On bounds for solutions of monotonic first order difference-differential systems

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Abstract

Many special functions are solutions of first order linear systems $y_n'(x) = a_n(x)y_n(x) + d_n(x)y_{n-1}(x)$, $y_{n-1}'(x) = b_n(x)y_{n-1}(x) + e_n(x)y_n(x)$. We obtain bounds for the ratios $y_n(x)/y_{n-1}(x)$ and the logarithmic derivatives of $y_n(x)$ for solutions of monotonic systems satisfying certain initial conditions. For the case $d_n(x)e_n(x) > 0$, sequences of upper and lower bounds can be obtained by iterating the recurrence relation; for minimal solutions of the recurrence these are convergent sequences. The bounds are related to the Liouville-Green approximation for the associated second order ODEs as well as to the asymptotic behavior of the associated three-term recurrence relation as $n \to +\infty$; the bounds are sharp both as a function of n and x. Many special functions are amenable to this analysis, and we give several examples of application: modified Bessel functions, parabolic cylinder functions, Legendre functions of imaginary variable and Laguerre functions. New Turán-type inequalities are established from the function ratio bounds. Bounds for monotonic systems with $d_n(x)e_n(x) < 0$ are also given, in particular for Hermite and Laguerre polynomials of real positive variable; in that case the bounds can be used for bounding the monotonic region (and then the extreme zeros).

Keywords: Monotonic difference-differential systems, Riccati equation, Three-term recurrence relation, Special function bounds, Turán-type inequalities, zeros of orthogonal polynomials

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1 Introduction

Many special functions, and in particular functions of hypergeometric type, satisfy first order differential systems of the form

$$\begin{aligned} y_n'(x) &= a_n(x)y_n(x) + d_n(x)y_{n-1}(x), \\ y_{n-1}'(x) &= b_n(x)y_{n-1}(x) + e_n(x)y_n(x). \end{aligned} \tag{1}$$

For the particular case of modified Bessel functions sharp bounds for function ratios $y_n(x)/y_{n-1}(x)$ and logarithmic derivatives $y_n'(x)/y_n(x)$, as well as Turán-type inequalities were recently obtained in [1]; the key ingredient in the analysis was the study of the qualitative behavior of the solutions of the Riccati equation satisfied by $h_n(x) = y_n(x)/y_{n-1}(x)$, together with the application of the three-term recurrence relation.

In this paper, the ideas in [1] are generalized and applied to a much broader set of functions. We analyze the qualitative behavior of the Riccati equation associated to the ratio $h_n(x) = y_n(x)/y_{n-1}(x)$,

$$h_n'(x) = d_n(x) - (b_n(x) - a_n(x))h_n(x) - e_n(x)h_n(x)^2, \tag{2}$$

in the general case in which the quadratic equation

$$e_n(x)\lambda_n(x)^2 + (b_n(x) - a_n(x))\lambda_n(x) - d_n(x) = 0 \tag{3}$$

has two distinct real roots $\lambda_n^{\pm}(x)$. This case corresponds to monotonic systems, with solutions which have one zero at most. As we will see, if the functions $\lambda_n^{\pm}(x)$ are monotonic, they are bounds for the ratios $h_n(x)$ satisfying certain initial value conditions.

The methods can be applied to many special functions, modified Bessel functions, parabolic cylinder functions, Legendre and Laguerre functions among them. Ratios of Bessel functions appear in a great number of applications, particularly as parameters of certain probability distributions (see, for instance, the examples mentioned in [1]). Parabolic cylinder ratios appear in the study of Ornstein-Uhlenbeck processes (see, for instance [2]), and other special function ratios (Whittaker, Legendre, Gauss hypergeometric functions) play similar roles as well [3, 4, 5]. In all these applications, a common characteristic is that the functions are real and the variables lie inside a monotonic region (region free of zeros). These are precisely the conditions under which our techniques can be applied.

In addition to direct applications in several areas, particularly in statistics and stochastic processes, the bounds on function ratios have implications in the construction of numerical algorithms. These techniques provide bounds for the region of computable parameters of a given function within the overflow and underflow limitations, and they also provide bounds for the condition numbers of the functions (see section 4.1.2 for the case of Parabolic Cylinder Functions). Additionally, as discussed for the particular case of modified Bessel functions [1], the bounds are useful for accelerating the convergence of certain continued fraction representations which are used in numerical algorithms; for instance, the algorithms in [6, 7] could be improved by using the bounds of sections 4.1.2 and 4.1.3 for accelerating the convergence.

We obtain upper and lower bounds for function ratios and logarithmic derivatives of the solutions of systems (1) with $d_n(x)e_n(x) > 0$. The bounds are accurate for large values of the variable x and the parameter n. This is a consequence of the connection between the bounds, the Liouville-Green approximation for the associated second order ODE and the asymptotic behavior of the associated three-term recurrence relation. We also give two examples of applications of the methods for the case $d_n(x)e_n(x) < 0$ (section

4.2) and use these results for bounding function ratios for Laguerre and Hermite polynomials in the real axis (but outside the oscillatory region). These bounds can be used for bounding the oscillatory region and, therefore, for bounding the extreme zeros.

The structure of the paper is as follows. In section 2 we analyze the conditions which guarantee that the roots of (3) are bounds for the ratios $y_n(x)/y_{n-1}(x)$. The dependence on n is analyzed in section 3. The use of the three-term recurrence relation allows us to obtain sequences of upper and lower bounds in the case $d_n(x)e_n(x) > 0$; Turán-type inequalities are also established, as well as bounds on the logarithmic derivatives. In section 4 the techniques are applied to parabolic cylinder, Legendre and Laguerre functions. Examples for the cases $d_n(x)e_n(x) > 0$ and $d_n(x)e_n(x) < 0$ are provided.

2 Qualitative behavior of Riccati equations

We consider first order differential systems (1) with differentiable coefficients, for which the ratio $h_n(x) = y_n(x)/y_{n-1}(x)$ satisfies the Riccati equation

$$h'(x) = d(x) - (b(x) - a(x))h(x) - e(x)h(x)^2. \tag{4}$$

The label n, which is common for h and the coefficients a, b, d and e, has been dropped in (4) for simplicity. The analysis in this section is valid for any system, depending or not on a parameter n. The explicit dependence on n will be recovered in the next section.

We have $h'(x) = 0$ when $h(x) = \lambda^{\pm}(x)$ with

$$\begin{aligned} &\lambda^{\pm}(x) = \operatorname{sign}(e(x))R(x)\left[-\eta(x) \pm \sqrt{\eta(x)^2 + s}\right], \\ &R(x) = \sqrt{\left|\frac{d(x)}{e(x)}\right|}, \eta(x) = \frac{b(x) - a(x)}{2\sqrt{|d(x)e(x)|}}, s = \operatorname{sign}(d(x)e(x)), \end{aligned} \tag{5}$$

We consider the case with real roots $\lambda^{\pm}(x)$. Two distinct situations may occur: either $d(x)e(x) > 0$, or $d(x)e(x) < 0$ but $|\eta(x)| > 1$.

The condition $d(x)e(x) > 0$ generally holds in the whole maximal interval of continuity of the functions because the coefficients $d(x)$ and $e(x)$ do not change sign under very general conditions (see, for instance, [8, lemma 2.1][1]). Contrarily, when $d(x)e(x) < 0$ the condition $|\eta(x)| > 1$ may hold only for a limited range of the variable x. In the first case ($d(x)e(x) > 0$) $h(x)$ may have one zero or one singularity, but not both ([8, lemma 2.4]), while in the second $h(x)$ may have both a zero and a singularity ([9, Theorem 2.1]). We analyze the case $d(x)e(x) > 0$, assuming that no change of sign of $h(x)$ occurs. For the case $d(x)e(x) < 0$, as the examples in section 4.2 will show, similar arguments can be applied.

In the sequel, we consider $d(x)e(x) > 0$. Without loss of generality, we take $d(x) > 0$, $e(x) > 0$ and then $\lambda^+(x) > 0$ and $\lambda^-(x) < 0$; if $d(x) < 0$, $e(x) < 0$ we can consider the replacement $y \to -y$ or $w \to -w$. In the next results, (a, b) is an interval where $h(x)$ and the coefficients of the system are differentiable; a or b could be $+\infty$ or $-\infty$. Depending on the value of $h(x)$ at a^+ or b^- different bounds can be established. First we consider $h(a^+) > 0$. We enunciate three results and give a common proof.

Lemma 1. *If $h(a^+) > 0$ then $h(x) > 0$ in (a, b)*

Theorem 1. *If $h(a^+) > 0$, $\lambda^+(x)$ is monotonic and $h'(a^+)\lambda^{+\prime}(a^+) > 0$ then $(h(x) - \lambda^+(x))\lambda^{+\prime}(x) < 0$ in (a, b).*

Theorem 2. *If $h(a^+) > 0$, $\lambda^+(x)$ is monotonic and $h'(a^+)\lambda^{+\prime}(a^+) < 0$ then either $h(x)$ reaches one relative extremum at $x_e \in (a, b)$ (a minimum if $\lambda^{+\prime}(x) > 0$ and a maximum if $\lambda^{+\prime}(x) < 0$) or $(h(x) - \lambda_+(x))\lambda^{+\prime}(x) > 0$ in (a, b).*

Proof. If $h(a^+) > 0$, then $h(x)$ can not change sign continuously: it can not become zero because $h'(x) > 0$ if $0 \leq h(x) < \lambda^+(x)$. On the other hand, it can not change sign discontinuously; for this, starting with $h(a^+) > 0$,

[1] All that is required is that the system is satisfied by two independent sets of functions

a value $x_\infty \in (a,b)$ should exist such that $h(x_\infty^-) = +\infty$ but this is not possible because $h'(x) < 0$ if $h(x) > \lambda^+(x)$.

Now, we consider that $\lambda^+(x)$ is monotonic. We take the case $\lambda^{+\prime}(x) > 0$; the case $\lambda^{+\prime}(x) < 0$ is analogous.

Assume first that $h'(a^+) > 0$; using (4) this means that $0 < h(a^+) < \lambda^+(a^+)$. And then, necessarily $h(x) < \lambda^+(x)$ in (a,b). Indeed, because $\lambda^+(x)$ is monotonically increasing and the graph of $h(x)$ is below the graph of $\lambda^+(x)$ close to $x = a$, the graph of $h(x)$ may touch the graph of $\lambda^+(x)$ at $x = x_e$ only if the first one has a larger slope at x_e, that is, if $h'(x_e) > \lambda^{+\prime}(x_e) > 0$; but if $h(x_e) = \lambda_+(x_e)$ then $h'(x_e) = 0$.

Assume now that $h'(a^+) < 0$. The graph of $h(x)$ lies above the graph of $\lambda^+(x)$ close to $x = a$ and there are two possibilities: either it remains above $\lambda^+(x)$ in all the interval or there is a point $x_e \in (a,b)$ where $h(x_e) = \lambda^+(x_e)$ and $h'(x_e) = 0$. The graph of $h(x)$ crosses the graph of $\lambda^+(x)$, which is an increasing function, and $h'(x) > 0$ for all $x > x_e$. Therefore there is a minimum at x_e. □

Figure 1 illustrates the situations described in Theorems 1 and 2.

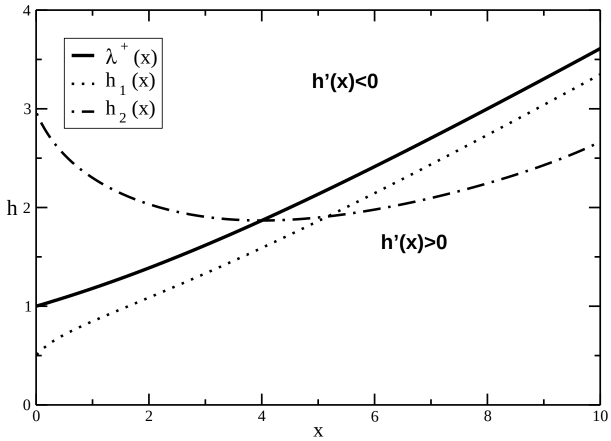

Figure 1: The characteristic root $\lambda^+(x)$ divides the plane in two regions: $h'(x) > 0$ if $0 < h(x) < \lambda^+(x)$ and $h'(x) < 0$ if $h(x) > \lambda^+(x)$. The graph of $h_1(x)$ corresponds to

the situation described in Theorem 1 while $h_2(x)$ corresponds to Theorem 2 when an extremum is reached.

If, differently from theorems 1 and 2, we have $h(a^+)<0$ then $h(x)$ may change sign once. But if it does not change sign and $h(b^-)<0$ we are in the previous situation. Indeed, with the change of variable $x\to -x$ and the change of function $w(x)\to -w(x)$, we have that the new ratio of functions $\tilde{h}(x)=-y(-x)/w(-x)$ is such that $\tilde{h}(\alpha^+)>0$ and the previous results hold in the interval $[\alpha,\beta]=[-b,-a]$. Then, we can write a common result for both cases. We only give the result corresponding to Theorem 1.

Theorem 3. *Let $h(x)$ be a solution of (4) with continuous coefficients and $d(x)>0$, $e(x)>0$. Suppose that either $h(a^+)>0$ or that $h(b^-)<0$ and take $s=+$, $c=a^+$ in the first case and $s=-$, $c=b^-$ in the second. Then, $h(x)$ does not change sign in (a,b), and if the characteristic root $\lambda^s(x)$ is monotonic and $\lambda^{s\prime}(c)h'(c)>0$ then*

$$(|h(x)|-|\lambda^s(x)|)\frac{d\lambda^s}{dx}<0\ \forall x\in(a,b)$$

Remark 1. *The condition $\lambda^{s\prime}(c)h'(c)>0$ is equivalent to*

$$(|h(c)|-|\lambda^s(c)|)\frac{d\lambda^s}{dx}(c)<0$$

3 Bounds for first order DDEs

Now, consider a first order difference-differential equation (1) and assume it holds for $n\geq n_0$ and that it is possible to make the shift $n\to n+1$ in (1). In this case the solutions of (1) are also solutions of a three-term recurrence relation

$$e_{n+1}y_{n+1}(x)+(b_{n+1}(x)-a_n(x))y_n(x)-d_ny_{n-1}(x)=0. \tag{6}$$

As in the previous section, we assume $d_n(x)e_n(x)>0$.

Let $\bar{\lambda}_n^{\pm}$ be the roots of the algebraic equation

$$e_{n+1}\bar{\lambda}_n^2+(b_{n+1}-a_n)\bar{\lambda_n}-d_n=0, \tag{7}$$

that is:

$$\begin{array}{l} \bar{\lambda}_n^{\pm} = R_n E_n(-\bar{\eta}_n \pm \sqrt{1+\bar{\eta}_n^2}), \\ R_n = \sqrt{d_n/e_n},\ E_n = \sqrt{e_n/e_{n+1}},\ \bar{\eta}_n = (b_{n+1}-a_n)/(2\sqrt{d_n e_{n+1}}) \end{array} \tag{8}$$

If $\lim_{n\to+\infty} \bar{\eta}_n \neq 0$ then $\lim_{n\to+\infty} |\bar{\lambda}_n^+/\bar{\lambda}_n^-| \neq 1$, and if the coefficients are of algebraic growth as a function of n, Perron-Kreuser theorem (see [10, Thm 4.5]) states that independent pairs of solutions $\{y_k^{(1)}, y_k^{(2)}\}$ exist such that

$$\lim_{n\to+\infty} \frac{1}{\bar{\lambda}_n^+}\frac{y_n^{(1)}}{y_{n-1}^{(1)}} = 1,\ \lim_{n\to+\infty} \frac{1}{\bar{\lambda}_n^-}\frac{y_n^{(2)}}{y_{n-1}^{(2)}} = 1. \tag{9}$$

If $\bar{\eta}_n > 0$ the minimal solution is $y_n^{(1)}$ and $y_n^{(2)}$ is dominant, and therefore $\lim_{n\to+\infty} y_n^{(1)}/y_n^{(2)} = 0$. If $\eta_n < 0$ the roles are reversed. In both cases we have, for sufficiently large n, $y_{n+1}^{(1)} y_n^{(1)} > 0$ and $y_{n+1}^{(2)} y_n^{(2)} < 0$.

Remark 2. *The minimal solution satisfies $\bar{\eta}_n y_n/y_{n-1} > 0$ for large n, while the dominant solutions are such that $\bar{\eta}_n y_n/y_{n-1} < 0$ for large n.*

Notice that the roots (8) are closely related to the characteristic roots of the Riccati equation (5):

$$\lambda_n^{\pm}(x) = \sqrt{\frac{d_n(x)}{e_n(x)}}(-\eta_n(x) \pm \sqrt{1+\eta_n(x)^2}),\ \eta_n(x) = \frac{b_n(x)-a_n(x)}{2\sqrt{d_n(x)e_n(x)}}. \tag{10}$$

As we have shown in the previous section, when $\lambda_n^{\pm}(x)$ are monotonic they provide bounds for some solutions. On the other hand, if $\lim_{n\to+\infty} \bar{\lambda}_n^{\pm}/\lambda_n^{\pm} = 1$ the function ratios have these bounds as limits. This explains why the bounds (10) tend to be sharper as n becomes larger. Because of this, we refer to these bounds as Perron-Kreuser bounds.

In section 3.2 we will obtain additional upper and lower sharp bounds starting from the bounds of Theorem 1 and using the three-term recurrence.

Before this, it is important to stress that for the Perron-Kreuser bounds to hold, it is crucial that the characteristic roots are monotonic as a function of x. This, however, is a quite general situation, as we next see.

3.1 Monotonicity of the characteristic roots

The next result relates the monotonicity properties of the characteristic roots with the monotonicity properties as a function of n.

Theorem 4. *Let $y_k(x)$, $k = n, n-1$, be solutions of second order ODEs $y_k''(x)+B_k(x)y_k'(x)+A_k(x)y_k(x) = 0$, with $A_k(x)$, $B_k(x)$ continuous in (a, b) and $B_n(x) = B_{n-1}(x)$. Assume that $y_n(x)$ and $y_{n-1}(x)$ satisfy a system (1) with $d_n(x)e_n(x) > 0$ and differentiable coefficients. Then, $e_n(x)/d_n(x)$ is constant as a function of x, and if $A_n(x) \neq A_{n-1}(x)$ the characteristic roots $\lambda_n^{\pm}(x)$ (10) are monotonic in (a, b). Furthermore, $d\lambda_n^{\pm}(x)/dx$ has the same sign as $A_{n-1}(x) - A_n(x)$ and $-\eta_n'(x)$.*

Proof. Differentiating the first equation of the system (1) and eliminating y_{n-1} and proceeding similarly with the second equation we have

$$y_k''(x) + B_k(x)y_k'(x) + A_k(x)y_k(x) = 0,\ k = n, n-1, \tag{11}$$

with coefficients satisfying:

$$\begin{aligned} B_n(x) - B_{n-1}(x) &= \frac{e_n'(x)}{e_n(x)} - \frac{d_n'(x)}{d_n(x)}, \\ A_n(x) - A_{n-1}(x) &= b_n'(x) - a_n'(x) - b_n(x)\frac{e_n'(x)}{e_n(x)} + a_n(x)\frac{d_n'(x)}{d_n(x)} \end{aligned} \tag{12}$$

Now, because we are assuming that $B_n(x) = B_{n-1}(x)$ the first equation implies that $d_n(x)/e_n(x)$ does not depend on x. Therefore, from the expression of the characteristic roots (8) we see that $d\lambda_n^{\pm}(x)/dx$ has the same sign as $-\eta_n'(x)$. All that remains to be proved is that $A_n(x) - A_{n-1}(x)$ has the same sign as $\eta_n'(x)$. But considering the second equation of (12) and using that $d_n'(x)/d_n(x) = e_n'(x)/e_n(x)$ one readily sees that $A_n(x) - A_{n-1}(x) = 2\sqrt{d_n(x)e_n(x)}\eta_n'(x)$, which proves the theorem.

□

Remark 3. *If $e_n(x)d_n(x) < 0$ and $\eta_n(x)^2 > 1$, it is also true that both roots are monotonic if $B_n(x) = B_{n-1}(x)$ and $A_n(x) \neq A_{n-1}(x)$, but $\lambda_n^+(x)\lambda_n^-(x) > 0$ and $\lambda_n^{+\prime}(x)\lambda_n^{-\prime}(x) < 0$ in this case.*

The case described in Theorem 4 is, for instance, the situation for Bessel functions, parabolic cylinder functions and the classical orthogonal polynomials when n is the degree of the polynomials.

3.2 Perron-Kreuser bounds

In the following, we assume that $\eta_n(x)$, $\bar{\eta}_n(x)$, $d_n(x)$, $e_n(x)$ and $h_n(x) = y_n(x)/y_{n-1}(x)$ do not change sign for large enough n (say $n \geq n_0$). Notice that the sign condition for $h_n(x)$ is satisfied for large enough n when Perron-Kreuser theorem holds. An immediate application of Theorem 3 gives:

Theorem 5 (First Perron-Kreuser bound). *Let $d_n(x) > 0$, $e_n(x) > 0$ and $h_n(x) = y_n(x)/y_{n-1}(x)$ with constant sign for $n \geq n_0$ and for any $x \in (a, b)$. Let $s = sign(h_n(x))$ and $\lambda_n^s(x)$ as in Eq. (10). Then, if $h_n(a^+) > 0$ and $h_n'(a^+)\lambda_n^{s\prime}(a^+) > 0$ or $h_n(b^-) > 0$ and $h_n'(b^-)\lambda_n^{s\prime}(b^-) > 0$ the following holds in (a, b):*

$$\left(|h_n(x)| - F_n^s(x)\right) \lambda_n^{s\prime}(x) < 0,\ n \geq n_0 \tag{13}$$

$$F_n^s(x) = R_n(x)(-s\eta_n(x) + \sqrt{1 + \eta_n(x)^2}) = \frac{R_n(x)}{s\eta_n(x) + \sqrt{1 + \eta_n(x)^2}} \tag{14}$$

Further bounds can be obtained by iteration of (6), which we write:

$$\frac{y_n(x)}{y_{n-1}(x)} = d_n \left(b_{n+1} - a_n + e_{n+1} \frac{y_{n+1}(x)}{y_n(x)} \right)^{-1}. \tag{15}$$

For minimal solutions we have that $\bar{\eta}_n(x) y_n(x)/y_{n-1}(x) > 0$ for large n. By substituting in the previous equation $y_{n+1}(x)/y_n(x)$ by a lower (upper) bound we get an upper (lower) bound for $y_n(x)/y_{n-1}(x)$. The process can be iterated to produce sequences of lower and upper bounds. We only give the first iteration.

Theorem 6 (Second Perron-Kreuser bound for minimal solutions). *Under the conditions of Theorem 5 and if $s\bar{\eta}_n > 0$, $s = sign(h_n)$ then*

$$\left(|h_n(x)| - S_n^{s+}\right) \lambda_n^{s\prime}(x) > 0,\ n \geq n_0 \tag{16}$$

where

$$S_n^{s+} = \frac{D_n E_n R_n}{s(2D_n\bar{\eta}_n - \eta_{n+1}) + \sqrt{1+\eta_{n+1}^2}} \tag{17}$$

$D_n = \sqrt{d_n/d_{n+1}}$*;* E_n*,* R_n *and* $\bar{\eta}_n$ *given by (8) and* η_n *by (10).*

The second superscript of the notation S_n^{s+} stands for the sign of $\bar{\eta}_n y_n/y_{n-1}$.

Notice that theorem 6 may be true for $n = n_0 - 1$ too, because Theorem 5 is used in the proof with the shift $n \to n+1$.

The similarity of the second expression of (14) with (17) indicates that for coefficients of algebraic growth we will generally have $\lim_{n\to+\infty} F_n^s/S_n^{s+} = 1$.

Further iterations are possible and this gives a convergent sequence of upper and lower bounds under the conditions of Theorem 5 and 6 and provided that Perron-Kreuser theorem holds (which implies that the recurrence admits a minimal solutions). We don't prove this result, but the convergence of the sequence of bounds for the minimal solution follows immediately by using the same arguments considered in [1] for the case of Modified Bessel functions of the first kind.

We can also obtain additional bounds for dominant solutions by writing

$$\frac{y_n(z)}{y_{n-1}(x)} = -\frac{b_n - a_{n-1}}{e_n} + \frac{d_{n-1}}{e_n}\frac{y_{n-2}(z)}{y_{n-1}(z)} \tag{18}$$

Differently from the case of minimal solutions, the sequence of bounds is not a convergent sequence. We give an explicit formula for the first iteration:

Theorem 7 (Second Perron-Kreuser bound for dominant solutions)**.** *Under the conditions of Theorem 5 and if* $s\bar{\eta}_{n-1} < 0$*,* $s = sign(h_n)$*,*

$$(|h_n(x)| - S_n^{s-})\lambda_n^{s\prime}(x) > 0,\ n \geq n_0 + 1 \tag{19}$$

where

$$S_n^{s-} = D_{n-1}E_{n-1}R_n\left(-s(2E_{n-1}^{-1}\bar{\eta}_{n-1} - \eta_{n-1}) + \sqrt{1+\eta_{n-1}^2}\right) \tag{20}$$

Notice that the previous theorem can only be guaranteed to be true for $n = n_0 + 1$, because Theorem 5 is used in the proof with the shift $n \to n-1$.

The similarity of the first expression in (14) with (20) is clear. For coefficients of algebraic growth we will generally have $\lim_{n\to+\infty} F_n^s/S_n^{s-} = 1$.

3.3 Turán-type inequalities

Turán-type properties for special functions have received a considerable attention in recent years; just to cite five different groups of researchers, we mention [11, 12, 13, 14, 1] (see also references cited therein). Turán-type inequalities can be obtained from the bounds on function ratios.

Indeed, because upper and lower bounds are available for $|y_n/y_{n-1}|$ both when y_n is a minimal or a dominant solution (Theorems 5, 6 and 7), upper and lower bounds for $|y_n/y_{n-1}||y_n/y_{n+1}|$ become available. The modulus can be skipped if y_n/y_{n-1} does not change sign (as assumed earlier). With this:

$$l_n \leq L_n(x) < \frac{y_n(x)}{y_{n+1}(x)}\frac{y_n(x)}{y_{n-1}(x)} < U_n(x) \leq u_n, \tag{21}$$

where $l_n = \min_x\{L_n(x)\}$ and $u_n = \max_x\{U_n(x)\}$. Many new Turán-type inequalities are found in section 4 by using this simple idea.

3.4 Bounds of Liouville-Green type

Using the difference-differential system (1) and the Perron-Kreuser bounds, bounds on the logarithmic derivatives can be established. We give the bounds obtained from the first Perron-Kreuser bound.

Theorem 8. *Under the hypothesis of Theorem 5 and if $d\lambda_n^s/dx > 0$ ($s = sign(y_n(x)/y_{n-1}(x))$):*

$$s\frac{y'_{n-1}(x)}{y_{n-1}(x)} < s\frac{a_n(x)+b_n(x)}{2} + \sqrt{d_n(x)e_n(x)}\sqrt{1+\eta_n(x)^2} < s\frac{y'_n(x)}{y_n(x)} \tag{22}$$

If $d\lambda_n^s/dx < 0$ the inequalities are reversed

Two consequences follow. First, we observe that the ratios $y_k'(x)/y_k(x)$ are monotonic as a function of the discrete variable k. Second, because we are assuming that the shift $n \to n+1$ is possible, we have both an upper and a lower bound for y_n'/y_n. Upper and lower bounds could also be obtained by considering both the first and second Perron-Kreuser bounds.

In the examples we will see that these bounds, after integrating the logarithmic derivative, are related to the Liouville-Green approximation for solutions of second order ODEs. In fact, using this analysis and by Liouville-transforming the first order system associated to the ODE $y''(x) + A(x)y(x) = 0$, conditions can be established under which the LG approximation for the solutions the ODE $y''(x) + A(x)y(x) = 0$ are bounds for some of the solutions. We leave this analysis for a future paper.

4 Applications

We give a number of examples of application of the techniques described in the paper. We focus on the case $d_n(x)e_n(x) > 0$, but examples of application for monotonic systems with $d_n(x)e_n(x) < 0$ are also given.

4.1 Cases with $d_n(x)e_n(x) > 0$

We analyze three families of functions, which have as particular cases some classical orthogonal polynomials outside the interval of orthogonality. In all cases Theorem 4 holds, with the exception of Laguerre functions of negative argument. In this case Theorem 4 can not be applied but the characteristic roots are still monotonic and the same analysis is therefore possible. Some monotonicity properties for the determinants of some of the functions analyzed were considered in [15].

4.1.1 Modified Bessel functions

These are solutions of $x^2y'' + xy' - (x^2 + \nu^2)y = 0$. This was the case considered in detail in [1], and most of the results obtained in that paper are direct consequences of the more general results of the present one.

4.1.2 Parabolic cylinder functions

The parabolic cylinder function $U(n,x)$ is a solution of the differential equation $y''(x) - (x^2/4 + n)y(x) = 0$, with coefficient $A(x) = -(x^2/4 + n)$ depending monotonically on the parameter n (Theorem 4 holds).

Considering the DDE satisfied by $U(n,x)$ [16, 12.8.2-3] and defining $y_n(x) = e^{i\pi n}U(n,x)$ [2] we have:

$$\begin{aligned} y_n'(x) &= \frac{x}{2}y_n(x) + y_{n-1}(x),\\ y_{n-1}'(x) &= -\frac{x}{2}y_{n-1}(x) + (n-1/2)y_n(x). \end{aligned} \tag{23}$$

where n will be real and positive. For this system

$$\eta_n(x) = -\frac{x}{2\sqrt{n-1/2}},\ \bar{\eta}_n(x) = \eta_{n+1}(x), \lambda_n^{\pm}(x) = \frac{-2}{x \mp \sqrt{4n-2+x^2}} \tag{24}$$

From [16, 12.9.1] we have $h_n(+\infty) = 0^-$ and $h_n'(+\infty) = 0^+$ and because $\lambda_n^-(+\infty) = 0^+$ then theorem 3 holds, as well as theorems 5 and 6. Therefore

Theorem 9. *For $n > 1/2$ and $x \geq 0$ the following holds*

$$\frac{2}{x + \sqrt{4n+2+x^2}} < \frac{U(n,x)}{U(n-1,x)} < \frac{2}{x + \sqrt{4n-2+x^2}} \tag{25}$$

The lower bound also holds if $n \in (-1/2, 1/2)$ and it turns to an equality if $n = -1/2$.

The lower bound is obtained from the upper bound and the application of the three-term recurrence relation: if $B_m(n,x)$ is a positive upper (lower) bound for $U(n,x)/U(n-1,x)$, $x > 0$, then

$$B_{m+1}(n,x) = 1/(x + (n+1/2)B_m(n+1,x)) \tag{26}$$

[2] It is not important that the new functions are complex, because we are dealing with ratios; an alternative definition could be $y_n(x) = (-1)^{\lfloor n \rfloor}U(n,x)$.

is a lower (upper) bound for the same ratio. The process can be continued as $m \to +\infty$ and the sequence is convergent (because $U(n,x)$ is minimal).

Now, consider $y_n(x) = U(n,-x)$, which is also solution of (23). Using the values of $U(n,0)$ and $U'(n,0)$ [16, 12.2.6-7] it is easy to prove that $h_n(0^+) > 0$, $h_n'(0^+) > 0$, $n > 1/2$, $x \geq 0$ and then $h_n'(0^+) d\lambda_n^+(0^+)/dx > 0$ and Theorem 1 holds. The corresponding Perron-Kreuser bounds (theorems 5 and 7), give:

Theorem 10. *For $n > 3/2$ and $x \geq 0$ the following holds*

$$\frac{x + \sqrt{4n - 6 + x^2}}{2n - 1} < \frac{U(n,-x)}{U(n-1,-x)} < \frac{x + \sqrt{4n - 2 + x^2}}{2n - 1} \tag{27}$$

The upper bound is also valid if $n \in (1/2, 3/2)$.

The upper bound in (25) has the same expression as (27) but with x in replaced by $-x$. Therefore:

Remark 4. *Theorems 9 and 10 hold for all real x, but for $x < 0$ the lower bound of Theorem 9 only holds for all $x < 0$ if $n > 1/2$. The lower bounds are sharper when $x > 0$.*

The following Turán-type inequalities are obtained from Theorems 9 and 10:

Theorem 11. *Let $F(x) = U(n,x)^2/(U(n-1,x)U(n+1,x))$. Then, for all real x:*

$$\sqrt{\frac{n - 3/2}{n + 1/2}} < \frac{n - 1/2}{n + 1/2} F(x) < 1 < F(x) < \sqrt{\frac{n + 3/2}{n - 1/2}} \tag{28}$$

The first inequality holds for $n > 3/2$ and the rest for $n > 1/2$. For $x < 0$ the third inequality also holds if $n \in (-1/2, 1/2)$.

Finally, considering Theorem 8 and writing together the results for $U(n,x)$ and $U(n,-x)$ we have the next result.

Theorem 12. *For all real x and $n \geq 1/2$ the following holds:*

$$-\sqrt{x^2/4+n+1/2} < \frac{U'(n,x)}{U(n,x)} < -\sqrt{x^2/4+n-1/2} \tag{29}$$

The left inequality also holds for $n > -1/2$.

These type of bounds are useful for studying the attainable accuracy of methods for computing the functions. In [17], the following estimation for large x and/or n was considered for the condition number with respect to x:

$$C_x(U(a,x)) = \left|xU'(a,x)/U(a,x)\right| \sim x\sqrt{x^2/4+a}, \tag{30}$$

and similarly for $V(a,x)$. The bounds (29) prove that this a good estimation because it lies between the upper and lower bounds. From the previous discussion on the $V(a,x)$ function, one can prove that similar bounds are valid for moderate x ($x > 1$ is enough); we consider later this function.

Integrating (29) we have

$$\begin{array}{l} F_{n+1/2}(x)/F_{n+1/2}(y) < \dfrac{U(n,y)}{U(n,x)} < F_{n-1/2}(x)/F_{n-1/2}(y), \\ F_\alpha(x) = \exp\left(\frac{x}{2}\sqrt{x^2/4+\alpha}\right)\left(x+2\sqrt{x^2/4+\alpha}\right)^{\alpha} \end{array} \tag{31}$$

and, in particular,

$$F_{n+1/2}(x) < \frac{U(a,x)}{U(a,0)} < F_{n-1/2}(x) \tag{32}$$

where

$$F_\alpha(x) = \exp\left(-\frac{x}{2}\sqrt{\frac{x^2}{4}+\alpha}\right)\left(\frac{x}{2\sqrt{\alpha}}+\sqrt{\frac{x^2}{4\alpha}+1}\right)^{-\alpha} \tag{33}$$

The bounds (32) are useful for obtaining the range of parameters for which function values are computable within the arithmetic capabilities of a computer (overflow and underflow limits). These results confirms the estimations based on the Liouville-Green approximation used in [18].

∘ **Iterated coerror functions and Mill's ratio:** In particular, considering Theorem 9 and the relation of parabolic cylinder functions $U(n+1/2, x)$ with the iterated coerror functions $i^n\mathrm{erfc}(x)$ [16, 12.7.7], $n \in \mathbb{N}$, the following follows:

$$M_{n+1}(x) < \frac{i^n\mathrm{erfc}(x)}{i^{n-1}\mathrm{erfc}(x)} < M_n(x),\ n = 1, 2, \ldots;\ M_n(x) = (x + \sqrt{2n + x^2})^{-1}. \tag{34}$$

These inequalities appear in [19].

Theorem 9 also gives bounds on Mill's ratio ($n = 1/2$). From the lower bound in Theorem (9) and the upper bound obtained by iterating with (26) we have

Theorem 13. *Let* $r(x) = e^{x^2/2}\int_x^{+\infty} e^{-t^2/2}dt$, *then*

$$\frac{2}{x + \sqrt{x^2 + 4}} < r(x) < \frac{4}{3x + \sqrt{x^2 + 8}} \tag{35}$$

The lower bound was obtained in [20] and the upper bound in [21]. In our case, these results follow from a more general result. See also [22] for an alternative proof.

Further iterations (see (26)) give additional sharper bounds:

Theorem 14.

$$R_{2k+1} < r(x) < R_{2k}(x) \tag{36}$$

$$R_n(x) = \frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\cdots\frac{n}{T_n(x)},\ T_n = (x + \sqrt{4n + x^2})/2 \tag{37}$$

where, as usual we denote $\frac{1}{a+}\frac{1}{b+}\ldots = 1/(a + 1/(b + \ldots))$

∘ **Hermite polynomials of imaginary variable** A similar analysis to that for $U(n, -x)$ can be carried for the PCF $V(n, x)$. Indeed, $y_n(x) = V(n, x)/\Gamma(n + 1/2)$ is a solution of (23) and $h_n(x) = y_n(x)/y_{n-1}(x)$ is such that $h_n(0^+) > 0$. Two situations take place depending on the values of n. First, if $n \in (2k - 1, 2k)$, $k \in \mathbb{N}$, then $h_n'(0^+) > 0$ and the upper bound of Theorem 10 holds for all $x > 0$ while the lower bound will hold for

$n \in (2k, 2k+1)$. Contrarily, if $n \in (2k, 2k+1)$ then $h_n'(0^+) < 0$, while $h_n(+\infty) > 0$, and the upper bound only holds for large enough x; a similar situation occurs with the lower bound when $n \in (2k-1, 2k)$.

We only consider the first case. Using the relation of $V(n+1/2, x)$, $n \in \mathbb{N}$, with Hermite polynomials [16, 12.7.3] we get:

Theorem 15.

$$\frac{V(n,x)}{V(n-1,x)} < \frac{x+\sqrt{4n-2+x^2}}{2},\ x>0,\ n \in (2k-1, 2k),\ k \in \mathbb{N} \tag{38}$$

$$\frac{x+\sqrt{4n-6+x^2}}{2} < \frac{V(n,x)}{V(n-1,x)},\ x>0,\ n \in (2k, 2k+1),\ k \in \mathbb{N} \tag{39}$$

$$-i\frac{H_{2k+1}(ix)}{H_{2k}(ix)} < x+\sqrt{4k+2+x^2},\ x>0,\ k=0,1,2\ldots \tag{40}$$

$$i\frac{H_{2k-1}(ix)}{H_{2k}(ix)} < (x+\sqrt{4k-2+x^2})^{-1},\ x>0,\ k \in \mathbb{N} \tag{41}$$

$$\frac{H_{2k}(ix)^2}{H_{2k-1}(ix)H_{2k+1}(ix)} > \sqrt{\frac{k-1/2}{k+1/2}},\ k \in \mathbb{N}, x \in \mathbb{R}. \tag{42}$$

Hermite polynomials of imaginary argument were also considered in [15]. The well-known Turán-type inequality for Hermite polynomials [23] $H_n(x)^2 - H_{n-1}(x)H_{n+1}(x) > 0$, $x \in \mathbb{R}$, does not hold on the imaginary axis, but a similar property $H_n(ix)^2 - \sqrt{(n-1)/(n+1)}H_{n-1}(ix)H_{n+1}(ix) > 0$ holds true for all $x > 0$ if n is even.

4.1.3 Oblate Legendre functions

These are Legendre functions of imaginary argument, which are functions appearing in the solution of Dirichlet problems in oblate spheroidal coordinates [6]. Denoting

$$p_n(x) = e^{-in\pi/2} P_n^m(ix) \tag{43}$$

and using the differential relations [16, 14.10.4-5] we have

$$\begin{aligned} p_n'(x) &= \frac{1}{1+x^2}\left\{nxp_\nu(x) + (n+m)p_{\nu-1}(x)\right\} \\ p_{n-1}'(x) &= \frac{1}{1+x^2}\left\{-nxp_{\nu-1}(x) + (n-m)p_\nu(x)\right\} \end{aligned} \tag{44}$$

and $q_n(x) = Q_n^m(ix)$, Q_n^m being the second kind Legendre function, satisfies the same system. We consider $n > m$ and $x > 0$. This is again an example for which Theorem 4 holds. The roles played in this case by the functions $Q_n^m(ix)$ and $P_n^m(ix)$ are very similar to the roles of $U(n,x)$ and $V(n,x)$ in the previous section. We omit details and only summarize the main results.

Theorem 16. *The following holds for $x > 0$ and real $n > m > 0$*

$$0 < i\frac{Q_n^m(ix)}{Q_{n-1}^m(ix)} < \frac{n+m}{n}\left[x + \sqrt{1 + x^2 - \frac{m^2}{n^2}}\right]^{-1} < \sqrt{\frac{n+m}{n-m}} \tag{45}$$

$$i\frac{Q_n^m(ix)}{Q_{n-1}^m(ix)} > \frac{n+m}{nx + (n+1)\sqrt{1 + x^2 - \frac{m^2}{(n+1)^2}}} \tag{46}$$

$$1 < \frac{n+m+1}{n+m}\frac{Q_n^m(ix)}{Q_{n-1}^m(ix)Q_{n+1}^m(ix)} < \sqrt{\frac{(n+2)^2 - m^2}{n^2 - m^2}} \tag{47}$$

Theorem 17. *The following holds for $x > 0$ and integer n, m, $n > m$:*

$$0 < -i\frac{P_n^m(ix)}{P_{n-1}^m(ix)} < \frac{n}{n-m}\left[x + \sqrt{1 + x^2 - \frac{m^2}{n^2}}\right],\ n-m \text{ odd} \tag{48}$$

$$\frac{1}{n-m}\left[nx + (n-1)\sqrt{1 + x^2 - \frac{m^2}{(n-1)^2}}\right] < -i\frac{P_n^m(ix)}{P_{n-1}^m(ix)},\ n-m \text{ even} \tag{49}$$

$$\frac{P_n^m(ix)^2}{P_{n-1}^m(ix)P_{n+1}^m(ix)} < 1 + \frac{1}{n-m},\ n-m \text{ odd} \tag{50}$$

For $m = 0$ we have Legendre polynomials. If n is odd, we have $P_n(ix)^2 < 0$ and therefore $P_n(ix)^2 - (1 + 1/n)P_{n-1}(ix)P_{n+1}(ix) > 0$. It appears, as numerical experiments show, that in this case the same Turán inequality that holds in the real interval $(-1, 1)$ [24] also holds in the imaginary axis if n is odd: $P_n(ix)^2 - P_{n-1}(ix)P_{n+1}(ix) > 0$; the same is not true if $m \neq 0$.

4.1.4 Laguerre functions of negative argument

Next we consider an example for which Theorem 4 can not be applied but the analysis is possible because the characteristic roots are monotonic.

Consider the Laguerre functions $y_{\nu,\alpha}(x) = L_{\nu}^{\alpha}(-x)$, $x > 0$. Using well known recurrences and differentiation formulas, we have

$$\begin{aligned} &y'_{\nu+1,\alpha-1}(x) = y_{\nu,\alpha}(x) \\ &xy'_{\nu,\alpha}(x) = -(\alpha + x)y_{\nu,\alpha}(x) + (\nu + 1)y_{\nu+1,\alpha-1} \end{aligned} \tag{51}$$

and

$$(\nu + 1)y_{\nu+1,\alpha-1}(x) = (\alpha + x)y_{\nu,\alpha}(x) + xy_{\nu-1,\alpha+1} \tag{52}$$

Considering [25, Theorem 2] it follows that $y_{\nu,\alpha}$ is a dominant solution of the recurrence (52) in the direction of increasing ν (and decreasing α).

With $h(x) = y_{\nu,\alpha}(x)/y_{\nu+1,\alpha-1}(x)$, the positive characteristic root $\lambda^{+}(x)$ of the associated Riccati equation turns out to be increasing if $\nu > -1$ and $\alpha > 0$. On the other hand, it is easy to check that for these values $h(0^{+}) > 0$ and $h'(0^{+}) > 0$. Theorem 1 holds and $\lambda^{+}(x)$ is a bound:

Theorem 18. *For any $\alpha > 0$, $\nu > -1$ and $x > 0$ the following holds*

$$0 < \frac{L_{\nu+1}^{\alpha-1}(-x)}{L_{\nu}^{\alpha}(-x)} < \frac{\alpha + x + \sqrt{(\alpha + x)^2 + 4(\nu + 1)x}}{2(\nu + 1)} \tag{53}$$

On the other hand, from the recurrence (52) we have

$$\frac{L_{\nu}^{\alpha}(-x)}{L_{\nu+1}^{\alpha-1}(-x)} = \left(\frac{\alpha + x}{\nu + 1} + \frac{x}{\nu + 1} \frac{L_{\nu-1}^{\alpha+1}(-x)}{L_{\nu}^{\alpha}(-x)} \right)^{-1} \tag{54}$$

and from this we obtain the second Perron-Kreuser bound:

Theorem 19. *For any $\alpha > -1$, $\nu > 0$ and $x > 0$ the following holds*

$$\frac{L_{\nu+1}^{\alpha-1}(-x)}{L_{\nu}^{\alpha}(-x)} > \frac{\alpha + x - 1 + \sqrt{(\alpha + x + 1)^2 + 4\nu x}}{2(\nu + 1)} \tag{55}$$

And from these bounds we get the following Turán-type inequalities:

Theorem 20. *For any $\nu \geq 0$ and $\alpha \geq 0$, $x > 0$ the following holds:*

$$\frac{\nu}{\nu+1}\frac{\alpha}{\alpha+1} < \frac{L_{\nu+1}^{\alpha-1}(-x)}{L_{\nu}^{\alpha}(-x)}\frac{L_{\nu-1}^{\alpha+1}(-x)}{L_{\nu}^{\alpha}(-x)} < \frac{\nu}{\nu+1} \tag{56}$$

A second independent solution of (51) which is a minimal solution of (52) as $\nu \to +\infty$ follows from [25, Theorem 2]. Bounds can be also obtained for this solution. We omit the details.

Other bounds and inequalities can be obtained using other recursions or using relations between contiguous functions. For example, using [16, 18.9.13], we have:

$$\frac{L_{\nu+1}^{\alpha-1}(x)}{L_{\nu}^{\alpha}(x)} = 1 + \frac{L_{\nu+1}^{\alpha}(x)}{L_{\nu}^{\alpha}(x)} \tag{57}$$

and upper and lower bounds for $L_n^{\alpha}(-x)/L_{n-1}^{\alpha}(-x)$ follow from the previous results. As a consequence of this new bounds, one can prove the following

Theorem 21.

$$\frac{\nu}{\nu+1} < \frac{L_{\nu-1}^{\alpha}(-x)}{L_{\nu}^{\alpha}(-x)}\frac{L_{\nu+1}^{\alpha}(-x)}{L_{\nu}^{\alpha}(-x)} < \frac{\nu}{\nu+1}\frac{\nu+\alpha+1}{\nu+\alpha-1} \tag{58}$$

where the first inequality holds for $\nu > 0$, $\alpha > -1$ and the second for $\nu > 0$, $\nu + \alpha > 1$.

For positive x, it is known that $L_{n-1}^{\alpha}(x)L_{n+1}^{\alpha}(x)/L_n^{\alpha}(x)^2 < 1$ [23]. For negative argument we have an upper bound greater that 1, which suggests that the Turán-type inequality for positive x does not hold for negative x, as numerical experiments show.

4.2 Two examples with $d_n(x)e_n(x) < 0$

The DDEs corresponding to a pair $\{p_n(x), p_{n-1}(x)\}$ of classical orthogonal polynomials satisfy $d_n(x)e_n(x) < 0$ in their interval of orthogonality because this is a necessary condition for oscillation [8, Lemma 2.4]. However, for values of the variable for which the polynomials are free of zeros, one can expect that $\eta_n(x)^2 > 1$ and that the DDE becomes monotonic ($\eta_n(x)^2 < 1$

is also a necessary condition for oscillation [9, Thm. 2.1]). This is the case of Laguerre and Hermite polynomials for large enough $x > 0$. We consider these two examples.

4.2.1 Hermite polynomials

Hermite polynomials satisfy

$$\begin{aligned} H_n'(x) &= 2nH_{n-1}(x), \\ H_{n-1}'(x) &= 2xH_{n-1} - H_n(x) \end{aligned} \tag{59}$$

We have $\eta_n(x) = x/\sqrt{2n}$ and $\eta_n(x) > 1$ if $x > \sqrt{2n}$ (monotonic case). The characteristic roots are both of them positive

$$\lambda_n^{\pm}(x) = x \pm \sqrt{x^2 - 2n}. \tag{60}$$

Defining $h_n(x) = H_n(x)/H_{n-1}(x)$ we have that $h_n(+\infty) = +\infty$ and $h_n'(+\infty) > 0$ because the coefficient of degree n of $H_n(x)$ is positive. Then $h_n(x) > \lambda_n^+(x)$ for enough $x > 0$ because $h_n'(x) > 0$ only if $h_n(x) < \lambda_n^-(x)$ or $h_n(x) > \lambda_n^+(x)$, but $h_n(+\infty) > \lambda_n^-(+\infty) = 0^+$, and therefore $h_n(x) > \lambda_n^+(x)$ for large x. And because $\lambda_n^{+\prime}(x) > 0$ if $x > \sqrt{2n}$, then, necessarily:

$$h_n(x) = \frac{H_n(x)}{H_{n-1}(x)} > x + \sqrt{x^2 - 2n}, x \geq \sqrt{2n}. \tag{61}$$

We can iterate the recurrence relation. Contrary to the case $e_n(x)d_n(x) > 0$, we will not obtain sequences of lower and upper bounds, but only lower bounds. Writing

$$h_{n+1}(x) = 2x - 2n/h_n(x) \tag{62}$$

and using (61) we get a lower bound for $h_{n+1}(x)$. We shift the parameter n and get

$$h_n(x) > x + \sqrt{x^2 - 2(n-1)}, x \geq \sqrt{2(n-1)}. \tag{63}$$

This improves Eq. (61) and enlarges the range of validity of the bound with respect to x, but reduces the range of validity with respect to n ($n \geq 2$).

The next iteration gives a bound for $n \geq 3$:

$$\begin{aligned} h_n(x) > \quad & F(n,x), x \geq \sqrt{2(n-2)} \\ & F(n,x) = (n-2)^{-1}[(n-3)x + (n-1)\sqrt{x^2 - 2(n-2)}] \end{aligned} \tag{64}$$

Because the largest zero of $H_n(x)$ is larger than that of $H_{n-1}(x)$, Eq. (64) implies that the largest zero of $H_n(x)$ is smaller than $\sqrt{2(n-2)}$, $n \geq 3$.

We consider just one more iteration and get

$$h_n(x) \geq 2x - 2(n-1)/F(n-1,x) = G(n,x),\ x > \sqrt{2(n-3)} \tag{65}$$

and if $G(n, \sqrt{2(n-3)}) > 0$ then $G(n,x) > 0$ if $x > \sqrt{2(n-3)}$, and the largest zero will be smaller than $\sqrt{2(n-3)}$; this condition is met if $n \geq 7$. A sharper bound has recently appeared in the literature [26] valid for all n. However, the result is sharper than previous results, like for instance those in [27], which is interesting given the simplicity of the analysis. This reflects the fact that the bounds on function ratios (our main topic) are sharp.

4.2.2 Laguerre polynomials

We give some results for Laguerre polynomials omitting details. Defining $h_n^{\alpha}(x) = -L_n^{\alpha}(x)/L_{n-1}^{\alpha}(x)$, we have $h_n^{\alpha}(+\infty) = +\infty$ and $h_n^{\alpha\prime}(+\infty) = +\infty$ and, proceeding similarly as before:

$$\begin{aligned} 2nh_n^{\alpha}(x) > \quad & x - (2n+\alpha) + \sqrt{(x-2n-\alpha)^2 - 4n(n+\alpha)}, \\ & x \geq 2n + \alpha + 2\sqrt{n(n+\alpha)} \end{aligned} \tag{66}$$

and after the first iteration of the recurrence we have:

$$\begin{aligned} & 2nh_n^{\alpha}(x) > f(x),\ x \geq 2n^* + \alpha + 2\sqrt{n^*(n^*+\alpha)},\ n^* = n-1, \\ & f(x) = x - (2n+\alpha) + \sqrt{(x-2n^*-\alpha)^2 - 4n^*(n^*+\alpha)}. \end{aligned} \tag{67}$$

This proves that the largest zero of $L_n^{\alpha}(x)$ is smaller than $x^* = 2n + \alpha - 2 + \sqrt{(n-1)(n-1+\alpha)}$, provided that $f(x^*) > 0$, which is true if $\alpha > (n-1)^{-1} - (n-1)$, $n \geq 2$; notice that values $\alpha < -1$ are allowed for large enough n. The bound in [28] is slightly sharper, and is improved in [26].

Further iterations are possible, but not so easy to analyze. The next iteration will give a bound

$$2nh_n^{\alpha}(x) > g(x),\ x \geq 2(n-2)+\alpha+2\sqrt{(n-2)(n-2+\alpha)} = x^* \qquad (68)$$

x^* is an upper bound for the largest zero provided that $g(x^*) > 0$. This condition is met for a larger range of values of α as n becomes larger. For $n \geq 10$, this holds for any $\alpha > -1$. The bound (68) is of more limited validity in terms of n but numerical experiments show that it is sharper than the bound in [26] for $\alpha \leq 12$

We expect that lower bounds for the smallest zero can be also obtained with a similar analysis.

The main message, as before, is that the bounds on function ratios are sharp for large x because they give the correct asymptotic behavior as $x \to +\infty$, but also for moderate x given the sharpness of the bounds on the largest zero.

Competing interests

The author declares that he has no competing interests.

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