi(z)=\log\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}.$$

In particular, for the J-Bessel function:

$$J_\nu(\nu z)\sim\frac{1}{\sqrt{2\nu\pi}}(1-z^2)^{-1/4}\exp\left\{-\nu\xi+\sum_{s=1}^{\infty}(-1)^s\frac{E_s(\xi)}{\nu^s}\right\},\ E_s(+i\infty)=0$$



The points where the relative error is greater than 10^{-12} are shown (for $\nu=100$ and 14 terms)

Airy-type expansions around the turning points

$$d^2w/dz^2 = \left\{u^2 f(z) + g(z)\right\} w,$$

where $u \to \infty$ and $f(z)$ has a simple zero (turning point) at $z = z_0$.
For an approximation around the turning point, we set

$$\zeta \left(\frac{d\zeta}{dz}\right)^2 = f(z) \to \frac{2}{3}\zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t)dt = \xi, \quad W = \zeta^{-1/4} f^{1/4}(z) w$$

and we have

$$\boldsymbol{d^2W/d\zeta^2 = \left\{u^2\zeta + \psi(\zeta)\right\} W},\ \psi(\zeta) = \frac{5}{16\zeta^2} + \zeta\phi(\xi)$$

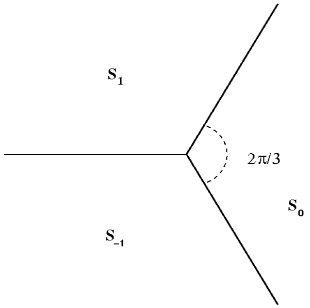
The natural approximants are **Airy functions**, solutions of $y''(x) = xy(x)$, and more specifically

$$\mathrm{Ai}_j(x) = \mathrm{Ai}(e^{-2\pi ij/3}x),\ j = 0, \pm 1,\ x = u^{2/3}\zeta$$

which are solutions of

$$\boldsymbol{d^2W/d\zeta^2 = u^2\zeta W}.$$

Airy functions: solutions of $y''(x) = xy(x)$. $\mathrm{Ai}_j(x) = \mathrm{Ai}(e^{-2\pi i j/3}x)$, $j = 0, \pm 1$, $x \in \mathbb{C}$
$\mathrm{Ai}_j(x)$ recessive as $x \to \infty$ in the sector S_j.



$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\chi} \sum_{s=0}^{\infty} (-1)^s c_s \chi^{-s}, \quad \chi = \tfrac{2}{3} x^{3/2}, \; |\arg x| \leq \pi - \delta$$

or alternatively

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-\chi + \sum_{i=1}^{\infty} (-1)^s \frac{a_s}{s\chi^s}\right), \; |\arg x| \leq \pi - \delta$$

where c_s and a_s are easily computable positive rational coefficients.

$$\frac{d^2W}{d\zeta^2} = u^2\zeta W \Rightarrow \mathrm{Ai}_j(u^{2/3}\zeta),\ j = 0, \pm 1$$

$$\frac{d^2W}{d\zeta^2} = \left\{u^2\zeta + \psi(\zeta)\right\} W \Rightarrow A(u,z)\mathrm{Ai}_j(u^{2/3}\zeta) + B(u,z)\mathrm{Ai}_j'(u^{2/3}\zeta)$$

Formal solutions

$$W_j(u,\zeta) \sim \mathrm{Ai}_j(u^{2/3}\zeta)\sum_{s=0}^{\infty} \frac{a_s(\zeta)}{u^{2s}} + \mathrm{Ai}_j'\left(u^{2/3}\zeta\right) u^{-4/3} \sum_{s=0}^{\infty} \frac{b_s(\zeta)}{u^{2s}},\ j = 0, \pm 1$$

The coefficients satisfy (see Olver's book, chap. 11):

$$b_s(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^{\zeta} \left\{\psi(t)a_s(t) - a_s''(t)\right\} \frac{dt}{t^{1/2}},$$

and

$$a_{s+1}(\zeta) = -\frac{1}{2} b_s'(\zeta) + \frac{1}{2} \int \psi(\zeta) b_s(\zeta) d\zeta.$$

Not easy! Instead of closed form expressions, normally one considers some approximations, like series around the turning point.

Alternative: compute directly $A(u,z)$ and $B(u,z)$ by some other means.

An iterative approach is considered in N. M. Temme, Numerical algorithms for uniform Airy-type asymptotic expansions, Numer. Algor. (1997).

The approach here is different. We start from

$$W_{-1}(u,z) = A(u,z)\mathrm{Ai}_{-1}(u^{2/3}\zeta) + B(u,z)\mathrm{Ai}'_{-1}(u^{2/3}\zeta),$$

$$W_{1}(u,z) = A(u,z)\mathrm{Ai}_{+1}(u^{2/3}\zeta) + B(u,z)\mathrm{Ai}'_{+1}(u^{2/3}\zeta).$$

Solve for the coefficients (using the Wronskian relation for Airy functions):

$$A(u,z) = -2\pi i \left\{ W_{-1}(u,z)\mathrm{Ai}'_{+1}(u^{2/3}\zeta) - W_{1}(u,z)\mathrm{Ai}'_{-1}(u^{2/3}\zeta) \right\},$$
$$B(u,z) = 2\pi i \left\{ W_{-1}(u,z)\mathrm{Ai}_{+1}(u^{2/3}\zeta) - W_{1}(u,z)\mathrm{Ai}_{-1}(\nu^{2/3}\zeta) \right\}.$$

We have $A(u,z)$ and $B(u,z)$. Problem: precisely we want to compute $W_j(u,z)$!!

But we know how to compute away from the turning point: use LG expansions.

Using LG asymptotics, we can write:

$$
\begin{aligned}
A(u,z) \sim \quad & \exp\left\{\sum_{j=1}^{\infty} \frac{E_{2j}(\xi) + \tilde{a}_{2j}\xi^{-2j}/(2j)}{u^{2j}}\right\} \\
& \times\cosh\left\{\sum_{j=0}^{\infty} \frac{E_{2j+1}(\xi) - \tilde{a}_{2j+1}\xi^{-2j-1}/(2j+1)}{u^{2j+1}}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
B(u,z) \sim \quad & \frac{1}{u^{1/3}\zeta^{1/2}} \exp\left\{\sum_{j=1}^{\infty} \frac{E_{2j}(\xi) + a_{2j}\xi^{-2j}/(2j)}{u^{2j}}\right\} \\
& \times\sinh\left\{\sum_{j=0}^{\infty} \frac{E_{2j+1}(\xi) - a_{2j+1}\xi^{-2j-1}/(2j+1)}{u^{2j+1}}\right\}
\end{aligned}
$$

The coefficients E_s and the constants a_s are those appearing before. The constants $\tilde{a}_s$ correspond to the LG asymptotics for the derivative of the Airy function.

Theorem

For the differential equation

$$d^2w/dz^2 = \left\{u^2 f(z) + g(z)\right\} w,$$

assume u *is positive and large,* $f(z)$ *has a simple zero at* $z = z_0$*, and* $f(z)$ *and* $g(z)$ *are analytic in a domain* D *containing* z_0*. Further assume that* $f(z)$ *does not vanish in the disk* $D(z_0, \rho) := \{z : 0 < |z - z_0| < \rho\} \subset D$*. Define variables* ξ *and* ζ *by*

$$\xi = \frac{2}{3}\zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t)dt,$$

and let $\mathrm{Ai}_j(u^{2/3}\zeta)$ $(j = 0, \pm 1)$ *denote the Airy functions* $\mathrm{Ai}(u^{2/3}\zeta e^{-2\pi ij/3})$*. Then there exist three numerically satisfactory solutions of the differential equation, given by*

$$w_j(u,z) = \zeta^{1/4} f^{-1/4}(z)\left\{\mathrm{Ai}_j(u^{2/3}\zeta)A(u,z) + \mathrm{Ai}_j'(u^{2/3}\zeta)B(u,z)\right\}.$$

In these, the coefficient functions $A(u,z)$ *and* $B(u,z)$ *are analytic at* $z = z_0$*, and possess the previous asymptotic expansions in a domain that includes* $D(z_0, \rho)$*. The integration constants for the odd coefficients must be selected so that* $\zeta^{1/2}E_{2j+1}(\xi)$ $(j = 0, 1, 2, \cdots)$ *is meromorphic as a function of* ζ *at* $\zeta = 0$*.*

We have LG expansions for the coefficients $A(u,z)$, $B(u,z)$, with which we can approximate

$$w_j(u,z) = \zeta^{1/4} f^{-1/4}(z) \left\{ A(u,z)\mathrm{Ai}_j(u^{2/3}\zeta) + B(u,z)\mathrm{Ai}'_j(u^{2/3}\zeta) \right\}.$$

But we can not compute close to the turning point with the LG expansions.

By now, we just have a **LG Airy-type** expansion (valid all around the TP, but not too close).

But...

But because the coefficients $A(u,z)$ and $B(u,z)$ are analytic in a domain containing the turning point we can consider the Cauchy integral formula

$$\boldsymbol{A(u,z)} = \frac{1}{2\pi i}\oint \frac{A(u,t)}{t-z}dt,\ \boldsymbol{B(u,z)} = \frac{1}{2\pi i}\oint \frac{B(u,t)}{t-z}dt,$$

with $A(u,t)$ and $B(u,t)$ approximated by LG-asymptotics along the Cauchy contour.

For the numerical integration the trapezoidal rule gives an exponential convergence rate.

Parametrization: $t(\theta) = z_c + Re^{i\theta}$, $\theta \in [0, 2\pi]$ (a circuit enclosing the turning point).

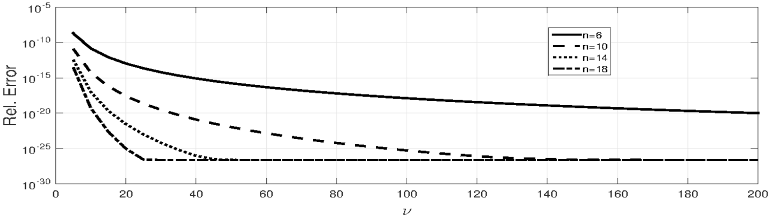
$$A(u,z) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)d\theta,\ F(\theta) = A(u,t(\theta))\frac{t(\theta)-z_c}{t(\theta)-z}$$

We write $\theta_j = 2\pi j/N$, $j = 0, \ldots N$ and then

$$A(u,z) \approx \frac{1}{N}\sum_{j=1}^{N} F(2\pi j/N)$$

Bessel functions

With the transformations and coefficients shown previously for the Bessel equation, the computation of the coefficients is straightforward.
We show an example of application of the Cauchy integral.



Error in the computation of $H_\nu^{(1)}(\nu z)$ for $z = 1 + 0.1i$ and various selections of the number of coefficients. The minimal possible error (close to 10^{-26}) is due to the discretization of the Cauchy integral with $N = 500$ points.

The trapezoidal rule has exponential convergence and the smallest reachable error roughly depends on the number of points over the Cauchy contour as $10^{-N/20}$.

Laguerre polynomials

$w = z^{(\alpha+1)/2} e^{-uz/2} L_n^{(\alpha)}(uz)$, $u = n + 1/2$ satisfies

$$\frac{d^2 w}{dz^2} = \left\{ u^2 f(z) + g(z) \right\},\ f(z) = \frac{(z - z_1)(z - z_2)}{4z^2},\ g(z) = -\frac{1}{4z^2}$$

where $\alpha = u(a^2 - 1)$ $(a \geq 0)$ and

$$z_1 = (a-1)^2,\ z_2 = (a+1)^2.$$

The TPs coalesce when $a = 0$ $(\alpha = -(n + 1/2))$. z_1 coalesces with $z = 0$ when $a = 1$ $(\alpha = 0)$.

We have considered two types of expansions:

1. **Expansions around z_1** with $1 + \delta \leq a \leq a_1 < \infty$. The domain contains $0 \leq z \leq z_2 - \delta$. The case of negative α (but z_1 no coalescing with $z = z_2$ or $z = 0$) has also been considered.
2. **Expansions around z_2** with $0 < a_0 \leq a \leq a_1 < \infty$: the TPs do not coalesce, but z_1 may coalesce with $z = 0$ (we therefore exclude it). Expansions hold in a complex domain including $z_1 + \delta \leq z < \infty$. This case can include negative α if $a_0 < 1$.

We give some results for the second case.

We compute expansions around z_2 for

$$\frac{d^2w}{dz^2} = \left\{u^2 f(z) + g(z)\right\},\ f(z) = \frac{(z-z_1)(z-z_2)}{4z^2},\ g(z) = -\frac{1}{4z^2}.$$

As before, we should consider:

1. The computation of the LG expansions (away from z_2). The transformation is the same, but with a different $f(z)$ (and therefore a different new variable). This provides the coefficients for the next step as well as LG expansions for $L_n^{(\alpha)}(z)$
2. The computation of the analytical (and slowly varying coefficients) $A(u,z)$ and $B(u,z)$ from the previous LG expansions. The expression for the coefficients is the same as before, but with different LG coefficients. This gives the LG Airy-type expansion away from the turning point.
3. The computation of the Airy-type expansion close to the turning point with Cauchy integrals.

We don't give all the details of the expansions.

The LG expansion (with matching at $+\infty$) is:

$$L_n^{(\alpha)}(uz) \sim \frac{(-1)^n u^{(u-1)/2}(u+\alpha)^{(u+\alpha)/2}}{e^u n!(euz)^{\alpha/2}\{(z-z_1)(z-z_2)\}^{1/4}} \exp\left\{\frac{1}{2}uz - u\xi + \sum_{s=1}^{\infty}(-1)^s \frac{\hat{E}_s(z)}{u^s}\right\}$$

with $\hat{E}_s(+\infty)=0$, and in terms of the corresponding new variables.

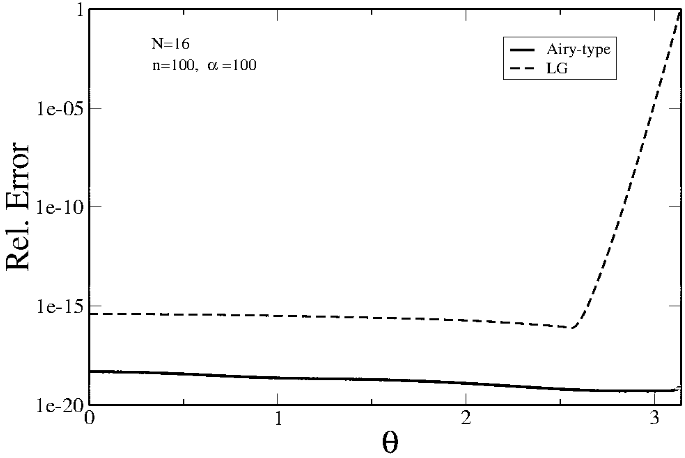
The LG Airy-type expansion away from the turning point

$$L_n^{(\alpha)}(uz) = H(u,z)\left\{\mathrm{Ai}(u^{2/3}\zeta)A(u,z) + \mathrm{Ai}'(u^{2/3}\zeta)B(u,z)\right\}$$

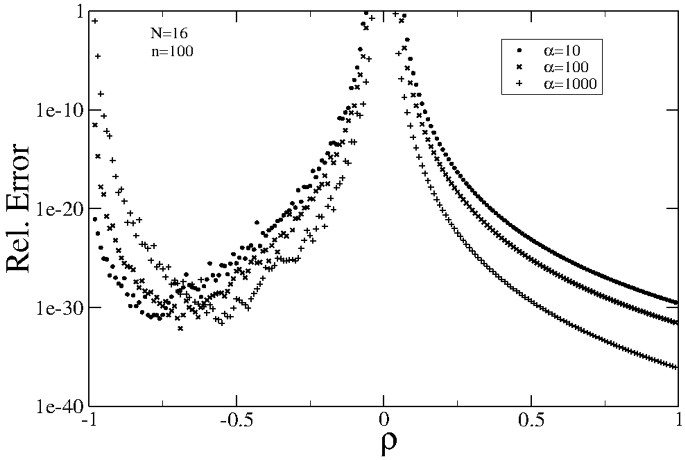
with $H(u,z)$ a known (lengthy but simple) function and $A(u,z)$ and $B(u,z)$ very similar to the Bessel case (but with different LG coefficients E_s).

And around the turning point Cauchy integrals are used.

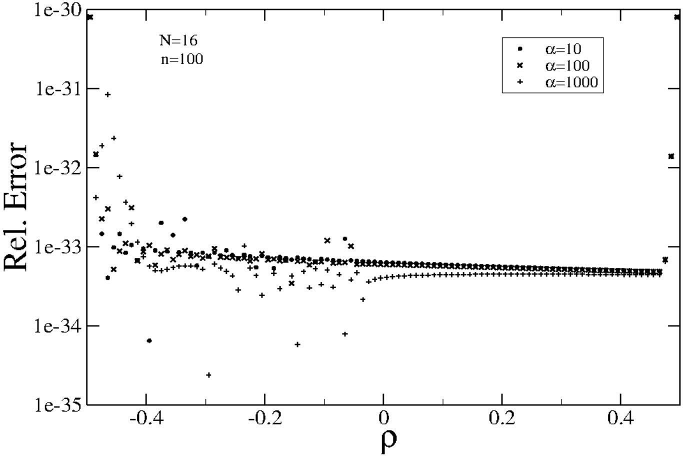
Let us illustrate the use of these expansions with some numerical results.



Relative errors ϵ for the LG expansion and the LG Airy-type expansion as a function of θ, where $z = z_2 + Re^{i\theta}$, $\theta \in [0, \pi]$ and $R = 0.25(z_2 - z_1)$.



Relative error of the LG Airy-type expansion for real values of z as a function of $\rho = (z - z_2)/(z_2 - z_1)$.



Relative error of the Airy-type expansion for real values of z as a function of $\rho = (z - z_2)/(z_2 - z_1)$, with the coefficients A and B computed by Cauchy integrals. The Cauchy contour is a circumference of radius $R = 0.7(z_2 - z_1)$ centered at z_2.

Conclusions:

- Methods for the computation of asymptotic approximations in the complex plane have been discussed.
- The methods relate two different types of asymptotic approximations (away or around turning points)
- Cauchy's integral formula for computing the coefficient functions provides this link.
- The new approach enlarges the range of validity of previous expansion (notably for Laguerre polynomials) and provides accurate approximations for complex variables near turning points.

For the future: sharp bounds, coalescing turning points, turning points and poles, ...

References

Computation of asymptotic expansions of turning point problems via Cauchy's integral formula: Bessel functions. T. M. Dunster, A. Gil, J. Segura. Constr. Approx. (2017). arXiv:1607.08269

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THANK YOU!