Counter-examples to the Hirsch conjecture

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Polyhedra and polytopes

Definition

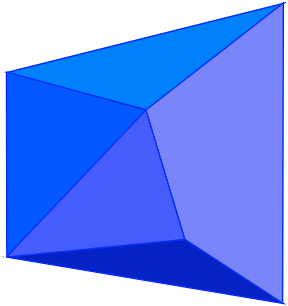
A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.



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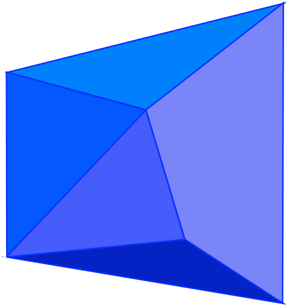
Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, but not conversely.



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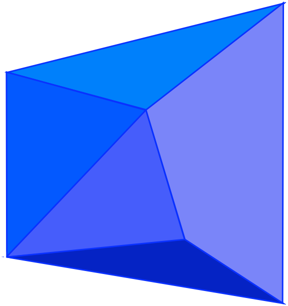

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Faces of P

Let P be a polytope (or polyhedron) and let

$$H = \{x \in \mathbb{R}^d : a_1 x_1 + \cdots a_d x_d \leq a_0\}$$

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If $P \subset H$ we say that $\partial H \cap P$ is a face of P.

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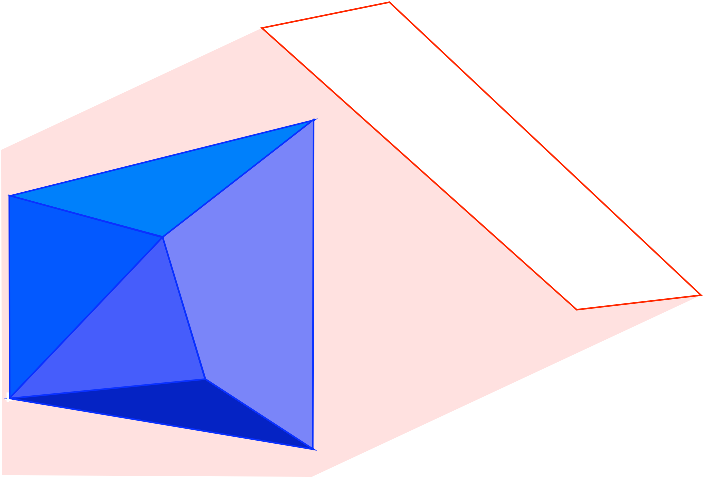
Faces of P

The “empty face” of P.



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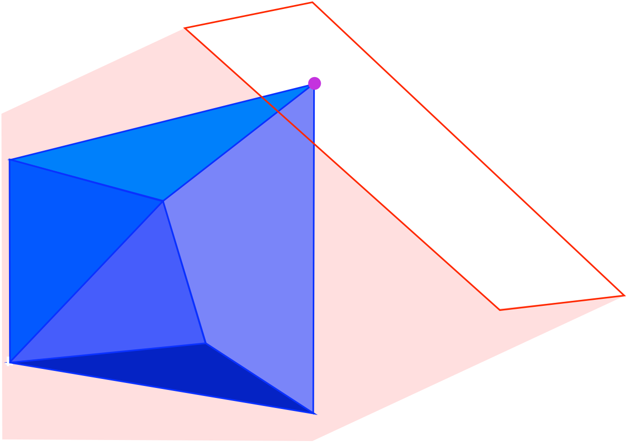
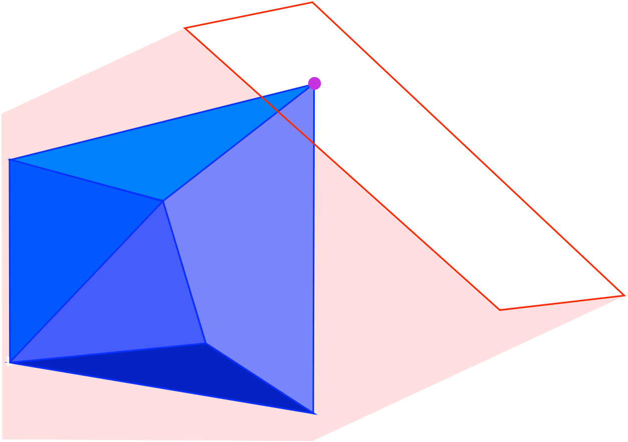


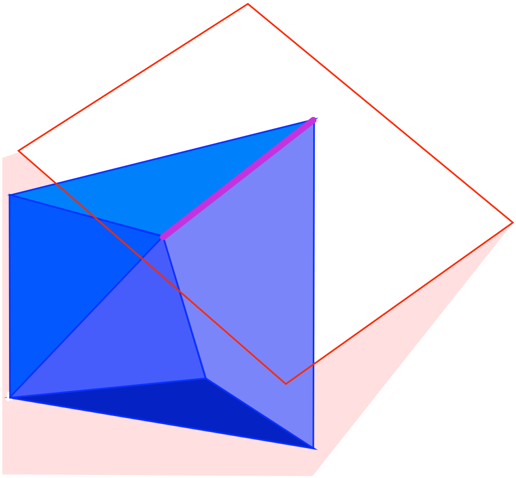
Faces of P

Faces of dimension 0 are called vertices.



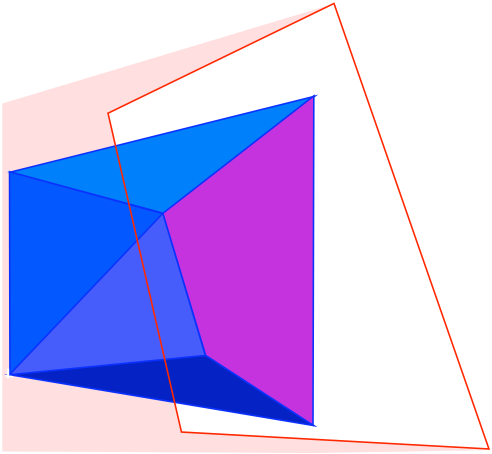
Faces of P

Faces of dimension 1 are called edges.



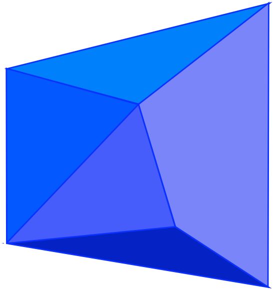
Faces of P

Faces of dimension $d-1$ (codimension 1) are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a (finite, undirected) graph.



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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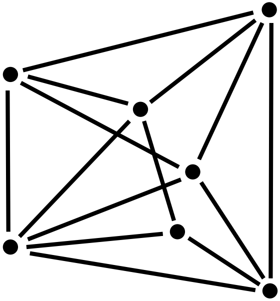

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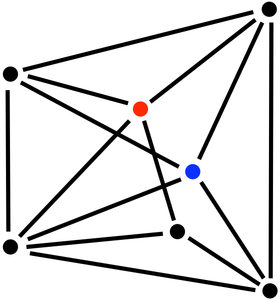

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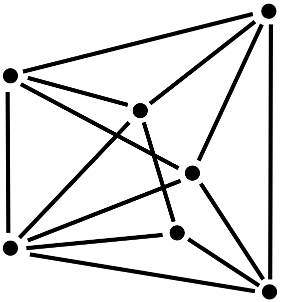

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) := \max\{d(u,v) : u,v \in V\}.$$

The Hirsch conjecture

Let $\delta(P)$ denote the diameter of the graph of a polytope P.

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

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Fifty three years later...

Theorem (S. 2010+)

There is a 43*-dim. polytope with* 86 *facets and diameter* $\geq$ 44*.*

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Corollary

There is an infinite family of non-Hirsch polytopes with diameter $\sim (1+\epsilon)n$, even in fixed dimension. (Best so far: $\epsilon = 1/20$).

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Remark

To this day, we do not know any polynomial upper bound for $\delta(P)$, in terms of n and d (polynomial Hirsch Conjecture)

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A **quasi**-polynomial bound, and a bound in fixed dimension

Theorem [Kalai-Kleitman 1992]

For every d-polytope with n facets:

$$\delta(P) \leq n^{\log_2 d+2}.$$

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Motivation: linear programming

A linear program is the problem of maximization (or minimization) of a linear functional subject to linear inequality constraints. That is: finding $\max\{c \cdot x : x \in \mathbb{R}^d, Mx \leq b\}$ for given $c \in \mathbb{R}^d, b \in \mathbb{R}^n, M \in \mathbb{R}^{d \times n}$.

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> *"If one would take statistics about which* **mathematical problem** *is using up* **most of the computer time in the world***, then (not including database handling problems like sorting and searching) the answer would probably be linear programming."*
>
> (László Lovász, 1980)

Connection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets and d dimensions.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves linear programming by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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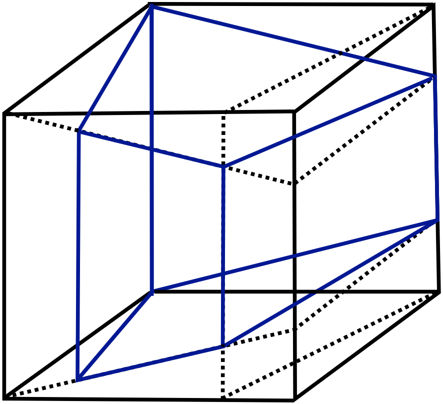

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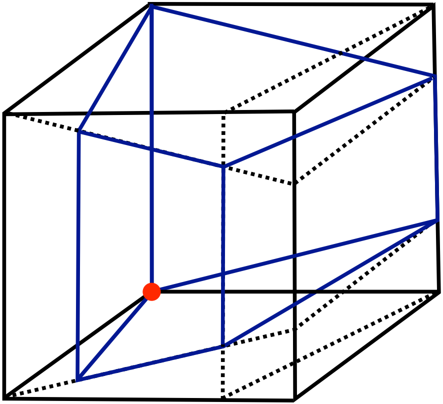

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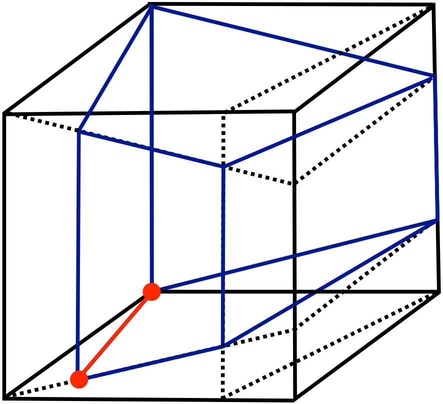

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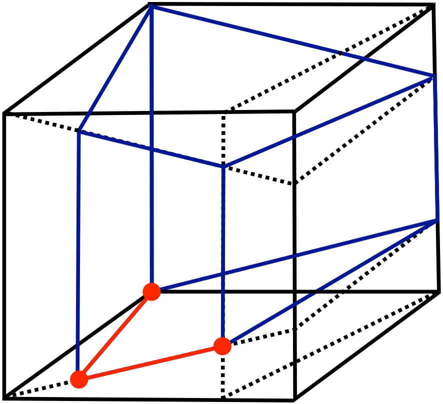

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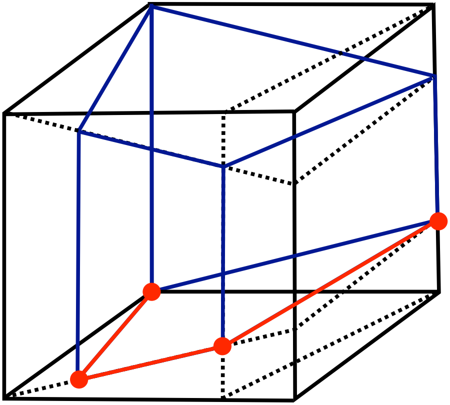

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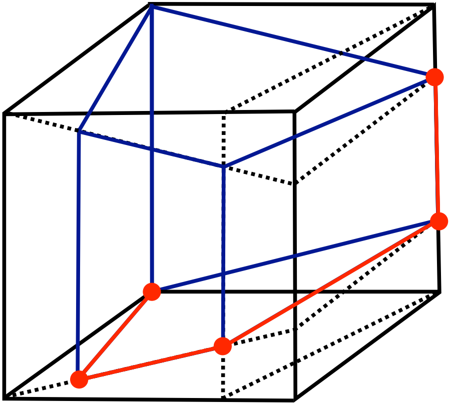

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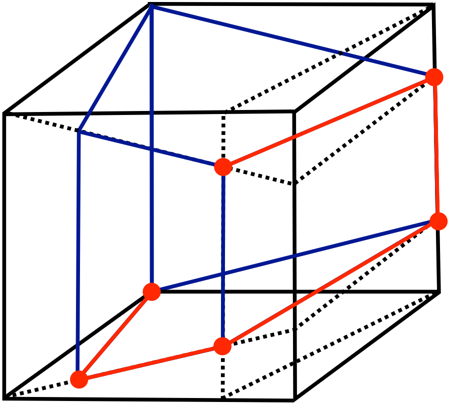

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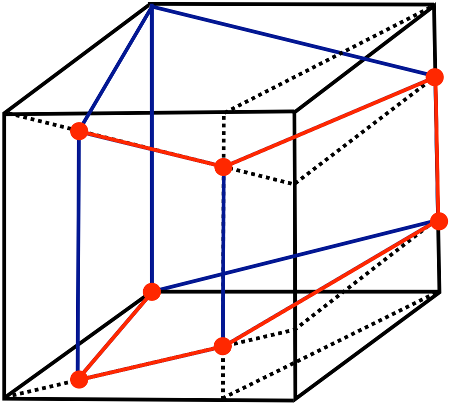

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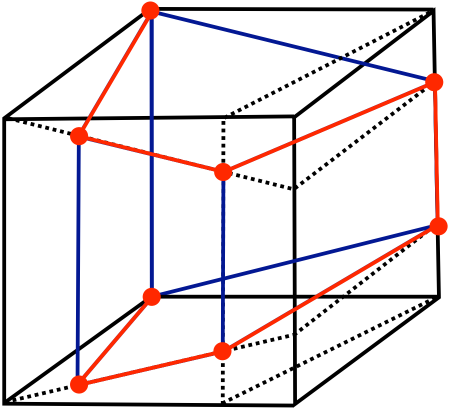

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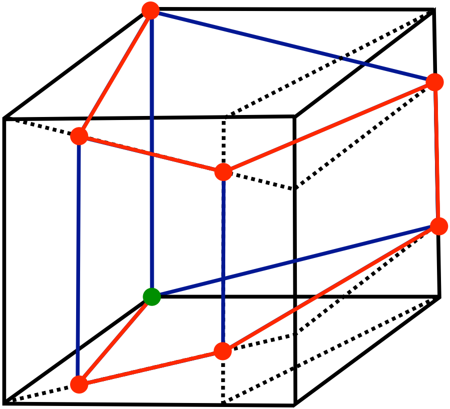

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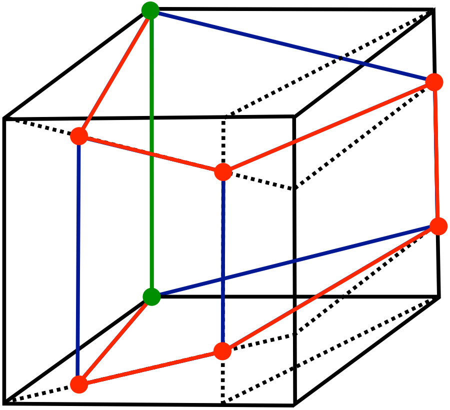

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The number of pivot steps [that the simplex method takes] to solve a problem with m equality constraints in n nonnegative variables is almost always at most a small multiple of m, say $3m$.

The simplex method has remained, if not the method of choice, a method of choice, usually competitive with, and on some classes of problems superior to, the more modern approaches.

(M. Todd, 2011)

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Besides, the known polynomial algorithms for linear programming known are not *strongly polynomial*: They are polynomial in the bit model of complexity (Turing machine) but not in the arithmetic model (real RAM machine).

Finding strongly polynomial algorithms for linear programming is one of the "mathematical problems for the 21st century" according to [Smale 2000]. A polynomial pivot rule would solve this problem in the affirmative.

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- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- It holds for all 0-1 polytopes [Naddef 1989] and for 3-polytopes [Klee 1966].
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

  For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).

- For every $n > d$, it is easy to construct unbounded polyhedra where the bound is tight.

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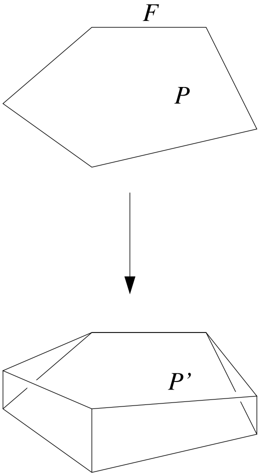
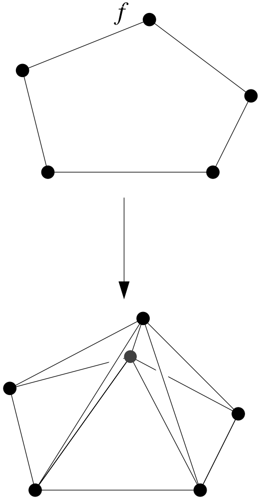
Wedging, a.k.a. one-point-suspension





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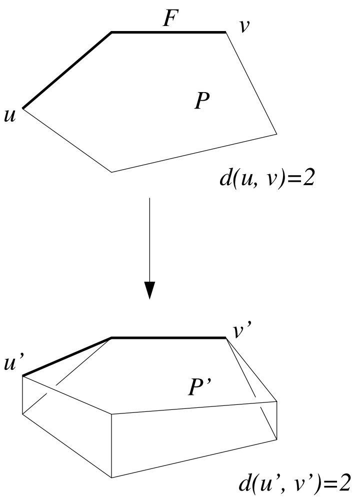

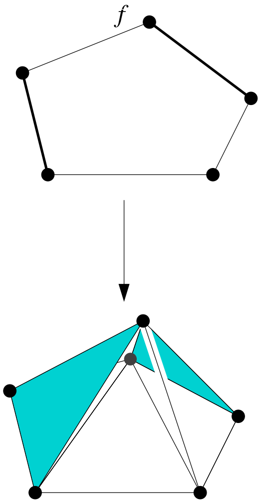

So, the d-step Theorem is based in the following lemma:

Lemma

Let P be a polytope of dimension d with and diameter λ. Then there is another polytope P' of dimension $d+1$, with $n+1$ facets and diameter λ.

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Our construction of counter-examples has two ingredients:

1. A strong d-step theorem for spindles/prismatoids.
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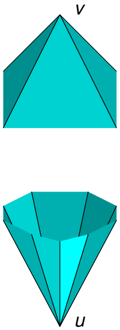
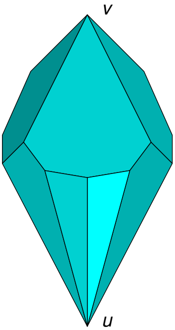
Spindles and prismatoids

Definition

A spindle is a polytope P with two distinguished vertices u and v such that every facet contains either u or v.





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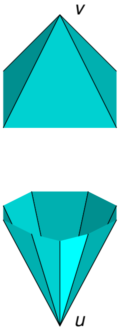

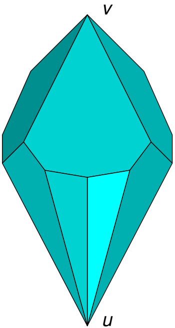

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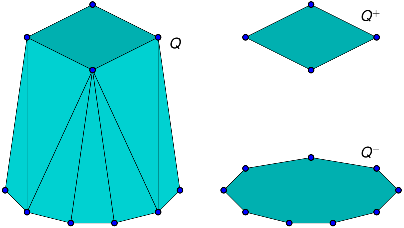
A prismatoid is a polytope Q with two facets Q^+ and Q^- containing all vertices.



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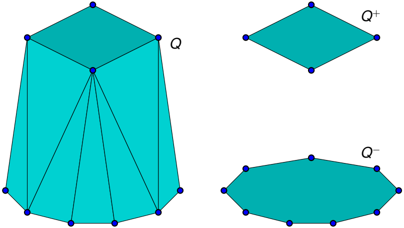

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The strong d-step Theorem

Theorem (Strong d-step, spindle version)

Let P be a spindle of dimension d, with $n > 2d$ facets, and with length δ. Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and with length $\delta+1$.

That is: we can increase the dimension, number of facets *and length* of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \delta + n - 2d > n - d$) that violates the Hirsch conjecture.

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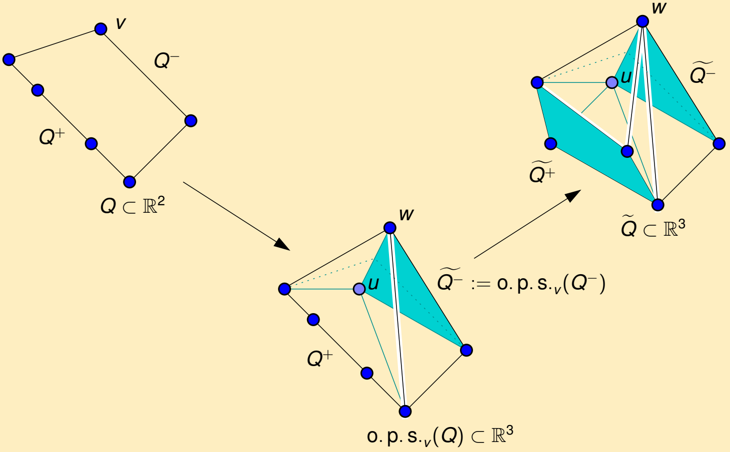

□

A 5-pismatoid of width six

Let Q be the polytope having as vertices the 48 rows of the following matrices:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} \pm 18 & 0 & 0 & 0 & 1 \\ 0 & \pm 18 & 0 & 0 & 1 \\ 0 & 0 & \pm 45 & 0 & 1 \\ 0 & 0 & 0 & \pm 45 & 1 \\ \pm 15 & \pm 15 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & \pm 30 & 1 \\ 0 & \pm 10 & \pm 40 & 0 & 1 \\ \pm 10 & 0 & 0 & \pm 40 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \pm 18 & -1 \\ 0 & 0 & \pm 18 & 0 & -1 \\ \pm 45 & 0 & 0 & 0 & -1 \\ 0 & \pm 45 & 0 & 0 & -1 \\ 0 & 0 & \pm 15 & \pm 15 & -1 \\ \pm 30 & \pm 30 & 0 & 0 & -1 \\ \pm 40 & 0 & \pm 10 & 0 & -1 \\ 0 & \pm 40 & 0 & \pm 10 & -1 \end{pmatrix} \end{array}$$

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\pm18 & 0 & 0 & 0 & 1\\
0 & \pm18 & 0 & 0 & 1\\
0 & 0 & \pm45 & 0 & 1\\
0 & 0 & 0 & \pm45 & 1\\
\pm15 & \pm15 & 0 & 0 & 1\\
0 & 0 & \pm30 & \pm30 & 1\\
0 & \pm10 & \pm40 & 0 & 1\\
\pm10 & 0 & 0 & \pm40 & 1
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
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$$
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Theorem

The prismatoid Q of the previous slide has width six.

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

A 5-pismatoid of width six

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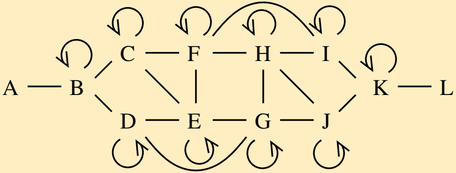
Proof 1 of the Theorem.

It has been verified with `polymake` that the dual graph of Q has the following structure:

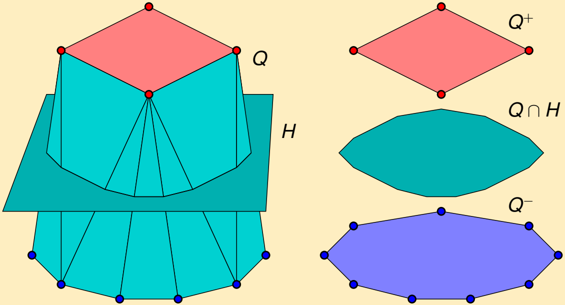


$\square$

Combinatorics of prismatoids

Proof 2 of the Theorem (idea).

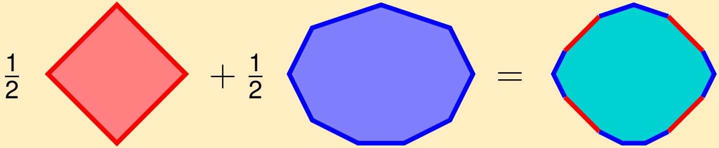
Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice …



Combinatorics of prismatoids

Proof 2 of the Theorem (idea).

…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.



Combinatorics of prismatoids

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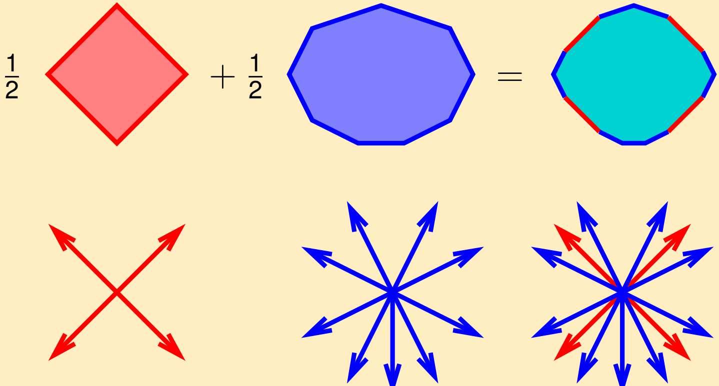
…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the "superposition" of those of Q^+ and Q^-.



Combinatorics of prismatoids

So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

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4-prismatoids $\Leftrightarrow$ pairs of maps in the 2-sphere.
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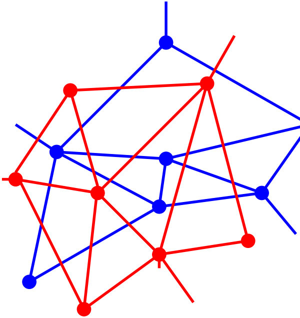

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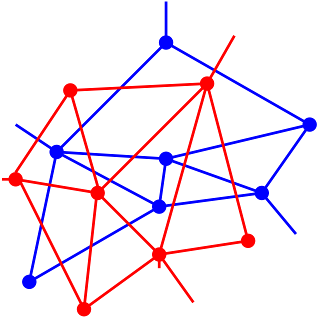

4-prismatoid of width > 4

$\Updownarrow$

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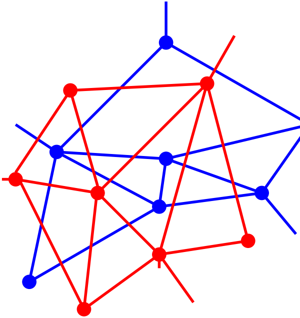


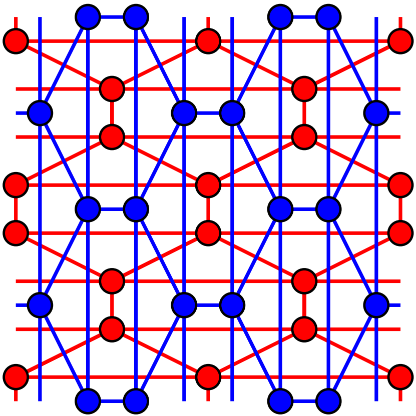
5-prismatoid of width > 5

$\Updownarrow$

pair of (geodesic, polytopal) maps in S^3 so that *three* steps do not let you go from a blue vertex to a red vertex.

A 4-dimensional prismatoid of width > 4?

Replicating the following basic structure we can get a "non-Hirsch" periodic pair of maps in the plane:



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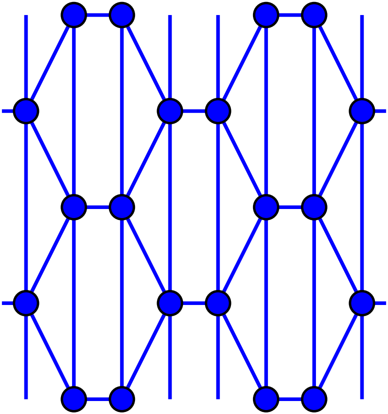

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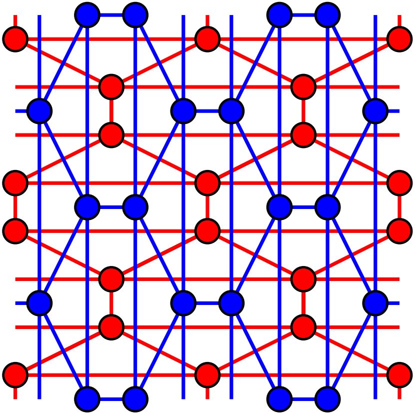

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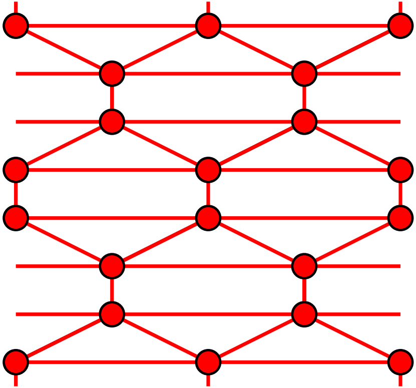

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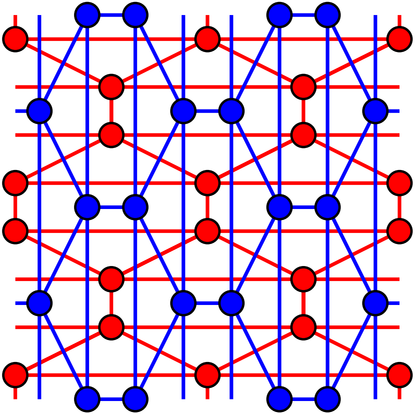

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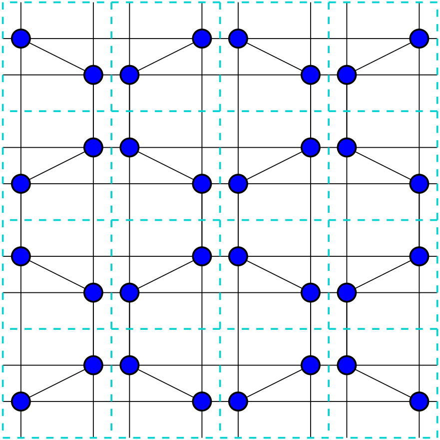
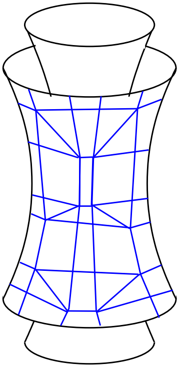
If this drawing was on a 2-sphere it would represent a 4-prismatoid of width 5.

This does not work, but putting the drawing in (two tori embedded in) S^3 does, and gives a prismatoid with 48 vertices.

A 5-prismatoid of width > 5





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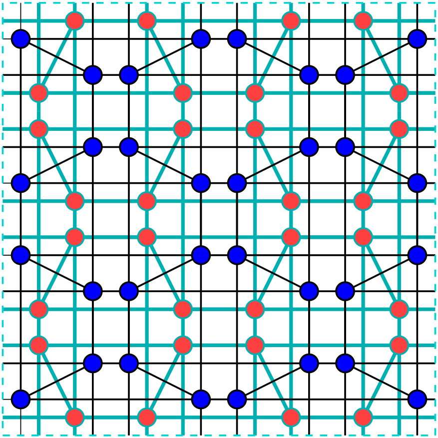

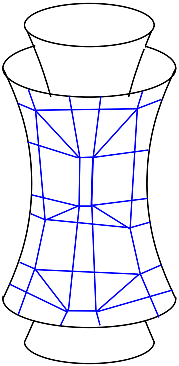

Smaller counter-examples

There are two ways in which a smaller non-Hirsch polytope could be obtained:

- Find a smaller 5-prismatoid of width > 5, or
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The latter is impossible:

Theorem (S.-Stephen-Thomas 2011)

Every prismatoid of dimension four has width ≤ 4.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1  1        0       0    21      0         0         0         0            0             0             0             0             0  0  0  1  1  0  0  0
1  1        0       0    16     -8         0         0         0            0             0             0             0             0  0  0  0  1  1  0  0
1  1        0       0     0    -25         0         0         0            0             0             0             0             0  0  0  0  0  1  1  0
1  1        0       0   -16    -10         0         0         0            0             0             0             0             0  0  0  0  0  0  1  1
1  1        0       0   -21      0         0         0         0            0             0             0             0             0  0  0  0  0  0  0  1
1  1        0       0   -20      3         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1  1        0       0     0     39         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1  1        0       0    19      7         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1  1    3/100   -1/50     0    -23         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1  1   -3/100   -1/50     0     37         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1  1  -3/2000  7/2000     0 388/10         0         0         0            0             0             0             0             0  0  1  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000  100000000000  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10 -10000000         0         0            0             0             0             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000 -10000000         0            0             0             0             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000 -10000000            0             0             0             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000 -10000000000             0             0             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000  10000000000 -100000000000             0             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000 -100000000000             0             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000 -100000000000             0  1  0  0  0  0  0  0
1  1   3/2000  7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000 -100000000000  1  0  0  0  0  0  0
1 -1       30       0     0      0         0         0         1            1             0             0             0             0  0  0  0  0  0  0  0
1 -1        4     -15     0      0         0         0         0            1             1             0             0             0  0  0  0  0  0  0  0
1 -1        0   -33/2     0      0         0         0         0            0             1             1             0             0  0  0  0  0  0  0  0
1 -1       -1     -16     0      0         0         0         0            0             0             1             1             0  0  0  0  0  0  0  0
1 -1    -55/2       0     0      0         0         0         0            0             0             0             1             1  0  0  0  0  0  0  0
1 -1      -17      18     0      0         0         0         0            0             0             0             0             1  0  0  0  0  0  0  0
1 -1        0      38     0      0         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1 -1       22      17     0      0         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1 -1      -10       0   1/5   -1/5         0         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1 -1 2999/100       0 -3/25   -1/5         0         0         1            0             0             0             0             0  0  0  0  0  0  0  0
1 -1 299999/10000 0       0  1/100         0         1         0            0             0             0             0             0  0  0  0  0  0  0  0
1 -1 -2745/100      0 1/5000 1/800         1         0         0            0             0             0             0             0  0  0  0  0  0  0  0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000  10000000  10000000  100000000  100000000  1000000000
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0  -100000         0         0         0          0          0           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000 -10000000         0         0          0          0           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000 -10000000         0          0          0           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000  10000000 -10000000          0          0           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000  10000000  10000000 -100000000          0           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000  10000000  10000000  100000000 -100000000           0
1 -1      -27       0 1/500  -1/80         0         0         0            0             0             0             0             0   100000  10000000  10000000  10000000  100000000  100000000 -1000000000
end
allbases
printcobasis
```

Asymptotic width in fixed dimension

If we fix the dimension d, the width of prismatoids is linear:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

Proof.

This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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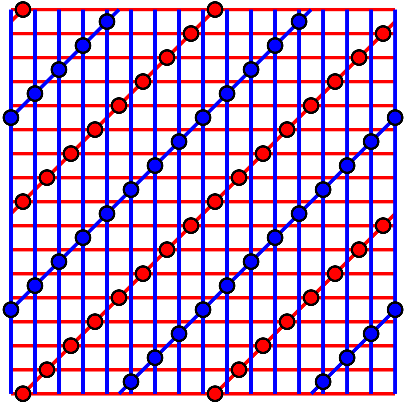

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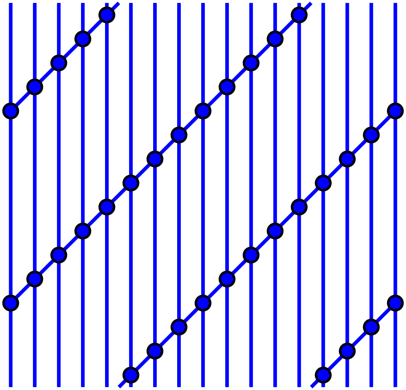

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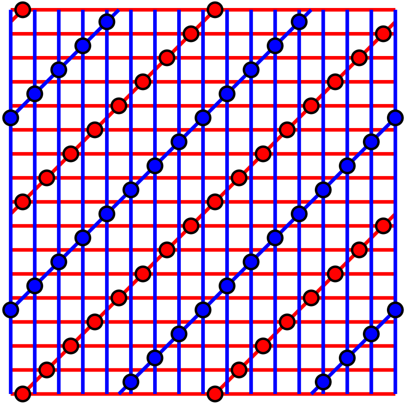

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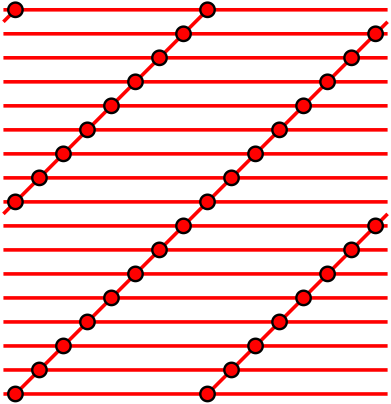

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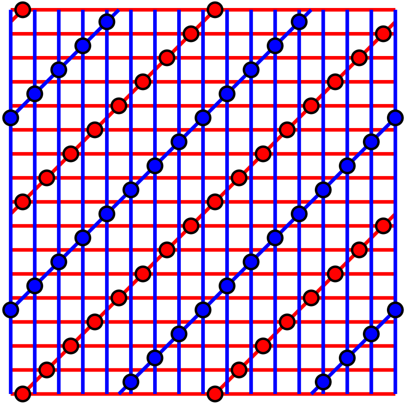

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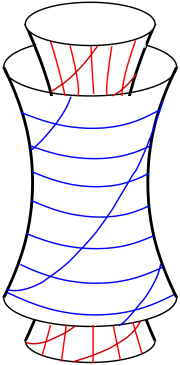
Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps.

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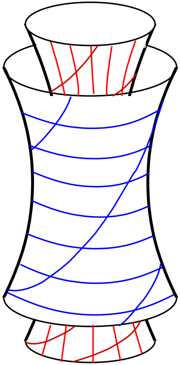

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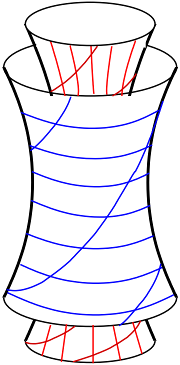

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- This breaks a “psychological barrier”, but for applications it is absolutely irrelevant.

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The diameter of a d-polytope with n-facets cannot exceed

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In fact, this conjecture is posed in a much more general setting (*connected layer families*, in the sense of Eisenbrand-Hähnle-Razborov-Rothvoss) which would include, for example, all polyhedral manifolds.

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Still, finding polytopes with diameter exceeding, say, $2(n-d)$ would be a breakthrough.

Conclusion

A proposal for the "next step":

Conjecture (Hähnle, 2010)

The diameter of a d-polytope with n-facets cannot exceed

$$d(n-d).$$

In fact, this conjecture is posed in a much more general setting (*connected layer families*, in the sense of Eisenbrand-Hähnle-Razborov-Rothvoss) which would include, for example, all polyhedral manifolds.

Still, finding polytopes with diameter exceeding, say, $2(n-d)$ would be a breakthrough.

The end

THANK YOU!