Reflexive Polytopes -

Combinatorics

and

Convex Geometry

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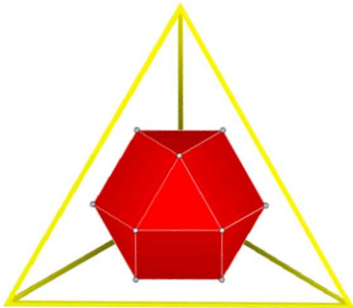


Topics of this talk

- What is a reflexive polytope?
- How many are there?
- What special properties do they have?
- What classification results exist?
- Bounds on invariants as the vertices or the volume?
- What can be said about the set of roots?

What is a reflexive polytope?

Lattice polytopes

Let M, N be dual lattices $\cong \mathbb{Z}^n$, with $\langle \cdot, \cdot \rangle$ the inner product.
We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Def.: A **lattice polytope** $P \subseteq M_{\mathbb{R}}$ is the convex hull of finitely many lattice points in M.

Two lattice polytopes are **isomorphic**, if there is a lattice isomorphism whose real extension maps the polytopes onto each other.

$\mathcal{V}(P)$ denotes the set of **vertices** of P.

The dual polytope

In this talk any lattice polytope P has full dimension n and contains the origin of the lattice in its interior.

The **dual polytope** of P is defined as

$$P^* := \{y \in N_{\mathbb{R}} \,:\, \langle x, y \rangle \geq -1 \;\forall\, x \in P\}.$$

P^* is also fully-dimensional and contains the origin in its interior. We have

$$\boxed{(P^*)^* = P.}$$

There is an inclusion-reversing combinatorial correspondence between the i-dimensional faces of P and the $(n-1-i)$-dimensional faces of P^*.

Reflexive polytopes

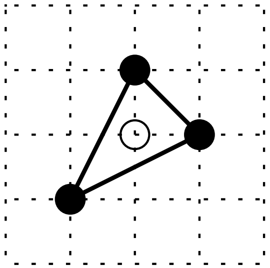
Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope (fully-dim., containing the origin in the interior).

Def.: P is a **reflexive** polytope iff P^* is a lattice polytope.

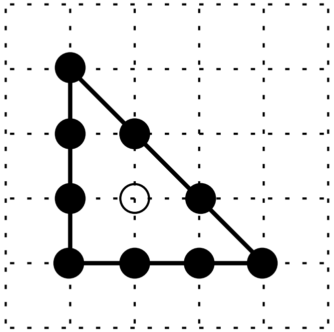
Hence there is a built-in *duality*:

P is **reflexive** iff P^* is **reflexive**

Reflexive:

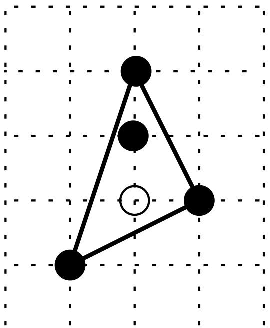


P

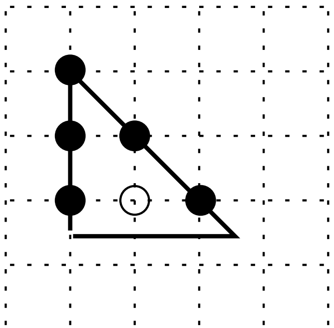


P*

Not reflexive:



P



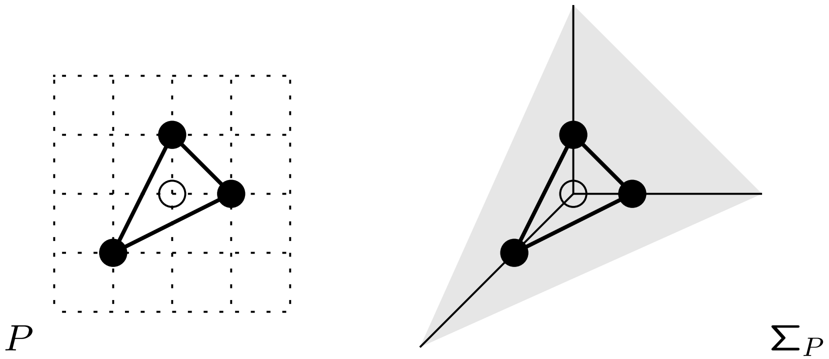
P*

Relevance for related fields

- **Algebraic geometry:**

  Any reflexive polytope P defines a fan Σ_P spanned by the faces of P, and hence an associated toric variety $X(M, \Sigma_P)$. By this construction reflexive polytopes are in 1-1-correspondence (up to isomorphism) with **Gorenstein toric Fano varieties**.

  e.g. $\mathbb{P}^2 = X(M, \Sigma_P)$:



  Here combinatorics can provide direct proofs in toric geometry and new conjectures on general Fano varieties with mild singularities.

- **Mirror symmetry:**

  Any dual pair P, P^* of reflexive polytopes corresponds to a "dual" pair of Gorenstein toric Fano varieties. Batyrev observed that general anticanonical hypersurfaces of a Gorenstein toric Fano variety are Calabi-Yau, and can be resolved to be smooth in up to 4-dim. space. This yields conjectural **Mirror pairs**.

  Here invariants of these Calabi-Yau varieties can be computed from the given reflexive polytope.

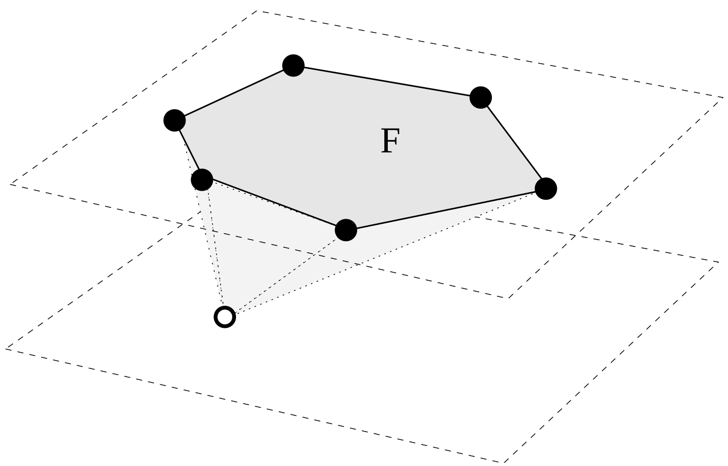
How many are there?

There are only finitely many ...

Thm.: In fixed dimension there are only **finitely** many isomorphism classes of reflexive polytopes.

For this we need the first important property of reflexive polytopes:

Lemma: P is reflexive $\iff$ for any facet F of P there is no lattice point lying between the affine hyperspace spanned by F and its parallel through the origin.



Cor.: The origin is *the only* lattice point in the interior of a reflexive polytope.

So we can apply

Thm.:(Lagarias/Ziegler 91): There are up to lattice isomorphisms only a *finite* number of n-dimensional lattice polytopes containing the origin as the only lattice point in the interior.

For $n = 2$ we find that any such lattice polytope is already reflexive, however this is *not* true for $n \geq 3$.

... but still a lot

While many people already classified the 16 isomorphism classes of reflexive polytopes for $n = 2$, in higher dimensions the use of a computer seems to be compulsory:

Thm.:(Kreuzer/Skarke 97-): There exists an **algorithm** for classifying reflexive polytopes. It yields a **computer database** of 4319 isomorphism classes for $n = 3$, and 473800776 for $n = 4$.

So there are *many* of them!

Thm.:(Haase/Melnikov 04): Any lattice polytope is isomorphic to the face of a reflexive polytope.

Basic properties

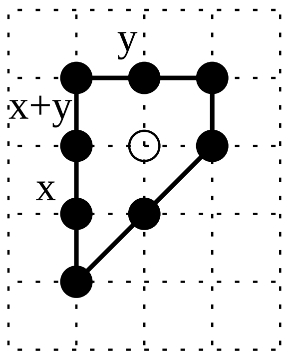
A very special feature

Let P be a reflexive polytope.

Reflexive polytopes exhibit many special behaviors, here we just present one fundamental property:

Prop. There is a **partial addition** on the set of lattice points in P.

If x, y are lattice points on the boundary of P and not contained in a common facet, then $x + y$ is a lattice point in P.



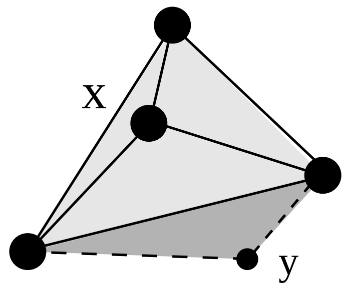
Cor.: One can "walk" from any lattice point on the boundary to any other by using at most three facets.

This yields constraints on the **combinatorics**:

Thm.: Let P be simplicial.

1. The diameter of the edge-graph is at most three.

2. If x is a vertex, then there are at most *three* vertices not lying in the star set of x, i.e., in a facet containing x.



A higher-dimensional classification result

Smooth Fano polytopes

Def.: P is called **smooth Fano polytope**, if P is simplicial and the vertices of any facet form a lattice basis.

Equivalently: $X(M, \Sigma_P)$ is a nonsingular toric Fano variety.

Smooth Fano polytopes were classified by Batyrev et.al. for $n \leq 4$.

As a generalization in $n = 3$ we would like to mention:

Thm.: There are 100 isomorphism classes of reflexive polytopes in dimension three such that any lattice point on the boundary is a vertex.

Equivalently: $X(M, \Sigma_P)$ is a Gorenstein toric Fano variety with terminal singularities.

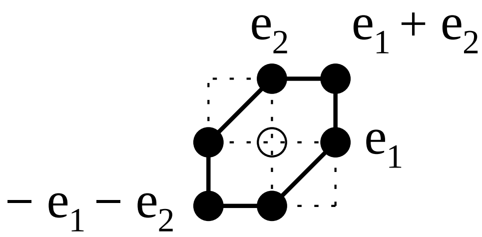
The theorem of Ewald

For this we need some notions:

Def.: Let $e_1, \ldots, e_n$ a lattice basis of M.

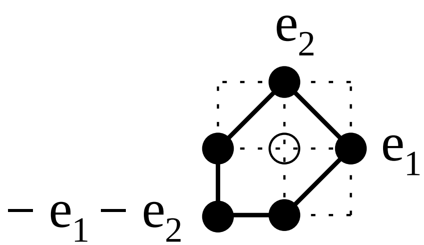
- P is called **centrally-symmetric**, if $-P = P$.
- P is called **del Pezzo polytope**, if n is even and $P \cong \text{conv}(\pm e_1, \ldots, \pm e_n, \pm(e_1 + \cdots + e_n))$.

  e.g. $n = 2$:



- P is called **facet-symmetric**, if there exists a pair of facets F and $-F$ of P.
- P is called **pseudo del Pezzo polytope**, if n is even and $P \cong \text{conv}(\pm e_1, \ldots, \pm e_n, -(e_1 + \cdots + e_n))$.

  e.g. $n = 2$:



Def.: P **splits** into Q,Q', if $P \cong \text{conv}(Q \times \{0\}, \{0\} \times Q')$.

Now we can formulate the following classical result:

Thm.:(Ewald) Any **facet-symmetric smooth Fano polytope splits** into copies of $[-1, 1]$, del Pezzo polytopes or pseudo del Pezzo polytopes.

Recently Casagrande showed that it is enough to have n linearly independent pairs of centrally-symmetric vertices.

The main result

For simplicial reflexive polytopes with few vertices and many symmetries some results had been proven by Ewald and his students (Wirth, Grabert). Here we present an (independent) generalization of the theorem of Ewald:

Thm.: Any **facet-symmetric simplicial reflexive polytope P splits uniquely** into

- copies of $[-1, 1]$, del Pezzo polytopes, pseudo del Pezzo polytopes,
- and a k-dimensional centrally-symmetric simplicial reflexive polytope P' with $2k$ vertices.

(Wirth, N.) Any such reflexive crosspolytope P' can be described by a suitable matrix normal form.

Cor.: For facet-symmetry the **combinatorics** of simplicial reflexive polytopes and smooth Fano polytopes is the same!

Cor.: Let $P \subseteq M_{\mathbb{R}}$ be a facet-symmetric simplicial reflexive polytope.

- Any two facets of P are isomorphic as lattice polytopes.
- P can be embedded in $[-1, 1]^n$.

Remarks:

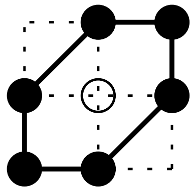
- There is a long-standing conjecture by Ewald that *any* smooth Fano polytope can be embedded in $[-1, 1]^n$.
- Under very mild assumptions one can embed n-dimensional reflexive polytopes into $n(w!)^2\,[-1, 1]^n$, where w is the so-called *width* of P.

Bounds on invariants

Vertices

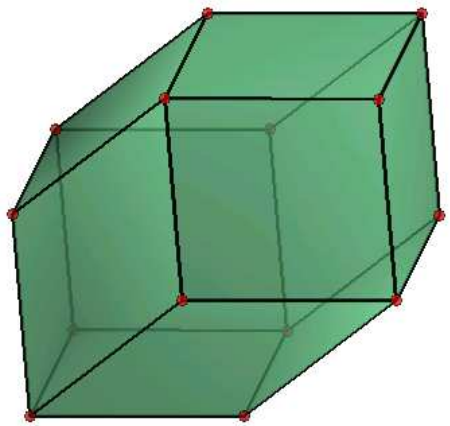
Observations in the computer database on the maximal number of vertices of an n-dimensional reflexive polytope:

- $n = 2$: 6 vertices



$\mathcal{H}$

- $n = 3$: 14 vertices



- $n = 4$: 36 vertices $\qquad \mathcal{H} \times \mathcal{H}$

Let $P \subseteq M_{\mathbb{R}}$ a reflexive polytope.

Conj. A: $|\mathcal{V}(P)| \leq 6^{n/2}$; equ. iff n even and $P \cong \mathcal{H}^{n/2}$.

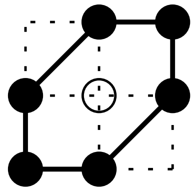
No proof known even for $n = 3$!

Only result known to be valid in any dimension:

Thm.: Conj. A holds for centrally-symmetric simple reflexive polytopes.

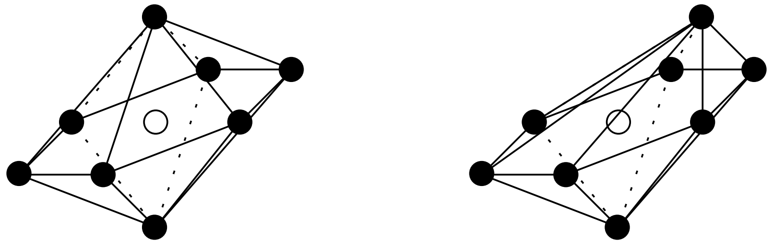
Observations in the computer database on the maximal number of vertices of an n-dimensional **simplicial** *reflexive polytope:*

- $n = 2$: 6 vertices



$\mathcal{H} \cong \mathcal{H}^*$

- $n = 3$: 8 vertices



- $n = 4$: 12 vertices $\quad (\mathcal{H} \times \mathcal{H})^* \cong \text{conv}(\mathcal{H} \times \{0\}, \{0\} \times \mathcal{H})$

Let $P \subseteq M_{\mathbb{R}}$ a simplicial reflexive polytope.

Conj. B: $|\mathcal{V}(P)| \leq 3n$; equ. iff n even and $P^* \cong \mathcal{H}^{n/2}$.

This upper bound was originally conjectured by Batyrev about 15(!) years ago for smooth Fano polytopes.

Thm.:(N. 5/04) If P^* contains a centrally-symmetric pair of vertices, then Conj. B holds.

Thm.:(Casagrande 11/04) Conj. B holds.

Open question for n odd:
Is the toric variety associated to a simplicial reflexive polytope with the maximal number of $3n-1$ vertices necessarily a (special) toric fibre bundle over $\mathbb{P}^1$?

Analyzing and modifying Casagrande's proof yields a generalization:

Cor.: Let P be a reflexive polytope.

Then

$$\boxed{|\mathcal{V}(P)| \leq 2\alpha + (\alpha - n + 1)\beta.}$$

with
α the maximal number of *vertices* of all facets,
β the maximal number of *facets* of all facets.

If P is simplicial, then $\alpha = n = \beta$, hence $|\mathcal{V}(P)| \leq 3n$.

Volume and lattice points

For any n there is a **special reflexive simplex** S_n defined using the Euclid/Sylvester sequence $2, 3, 7, 43, \ldots$ that is conjectured to have for $n \geq 4$ **the largest volume** and **the most lattice points** solely among all reflexive polytopes.

The following affirmative results could be proven:

Thm.:

- S_n has for $n \geq 4$ among all reflexive simplices solely the largest volume.
- S_n has for $n \geq 2$ among all reflexive simplices solely the most lattice points on an edge.

The proof depends on number-theoretic results on n-tuples $k_0, \ldots, k_n$ with

$$\frac{1}{k_0} + \cdots + \frac{1}{k_n} = 1.$$

Roots

The set of roots

Let $P \subseteq M_{\mathbb{R}}$ a reflexive polytope.

Def.:

- The set $\mathcal{R}$ of lattice points in interior of facets of P is called the set of **roots** of P.
- $\mathcal{S} := \mathcal{R} \cap (-\mathcal{R})$ is the set of **semisimple** roots.

$\mathcal{R}$ is the set of Demazure roots of the projective toric variety

$$X_P := X(N, \Sigma_{P^*})$$

associated to the dual polytope. We have

$$\dim \mathrm{Aut}(X_P) = n + |\mathcal{R}|.$$

Two questions with answers

1. *What is the maximal number of facets containing roots?*

   Showing the existence of special 'orthogonal' families of roots yields:

   Thm.:
   There are at most $2n$ facets containing roots;
   equality implies $P \cong [-1,1]^n$.

   As an application (of calculating modulo 3) we can prove:

   Thm.:
   If P is centrally-symmetric, then $|\,P \cap M\,| \leq 3^n$;
   equality implies $P \cong [-1,1]^n$.

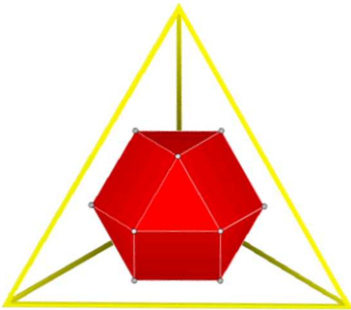
2. *What is the maximal number of semisimple roots?*

Prop.: $|\mathcal{S}| \leq n^2 + n$; equality implies $X_P \cong \mathbb{P}^n$.

Moreover we can prove the following structure theorem by
using extensively the partial addition on P:

Thm.: The intersection P' of P with the linear subspace generated by $\mathcal{S}$ is again a reflexive polytope s.t. $X_{P'}$ is a product of projective spaces.

The convex hull of semisimple roots forms again a reflexive polytope,
e.g., for $X_P = \mathbb{P}^3$:



Semisimple reflexive polytopes

Def.: We say P is **semisimple**, if any root is semisimple.

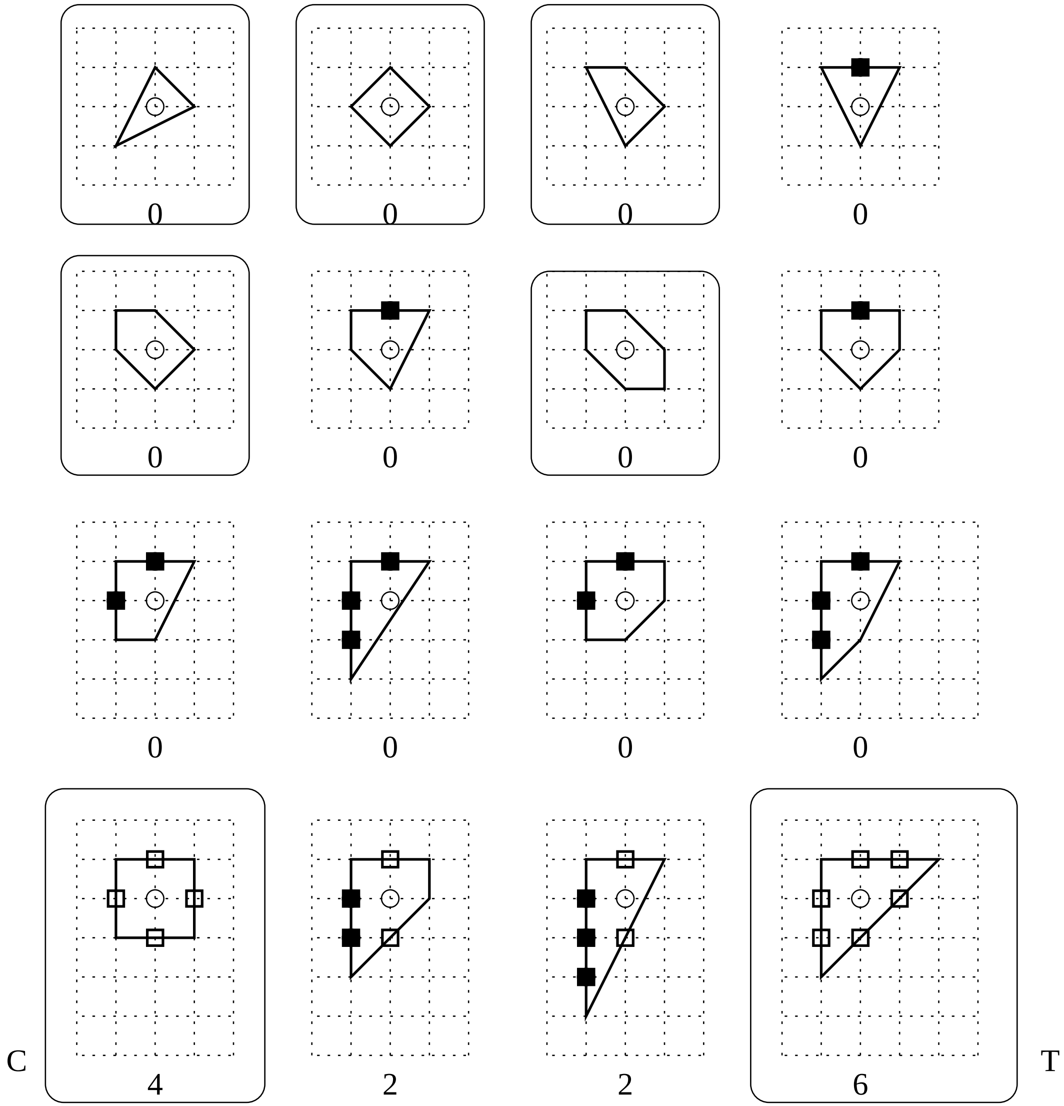
Equivalently: Aut(X_P) is a reductive algebraic group.

Here we are interested in the following two problems:

1. Finding constraints on the number of roots of a semisimple polytope
2. Finding sufficient criteria for semisimplicity

Semisimple reflexive polygons with roots:

There is no semisimple reflexive polygon with precisely 2 roots!



(sqares=roots, white squares=semisimple roots)

$X_C \cong \mathbb{P}^1 \times \mathbb{P}^1$
(most facets with roots)

$X_T \cong \mathbb{P}^2$
(most semisimple roots)

1. A sharp upper bound on the number of roots:

Thm.: Let X_P be <u>not</u> a product of projective spaces . Then

$$P \text{ is semisimple} \Rightarrow |\mathcal{R}| \leq \begin{cases} 0 & , \quad n = 2 \\ n^2 - 3n + 4 & , \quad n \geq 3 \end{cases}$$

In particular a semisimple reflexive polygon P

- either has no roots (i.e., P is a smooth Fano polytope)
- or has 4 roots and is isomorphic to C (i.e., $X_P \cong \mathbb{P}^1 \times \mathbb{P}^1$)
- or has 6 roots and is isomorphic to T (i.e., $X_P \cong \mathbb{P}^2$)

Note: These results have been generalized to complete toric varieties X by using results of Cox et.al. on the homogeneous coordinate ring and Aut(X).

2. Sufficient criteria for semisimplicity

Let $P \subseteq M_{\mathbb{R}}$ a reflexive polytope.

Thm.: Let X_P be a *nonsingular* toric Fano variety.

X_P is symmetric, i.e., $\text{Aut}_M(P)$ has no non-trivial fixpoints

$\Rightarrow$ (Batyrev/Selivanova 99)

X_P admits an Einstein-Kähler metrics

$\Longleftrightarrow$ (Futaki 83, Mabuchi 87, Wang/Zhu 03)

the barycenter of P is the origin

$\Rightarrow$ (Matsushima 57)

$\text{Aut}(X_P)$ is reductive (i.e., P is **semisimple**)

Questions:
· Combinatorial proofs for **combinatorial** implications?
· Also true in the reflexive case in general?

Thm.:

P is *semisimple*, if one of the following conditions holds:

· the barycenter of P is the origin,

· the barycenter of P^* is the origin,

· the sum over all lattice points of P is the origin,

· the sum over all lattice points of P^* is the origin,

· the sum over all vertices of P^* is the origin,

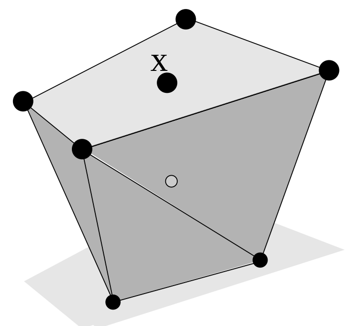
· all facets of P have the same number of lattice pts,

· all facets of P have the same volume.

The proof is purely convex-geometric.
Here is the basic idea for the proof of the first condition:

Prop. When projecting P along a lattice pt. x on the boundary it is enough to consider the star set of x, i.e., the union of the facets containing x.

Hence for a root x we get the following picture:



References

- Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Algebr. Geom. **3**, 493-535 (1994)
- Casagrande, C.: The number of vertices of a Fano polytope. Preprint, math.AG/0411073 (2004)
- Haase, C.; Melnikov, I.V.: The reflexive dimension of a lattice polytope. Preprint, math.CO/0406485 (2004)
- Kreuzer, M.; Skarke, H.: PALP: A package for analyzing lattice polytopes with applications to toric geometry. Computer Phys. Comm., **157**, 87-106 (2004)
- Nill, B.: Gorenstein toric Fano varieties. Manuscr. Math. **116**, 183-210 (2005)
- Nill, B.: Complete toric varieties with reductive automorphism group. Preprint, math.AG/0407491 (2004)
- Nill, B.: Volume and lattice points of reflexive simplices. Preprint, math.AG/0412480 (2004)